ON THE ANALYSIS OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS IN THE REGULAR CASE

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Summary

Analysis of a regular partially balanced incomplete block design is investigated in connection with its association algebra and relationship algebra. Properties of the association algebra and the relationship algebra are summarized in so far as they are useful for the later discussions. Although the properties of the association algebra have been known already in another expression,* those of the relationship algebra of a PBIBD are

*Bose, R. C. and D. M. Mesner (1959) On linear associative algebra corresponding to association schemes of partially balanced designs, Ann. Math. Stat. 30 21-38.

believed to be new. Partitions of the sum of squares due to treatments adjusted pertinent to the association being considered are given, and these are supposed to be new. As examples, explicit expressions of the partitions are given for PBIBD's of certain types. Finally a numerical example is presented for illustration.

§1. Partially balanced incomplete block design. For the sake of reader's convenience, we gave a brief description of a partially balanced incomplete block design. Reference should be made to [1], [2], [3], and [7].

Given v treatments $\varphi_1, \varphi_2, \ldots, \varphi_v$, a relation among them satisfying the following 3 conditions is called an association with m associate classes:

(a) any two treatments are either 1st, or 2nd, ..., or m-th associates,

(b) each treatment has n_i i-th associates, $i = 1, 2, \ldots, m$, and

(c) for each pair of treatments which are i-th associates, there are p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$ treatments which are j-th associates of the one treatment of the pair and at the same time k-th associates of the other.

We have a partially balanced incomplete block design--PBIBD in short, if there are b blocks each containing k experimental units in such a way that

(1) each block contains k $(\leq v)$ different treatments,

(2) each treatment occurs in r blocks, and

(3) any two treatments which are i-th associates occur together in λ_i blocks, $i = 1, 2, \ldots, m$.

In a degenerate case when m = 1, a PBIBD reduces to a BIBD. Certain cases in which m = 2 have been useful in practical applications.

Parameters describing an association are

$$v,\ n_i\ (i = 1, 2, \ldots, m),\ p^i_{jk}\ (i, j, k = 1, 2, \ldots, m)$$

and additional design parameters are

$$b, r, k, \lambda_i \ (i = 1, 2, \ldots, m).$$

It should be that

(1.1) $$n_1 + n_2 + \ldots + n_m = v-1 ,$$

and

(1.2) $$p^i_{jk} = p^i_{kj} \text{ (symmetry with respect to subscripts).}$$

Further it can be shown that

(1.3) $$\sum_{k=1}^{m} p^i_{jk} = n_j - \delta_{ij} ,$$

and

(1.4) $$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij} ,$$

where δ_{ij} stands for the Kronecker delta.

There are $r(k-1)$ treatments occuring in the blocks in which a fixed treatment φ occurs and they are classified into m associate classes with respect to φ. On the other hand, since there are n_i i-th associates of φ occuring in λ_i blocks, it follows that

(1.5) $$n_1\lambda_1 + n_2\lambda_2 + \ldots + n_m\lambda_m = r(k-1) .$$

A treatment may be regarded as the 0-th associate of its own. Thus we add the following conventional notations:

$$n_o = 1, \qquad \lambda_o = r ,$$

(1.6) $$p^o_{jk} = n_j\delta_{jk}, \ p^i_{ok} = p^i_{ko} = \delta_{ik} .$$

Under these notations, we have the following relations

(1.7) $$\sum_{i=o}^{m} n_i = v,$$

(1.8) $$\sum_{k=o}^{m} p^i_{jk} = n_j,$$

and

(1.9) $$\sum_{i=0}^{m} n_i \lambda_i = rk \; .$$

Let A_o be the unit matrix of order v. Also let A_i be a symmetric matrix of order v such that its element $a_{\alpha i}^{\beta}$ in the α-th row and in the β-th column is 1 if φ_α and φ_β are i-th associates, and is 0 otherwise, i.e.,

(1.10) $$A_i = \begin{Vmatrix} a_{1i}^{1} & a_{1i}^{2} & \cdots\cdots & a_{1i}^{v} \\ a_{2i}^{1} & a_{2i}^{2} & \cdots\cdots & a_{2i}^{v} \\ \cdots & \cdots & \cdots & \cdots \\ a_{vi}^{1} & a_{vi}^{2} & \cdots\cdots & a_{vi}^{v} \end{Vmatrix} \; , \; i = 1, 2, \ldots, m,$$

where

$$a_{\alpha i}^{\beta} = \begin{cases} 1, \text{ if } \varphi_\alpha \text{ and } \varphi_\beta \text{ are i-th associates} \\ 0, \text{ otherwise.} \end{cases}$$

$A_o, A_1, A_2, \ldots, A_m$ are called the <u>association matrices</u>. It can be seen that

(1.11) $$A_o + A_1 + A_2 + \ldots + A_m = G_v$$

where G_v stands for the square matrix of order v whose elements are all unity. Hence $A_o, A_1, A_2, \ldots, A_m$ are linearly independent with respect to the field of all real numbers.

Furthermore we have

(1.12) $$A_j A_k = A_k A_j = \sum_{i=0}^{m} p_{jk}^{i} A_i, \; (j, k = 0, 1, 2, \ldots, m)$$

Thus the linear closure of the matrix set $\{A_o, A_1, A_2, \ldots, A_m\}$ with respect to the field of all real numbers is a linear associative and commutative algebra $\mathfrak{A}$, called the <u>association algebra</u>. The abstract counterpart of the matrix algebra $\mathfrak{A}$ is denoted by $\mathcal{O}\!\iota$.

Let

(1.13) $$\rho_k = \left\| \begin{matrix} p^{o}_{ok} & p^{1}_{ok} & \cdots\cdots & p^{m}_{ok} \\ p^{o}_{1k} & p^{1}_{1k} & \cdots\cdots & p^{m}_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ p^{o}_{mk} & p^{1}_{jk} & \cdots\cdots & p^{m}_{mk} \end{matrix} \right\| , \quad k = 0, 1, 2, \ldots, m,$$

then (1.12) may be rewritten as follows:

(1.14) $$\begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} A_k = \rho_k \begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} , \qquad k = 0, 1, 2, \ldots, m.$$

Thus it follows immediately that

(1.15) $$\begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} (A_i + A_k) = (\rho_i + \rho_k) \begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} ,$$

and

(1.16) $$\begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} A_i A_k = \rho_i \rho_k \begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix}$$

In other words, the mapping of $\mathfrak{A}$ into the ring of all matrices of order (m+1) generated by

(1.17) $$A_i \longrightarrow \rho_i \quad , \qquad i = 0, 1, 2, \ldots, m$$

gives the <u>regular representation</u> ($\mathfrak{R}$) of the association algebra $\mathfrak{A}$.

§2. Properties of the association algebra. The abstract algebra $\mathfrak{A}$ is completely reducible in the field of all rational numbers [10], hence it is completely reducible in any number field.

On the other hand, Shur's lemma [11] shows us that any irreducible representation of a commutative algebra in an algebraically closed number field must be linear. Hence any irreducible representation of a commutative matrix algebra in a field containing all characteristic roots of the matrices must be linear.

From the general theory of algebra [12], we know that any representation of a completely reducible algebra decomposes into irreducible representations, each of which is equivalent to one of the irreducible constituents of the regular representation of the algebra.

Since the rank of $\mathfrak{A}$ is m+1, the regular representation $(\mathfrak{A})$ decomposes into m+1 inequivalent and linear representations in the field of all complex numbers. Since these linear representations are the characteristic roots of symmetric matrices, they must be all real. Thus the regular representation $(\mathfrak{A})$ decomposed into m+1 inequivalent and linear representations in the field of all real numbers. Even more, if the characteristic roots of all association matrices are rational as in the cases which will be considered in the next section, then $(\mathfrak{A})$ decomposes into m+1 inequivalent and linear representations in the field of all rational numbers.

On account of the fact that

$$(2.1) \qquad A_k G_v = G_v A_i = n_i G_v, \qquad i = 1, 2, \ldots, m$$

we can choose a non-singular matrix C of order m+1 in the field of all real numbers, being of the form

$$(2.2) \qquad C = \begin{Vmatrix} c_{oo} & c_{o1} & \cdots & c_{om} \\ c_{1o} & c_{11} & \cdots & c_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{mo} & c_{m1} & \cdots & c_{mm} \end{Vmatrix} ,$$

with $\qquad c_{oo} = c_{o1} = \cdots\cdots = c_{om} = 1,$

in such a way that we have simultaneously

$$(2.3) \qquad C \begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} A_u = C\mathcal{P}_u C^{-1}.C \begin{bmatrix} A_o \\ A_1 \\ \vdots \\ A_m \end{bmatrix} , \; u = 0, 1, 2, \ldots, m,$$

where

$$(2.4) \qquad C\mathcal{P}_u C^{-1} = \begin{Vmatrix} z_{ou} & & & 0 \\ & z_{1u} & & \\ & & \ddots & \\ 0 & & & z_{mu} \end{Vmatrix} , \quad z_{ou} = n_u ; \; u = 0, 1, 2, \ldots, m$$

Let

$$(2.5) \qquad A^*_u = \sum_{t=0}^{m} c_{ut} A_t ,$$

then

$$A_i A^*_u = \sum_{t=o}^{m} c_{ut} A_i A_t = \sum_{k=o}^{m} \left(\sum_{t=o}^{m} c_{ut} p^k_{ti} \right) A_k$$

$$= \sum_{k=o}^{m} \left(\sum_{t=o}^{m} c_{ut} p^k_{ti} \right) \sum_{\ell=o}^{m} c^{k\ell} A^*_\ell$$

$$= \sum_{\ell=o}^{m} \left(\sum_{k=o}^{m} \sum_{t=o}^{m} c_{ut} p^k_{ti} c^{k\ell} \right) A^*_\ell$$

$$= \sum_{\ell=0}^{m} z_{ui} \delta_{u\ell} A_{\ell}^{*} = z_{ui} A_u^{*}$$

where we have put

$$C^{-1} = || c^{ij} || .$$

Multiplying (2.6) by c_{wi} and summing up with respect to i, we get

(2.7) $$A_w^{*} A_u^{*} = \sum_{i=0}^{m} c_{wi} z_{ui} A_u^{*} ,$$

and similarly

(2.8) $$A_u^{*} A_w^{*} = \sum_{i=0}^{m} c_{ui} z_{wi} A_w^{*} .$$

Thus we get

(2.9) $$\sum_{i=0}^{m} c_{ui} z_{wi} = 0 \quad \text{if } u \neq w ,$$

and

(2.10) $$A_u^{\#} = \left(\sum_{i=0}^{m} c_{ui} z_{ui} \right)^{-1} . A_u^{*} , \quad u = 0, 1, 2, \ldots, m$$

are orthogonal system of idempotent matrices. It is clear that

(2.11) $$A_0^{\#} = v^{-1} . G_v ,$$

and

(2.12) $$1_v = A_0^{\#} \quad A_1^{\#} + \ldots + A_m^{\#} .$$

Now, since the matrix algebra $\mathfrak{A}$ is also a representation of $\mathfrak{a}$, we are going to find out the multiplicities of linear representations in $\mathfrak{A}$.

Let $\alpha_0, \alpha_1, \ldots \alpha_m$ be the respective multiplicities of m+1 linear representations in $\mathfrak{A}$. First, by considering the trace of G_v, we find that

(2.13) $$\alpha_o = 1.$$

Next, by considering traces of A_o, A_1, ..., A_m, we get the following system of linear equations.

$$(2.14)\quad \begin{aligned} &\alpha_o + \alpha_1 + \cdots\cdots + \alpha_m = v, \\ &\alpha_o n_1 + \alpha_1 z_{11} + \cdots\cdots + \alpha_m z_{m1} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\alpha_o n_m + \alpha_1 z_{m1} + \cdots\cdots + \alpha_m z_{mm} = 0. \end{aligned}$$

Since m+1 vectors of dimension (m+1)

$$(1, n_1, n_2, \ldots, n_m), (1, z_{11}, z_{12}, \ldots, z_{1m}), \ldots, (1, z_{m1}, z_{m2}, \ldots, z_{mm})$$

are linearly independent, the coefficient matrix of (2.14) is non-singular, and consequently the multiplicities α_o, α_1, ..., α_m are determined uniquely by the equations (2.14).

§3. Examples of associations of certain types.

(1) Group divisible type: The number of treatments is $v = mn$, where m and n are positive integers. They can be divided into m groups of n elements each, such that any two treatments in the same group are 1st associates and two treatments in different groups are 2nd associates.

If the whole treatments are numbered lexicographically with respect to the order of groups and then with respect to the order of treatments in groups, i.e., the j-th treatment in the i-th group bears the number $(i-1)n + j$, then the association matrices have simple forms as follows:

$$(3.1)\qquad A_o + A_1 = \begin{Vmatrix} G_n & & & 0 \\ & G_n & m & \\ & & \ddots & \\ 0 & & & G_n \end{Vmatrix} \qquad \text{and } A_1 = G_v - A_o - A_1,$$

and

$$(3.2)\qquad n_1 = n-1, \qquad n_2 = (m-1)n.$$

The regular representation $(\mathfrak{a})$ of the association algebra is generated by the mappings

$$A_o \longrightarrow \rho_o = I_3,$$

$$(3.3)\qquad A_1 \longrightarrow \rho_1 = \begin{Vmatrix} 0 & 1 & 0 \\ n-1 & n-2 & 0 \\ 0 & 0 & n-1 \end{Vmatrix},$$

$$A_1 \longrightarrow \rho_1 = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & n-1 \\ (m-1)n & (m-1)n & (m-2)n \end{Vmatrix}.$$

Transforming by a non-singular matrix

$$(3.4)\qquad C = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{-1}{m-1} \\ 1 & \frac{-1}{n-1} & 0 \end{Vmatrix},$$

we get

$$(3.5)\qquad C\rho_1 C^{-1} = \begin{Vmatrix} n-1 & 0 & 0 \\ 0 & n-1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad C\rho_2 C^{-1} = \begin{Vmatrix} (m-1)n & 0 & 0 \\ 0 & -n & 0 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Therefore

$$(3.6)\qquad \begin{aligned} \sum_{j=o}^{2} c_{1j} z_{1j} &= 1.1 + 1.(n-1) = \frac{1}{m-1}(-n) = \frac{mn}{m-1} \\ \sum_{j=o}^{2} c_{2j} z_{2j} &= 1.1 = \frac{1}{n-1}(-1) + 0.0 = \frac{n}{n-1} \end{aligned}$$

Thus the three orthogonal idempotents are given by

$$
\begin{aligned}
(3.7) \qquad A_o^{\#} &= (mn)^{-1} [\quad A_o \quad + A_1 + A_2], \\
A_1^{\#} &= (mn)^{-1} [(m-1)A_o+(m-1)A_1 - A_2], \\
A_1^{\#} &= \; n^{-1} [(n-1)A_o \quad - A_1 \quad].
\end{aligned}
$$

The multiplicities of the linear representations induced by those idempotents in the matrix association algebra are

$$(3.8) \qquad \alpha_o = 1, \quad \alpha_1 = m-1, \quad \alpha_2 = m(n-1)$$

respectively, and they are nothing but the ranks of those idempotents.

(2) Triangular type: The number of treatments is $v = n(n-1)/2$, where n is a positive integer. We take an n/n square, and fill the $n(n-1)/2$ positions above the main diagonal by different $n(n-1)/2$ treatments, taken in order. The positions in the main diagonal are left blank, while the positions below the main diagonal are filled so that the scheme is symmetrical with respect to the main diagonal--see the following figure. Two treatments in the same column are 1st associates, whereas two treatments which do not occur in the same column are 2nd associates.

Fig. $n = 5$, $v = 10$

	1	2	3	4
1		5	6	7
2	5		8	9
3	6	8		10
4	7	9	10	

Hence

$$(3.9) \qquad n_1 = 2n - 4, \qquad n_2 = (n-2)(n-3)/2 \; .$$

The regular representation of the association algebra in this case is given by

$$
(3.10)\ (\ ): \quad
\begin{aligned}
A_0 &\longrightarrow \rho_0 = I_3, \\
A_1 &\longrightarrow \rho_1 = \left\| \begin{matrix} 0 & 1 & 0 \\ 2n-4 & n-2 & 4 \\ 0 & n-3 & 2n-8 \end{matrix} \right\| \\
A_1 &\longrightarrow \rho_2 = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & n-3 & 2n-8 \\ (n-2)(n-3)/2 & (n-3)(n-4)/2 & (n-4)(n-5)/2 \end{matrix} \right\|
\end{aligned}
$$

Transforming by a non-singular matrix

$$
(3.11) \qquad C = \left\| \begin{matrix} 1 & 1 & 1 \\ 2n-4 & n-4 & -4 \\ -(n-2)(n-3) & n-3 & -2 \end{matrix} \right\| .
$$

we get

$$
(3.12) \quad \rho_1 C^{-1} = \left\| \begin{matrix} 2n-4 & 0 & 0 \\ 0 & n-4 & 0 \\ 0 & 0 & -2 \end{matrix} \right\| , \quad \rho_2 C^{-1} = \left\| \begin{matrix} (n-2)(n-3)/2 & 0 & 0 \\ 0 & -(n-3) & 0 \\ 0 & 0 & 1 \end{matrix} \right\| .
$$

Therefore

$$
(3.13) \quad
\begin{aligned}
\sum_{j=0}^{2} c_{1j} z_{1j} &= (2n-4).1 + (n-4).(n-4) + (-4).[-(n-3)] = n(n-2) \\
\sum_{j=0}^{2} c_{2j} z_{2j} &= -(n-2)(n-3).1 + (n-3).(-2) + (-2).1 = 1(n-1)(n-2).
\end{aligned}
$$

Hence we get the three orthogonal idempotents as follows:

$$
(3.14) \quad
\begin{aligned}
A_0^{\#} &= \frac{2}{n(n-1)} [A_0 + A_1 + A_2], \\
A_1^{\#} &= \frac{1}{n(n-1)} [(2n-4) A_0 + (n-4) A_1 - 4A_2], \\
A_2^{\#} &= \frac{1}{(n-1)(n-2)} [(n-2)(n-3) A_0 - (n-3) A_1 + 2A_2].
\end{aligned}
$$

with respective ranks

$$(3.15) \qquad \alpha_o = \text{tr } A_o^{\#} = 1,\ \alpha_1 = \text{tr } A_1^{\#} = n-1,\ \alpha_2 = \text{tr } A_2^{\#} = n(n-3)/2 .$$

<u>(3) G_3 type:</u> The number of treatments is v = mn, where m and n are positive integers. They can be arranged in the form of m x n treatments in the same column and the remaining (m-1)(n-1) treatments are the 3rd associates [8]. Hence

$$(3.16) \qquad n_1 = n-1, \quad n_2 = m-1, \quad n_3 = (m-1)(n-1).$$

The regular representation of the association algebra is generated by

$$(3.17) \qquad \begin{aligned} & A_o \longrightarrow \rho_o = I_4 \\ & A_1 \longrightarrow \rho_1 = \left\| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ n-1 & n-2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & n-1 & n-2 \end{array} \right\| , \\ & A_1 \longrightarrow \rho_2 = \left\| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ m-1 & 0 & m-2 & 0 \\ 0 & m-1 & 0 & m-2 \end{array} \right\| , \\ & A_3 \longrightarrow \rho_3 = \left\| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & n-1 & n-2 \\ 0 & m-1 & 0 & m-2 \\ (n-1)(m-1) & (n-2)(m-1) & (n-1)(m-2) & (n-2)(m-2) \end{array} \right\| . \end{aligned}$$

Transforming by a non-singular matrix

$$(3.18) \qquad C = \left\| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ n-1 & -1 & n-1 & -1 \\ m-1 & m-1 & -1 & -1 \\ (n-1)(m-1) & -(m-1) & -(n-1) & 1 \end{array} \right\| ,$$

we get

$$
(3.19)\qquad
\begin{array}{llll}
z_{oo} = 1, & z_{o1} = n-1, & z_{o2} = m-1, & z_{o3} = (n-1)(m-1), \\
z_{1o} = 1, & z_{11} = -1, & z_{12} = m-1, & z_{13} = -(m-1), \\
z_{2o} = 1, & z_{21} = n-1, & z_{22} = -1, & z_{23} = -(n-1), \\
z_{3o} = 1, & z_{31} = -1, & z_{32} = -1, & z_{33} = 1.
\end{array}
$$

Hence we have four orthogonal idempotents

$$
(3.20)\qquad
\begin{aligned}
A_o^{\#} &= \frac{1}{mn}[\qquad A_o + A_1 + A_2 + A_3], \\
A_1^{\#} &= \frac{1}{mn}[(n-1)A_o - A_1+(n-1)A_2 - A_3], \\
A_2^{\#} &= \frac{1}{mn}[(m-1)A_o+(m-1)A_1 - A_2 - A_3], \\
A_3^{\#} &= \frac{1}{mn}[(n-1)(m-1)A_o-(m-1)A_1-(n-1)A_2 + A_3],
\end{aligned}
$$

with respective ranks

$$
(3.21)\qquad
\begin{aligned}
&\alpha_o = \mathrm{tr}\, A_o^{\#} = 1, \quad \alpha_1 = \mathrm{tr}\, A_1^{\#} = n-1, \\
&\alpha_2 = \mathrm{tr}\, A_1^{\#} = m-1, \ \alpha_3 = \mathrm{tr}\, A_1^{\#} = (n-1)(m-1).
\end{aligned}
$$

§4. The relationship algebra of a PBIBD. We now define the so-called relationship matrices of a PBIBD. There are $W = bk = vr$ experimental units in the whole and they are numbered from 1 through W in any way but once for all.

(1) Identity relation: Corresponding to this relation, we take $I = I_w$, i.e., the unit matrix of order W.

(2) Universal relation: Corresponding to this relation, we take $G = G_w$, where G_w stands for the matrix of order W whose elements are all unity.

(3) Block relation: Let the incidence matrix of blocks be

$$\psi = \|\eta_1 \eta_2 \ldots \eta_b\| ,$$

where

$$\eta'_a = (\eta_a, \eta_{a1} --- \eta_{aw}) \text{ with } \eta_{af} = \begin{cases} 1, & \text{if the f-th unit belongs to the a-th block,} \\ 0, & \text{otherwise,} \end{cases}$$

then the block relation is represented by a WxW matrix

$$(4.1) \qquad B = \psi\psi' .$$

(4) Treatment relation: Let the incidence matrix of treatments be

$$\Phi = \|\zeta_1 \zeta_2 ---- \zeta_v\| ,$$

where

$$\zeta'_\alpha = (\zeta_{\alpha 1}\ \zeta_{\alpha 2}\ ----- \ \zeta_{\alpha w}) \text{ with } \zeta_{\alpha f} = \begin{cases} 1, & \text{if the } \alpha\text{-th treatment occurs at the f-th unit,} \\ 0, & \text{otherwise,} \end{cases}$$

then the treatment relation is represented by the following m+1 matrices of order W:

$$(4.2) \qquad T = T_o, T_1, T_2, \ldots, T_m$$

where

$$(4.3) \qquad T_u \equiv \|t^u_{fg}\| = \Phi A_u \Phi' \quad , u = 0, 1, \ldots, m.$$

It is seen immediately that

$$(4.4) \qquad \sum_{u=o}^{m} T_u = G_1 .$$

Also

$$(4.5) \qquad G^2 = wG, \; BG = GB = kG, \; B^2 = kB,$$

and

$$(4.6) \qquad GT_u = T_uG = rn_uG, \qquad u = 0, 1, \ldots, m.$$

Let $N = \| n_{\alpha a} \|$ be the incidence matrix of the design, then, since

$$N = \Phi'\psi,$$

and

(4.7) $$NN' = \sum_{u=0}^{m} \lambda_u A_u,$$

it follows that

(4.8) $$NN' = \sum_{u=0}^{m} \rho_u A_u^{\#},$$

where

(4.9) $$\rho_o = r + n_1\lambda_1 + \dots + n_m\lambda_m = rk , \quad \rho_u = \sum_{j=0}^{m} z_{uj} \lambda_j,$$

$$u = 1, 2, \dots, m.$$

ρ_u are the characteristic roots of the matrix NN' with multiplicities α_u respectively. (4.8) is the spectral decomposition of NN'. The design is said to be regular, if all ρ's are positive. We shall be mainly concerned with regular PBIBDs.

It can be shown that

(4.10) $$TBT = \Phi NN'\Phi' = \sum_{u=0}^{m} \lambda_u T_u,$$

(4.11) $$T_u B T_w = \sum_{t=0}^{m} \left(\sum_{k,\ell=0}^{m} \lambda_k p_{uk}^{\ell} p_{\ell w}^{t} \right) T_t,$$

and

(4.12) $$T_u T_w = \sum_{t=0}^{m} p_{uw}^{t} T_t.$$

Hence the linear closure with respect to the field of all real numbers of the set of the following 4m+3 matrices

$$(4.13) \qquad I,\ G,\ B,\ T_u,\ T_uB,\ BT_u,\ BT_uB, \qquad u = 1, 2, \ldots, m,$$

is a linear associative algebra $\mathfrak{R}$, which is called the relationship algebra of the PBIBD.

The relationship algebra $\mathfrak{R}$ contains a subalgebra $\mathfrak{R}^*$ generated by T_u, u = 0, 1, ..., m, which is isomorphic to the association algebra $\mathfrak{A}$.

As a special case, we get the relationship algebra of a balanced incomplete block design--BIBD in short--as the linear closure [I, G, B, T, TB, BT, BTB]. This algebra was investigated by A. T. James [4] in detail. H. B. Mann [6] exploited more general algebra which is associated with linear hypotheses. The relationship algebra $\mathfrak{R}$ of a PBIBD is located in between James' algebra and Mann's algebra, so to speak.

The relationship algebra $\mathfrak{R}$ of a PBIBD is not commutative in general. Since $\mathfrak{R}$ is generated by symmetric matrices, it is completely reducible. Hence all irreducible representations of $\mathfrak{R}$ are obtained by reducing its regular representation.

[G], the totality of multiples of G, is a one-dimensional two-sided ideal of $\mathfrak{R}$ and

$$G^2 = wG,\ BG = GB = kG,\ T_uG = GT_u = rn_uG .$$

Hence we obtain a linear representation $\mathfrak{R}_G^{(1)}$ induced by [G] as follows:

$$(4.14) \qquad \mathfrak{R}_G^{(1)}:\ \ I \longrightarrow 1,\ G \longrightarrow w,\ B \longrightarrow k,\ T_u \longrightarrow rn_u .$$

Next we shall consider the factor algebra $\mathfrak{R}/[G]$, i.e., consider the algebra $\mathfrak{R}$ mod G. To this end, it is convenient to change the basis of $\mathfrak{R}$ into [I, G, B, T_u^*, I_u^*B, BT_u^*, BT_u^*B, u = 1, 2, ..., m].

$$(4.15) \qquad T_u^* = \Phi A_u^* \Phi = \sum_{j=0}^{m} c_{uj}T_j, \qquad u = 1, \ldots, m,$$

$$(4.16) \qquad T_u^* T_w^* = \Phi A_u^* \Phi' \Phi A_w^* \Phi' = r\Phi A_u^* A_w^* \Phi' = r(\sum_{i=0}^{m} c_{ui} z_{ui}) \delta_{uw} T_u^*,$$

and

$$T_u^* B T_w^* = \Phi A_u^* \Phi' \psi \psi' \Phi A_w^* \Phi' = \Phi A_u^* N N' A_w^* \Phi'$$

$$(4.17) \qquad = (\sum_{i=0}^{m} z_{ui} \lambda_i) \Phi A_u^* A_w^* \Phi' = \rho_u (\sum_{i=0}^{m} c_{ui} z_{ui}) \delta_{uw} T_u^*.$$

The following m subalgebras

$$(4.18) \qquad [T_u^*, BT_u^*, T_u^*B, BT_u^*B] \quad \text{mod. } G, \ u = 1, 2, \ldots, m$$

are two-sided ideals of $\mathfrak{M}$ mod G, and they are annihilating each other. Indeed, for instance,

$$T_u^* B \cdot T_w^* B = T_u^* B T_w^* \cdot B = \rho_u \cdot (\sum_{i=0}^{m} c_{ui} z_{ui}) \delta_{uw} T_u^* B .$$

If $\rho_u = 0$, then

$$BT_u^* = T_u^* B = BT_u^* B = 0,$$

and consequently the subalgebra reduces to $[T_u^*]$ mod. G.

Thus in the regular case, there are m inequivalent irreducible representations of the 2nd degree, each of which being induced by a one-sided ideal of the above two-sided ideals.

Now by direct calculations, we obtain

$$(4.19) \qquad T_i [T_u^*, BT_u^*, T_u^*B, BT_u^*B] = [T_u^*, BT_u^* T_u^* B, BT_u^* B] \left\| \begin{matrix} rz_{ui} & \rho_u z_{ui} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & rz_{ui} & \rho_u z_{ui} \\ 0 & 0 & 0 & 0 \end{matrix} \right\|,$$

$$B[T_u^*, BT_u^*, T_u^*B, BT_u^*B] = [T_u^*, BT_u^*, T_u^*B, BT_u^*B] \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & k \end{Vmatrix} .$$

Thus we get m irreducible representations of the 2nd degree:

$$(4.20) \qquad \mathfrak{R}_u^{(2)}: \quad \begin{matrix} 1 \longrightarrow \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad G \longrightarrow \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad B \longrightarrow \begin{Vmatrix} 0 & 0 \\ 1 & k \end{Vmatrix}, \\ T_i \longrightarrow \begin{Vmatrix} rz_{ui} & \rho_u z_{ui} \\ 0 & 0 \end{Vmatrix}, \quad i = 1, 2, \ldots, m \end{matrix} \quad ; u = 1, 2, \ldots, m$$

Other irreducible representations of $\mathfrak{R}$ are obtained by considering the factor algebra

$$\mathfrak{R}/[G]/[T_u^*, BT_u^*, T_u^*B, BT_u^*B, \; u = o, 2, \ldots, m]/[G].$$

They are given by

$$(4.21) \qquad \mathfrak{R}_o^{(1)}: \quad I \longrightarrow 1, \; G \longrightarrow o, \; B \longrightarrow o, \; T_i \longrightarrow o, \; i = 1, 2, \ldots, m$$

and

$$(4.22) \qquad \mathfrak{R}_1^{(1)}: \quad I \longrightarrow 1, \; G \longrightarrow o, \; B \longrightarrow k, \; T_i \longrightarrow o, \quad i = 1, 2, \ldots, m \; .$$

Since

$$1^2 + 1^2 + 1^2 + m\, 2^2 = 4m + 3,$$

there can be no other irreducible representations of $\mathfrak{R}$.

We shall show that

$$(4.23) \qquad \mathfrak{R} \sim (W-b-v+1)\mathfrak{R}_o^{(1)} + (b-v)\mathfrak{R}_1^{(1)} + \mathfrak{R}_G^{(1)} + \sum_{u=1}^{m} \alpha_u \mathfrak{R}_u^{(1)}$$

if the design is regular and therefore $v \leq b$. If $v > b$, then NN' must be singular and hence at least one of ρ_u should vanish.

Let

$$\mathfrak{R} \sim \gamma_o \mathfrak{R}_o^{(1)} + \gamma_1 \mathfrak{R}_1^{(1)} + \gamma_G \mathfrak{R}_G^{(1)} + \sum_{u=1}^{m} \beta_u \mathfrak{R}_u^{(1)} ,$$

then we get

$$\begin{aligned} \operatorname{tr} I = W &= \gamma_o + \gamma_1 + \gamma_G + 2 \sum_{u=1}^{m} \beta_u , \\ \operatorname{tr} G = W &= W\gamma_G, \\ \operatorname{tr} B = W &= k\gamma_1 + k\gamma_G + k \sum_{u=1}^{m} \beta_u , \\ \operatorname{tr} T_i = 0 &= rn_i\gamma_G + r \sum_{u=1}^{m} \beta_u z_{ui}, \quad i = 1, 2, \ldots, m. \end{aligned}$$

From the first three equations of (4.24), we get

$$\gamma_G = 1, \qquad \gamma_o = W - b - \sum_{u=1}^{m} \beta_u .$$

Comparing the last m equations of (4.24) with those of (2.14), we get

$$\beta_u = \alpha_u, \qquad u = 1, 2, \ldots, m.$$

Consequently it follows that

$$\gamma_o = W - b - v + 1 .$$

Let

(4.25) $$T_u^{\#} = \Phi A_u^{\#} \Psi, \qquad u = 1, 2, \ldots, m,$$

then it is clear that

(4.26) $$T_u^{\#} T_w^{\#} = r\delta_{uw} T_u^{\#} .$$

Now, let us consider the following m matrices:

(4.27) $$V_u = (T_u^{\#} - \frac{1}{k} BT_u^{\#})(T_u^{\#} - \frac{1}{k} T_u^{\#}B), \quad u = 1, 2, \ldots, m.$$

It can be shown that

$$V_u V_w = \delta_{uw} r(r - \frac{\rho_u}{k}).V_u ,$$

and

$$\mathrm{tr}\, V_u = r \;.\; \mathrm{tr}(I - \frac{1}{k} B)\Phi A_u^{\#}\Phi'(I - \frac{1}{k} B)$$

$$= r \;.\; \mathrm{tr}A_u^{\#}\Phi'(I - \frac{1}{k} B)\Phi = r \;.\; \mathrm{tr}A_u^{\#}(rI - \frac{1}{k} NN')$$

$$= r(r - \frac{\rho_u}{k}).\; \mathrm{tr}A_u^{\#} = r(r - \frac{\rho_u}{k})\; \alpha_u.$$

In other words, m matrices

(4.29) $$V_u^{\#} = \frac{k}{r(rk-\rho_u)} (T_u^{\#} - \frac{1}{k} BT_u^{\#})(T_u^{\#} - \frac{1}{k} T_u^{\#}B), \quad u = 1, 2, \ldots, m$$

are mutually orthogonal idempotents of rank α_u, u = 1, 2, ..., m respectively. Thus the following expression

(4.30) $$I = (I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#}) + (\frac{1}{k} B - \frac{1}{w} G) + \frac{1}{w} G + \sum_{u=1}^{m} V_u^{\#}$$

is the decomposition of the unit of $\mathfrak{R}$ into mutually orthogonal idempotents, which will be shown to be useful from the point of view of analysis of variance.

§5. Analysis of PBIBD. We are concerned with the linear model which is often called the intra-block model.

$$x = \gamma 1 + \Phi_\tau + \Psi\beta + e,$$

where $x' = (x_1, \ldots, x_W)$ stands for the observation vector, γ is the general mean, $\tau' = (\tau_1, \ldots, \tau_v)$ and $\beta' = (\beta_1, \ldots, \beta_b)$ are treatment and block effects being subjected to the restrictions

$$\sum_{\alpha=1}^{v} \tau_\alpha = \sum_{a=1}^{b} \beta_a = 0$$

respectively, and finally $e' = (e_1, \ldots, e_W)$ is the error being distributed as $N(0, \sigma^2 I)$. We have the adjusted normal equation [2]

$$(5.1) \qquad [r(1 - \frac{1}{k}) A_o - \frac{\lambda_1}{k} A_1 - \ldots - \frac{\lambda_m}{k} A_m] \underline{t} = \underline{Q} ,$$

or

$$(5.2) \qquad \sum_{u=1}^{m} \frac{rk-\rho_u}{k} A_u^{\#} \underline{t} = \underline{Q} .$$

Let α_u linearly independent column vectors of $A_u^{\#}$ be

$$\underline{a}^{(u)}_{\nu_u+1}, \underline{a}^{(u)}_{\nu_u+2}, \ldots, \underline{a}^{(u)}_{\nu_u+\alpha_u}, \nu_u = 1+\alpha_1 + \ldots + \alpha_{u-1} ,$$

then

$$(5.3) \qquad \underline{a}^{(u)'}_{\nu_u+\alpha} \underline{Q} = \frac{rk-\rho_u}{k} \underline{a}^{(u)'}_{\nu_u+\alpha} \underline{t} , \quad \alpha = 1, 2, \ldots, \alpha_u$$

Hence

$$(5.4) \quad \underline{a}^{(u)'}_{\nu_u+\alpha}\underline{t} = \frac{k}{rk-\rho_u} \underline{a}^{(u)'}_{\nu_u+\alpha} \underline{Q}, \ E(\underline{a}^{(u)'}_{\nu_u+\alpha}\underline{t}) = \underline{a}^{(u)'}_{\nu_u+\alpha}\underline{\tau} , \ V(\underline{a}^{(u)'}_{\nu_u+\alpha}\underline{t}) = \frac{k}{rk-\rho_u} \sigma^2 a^{(u)}_{\nu_u+\alpha, \nu_u+\alpha}$$

Balance is achieved over the set of normalized contrasts $(a^{(u)'}_{\nu_u+\alpha}\underline{\tau} / \sqrt{a^{(u)}_{\nu_u+\alpha, \nu_u+\alpha}}$, $\alpha = 1, \ldots, \alpha_u)$

Since

$$(5.5) \qquad \underline{x}' V_u^{\#} \underline{x} = \frac{k}{rk-\rho_u} \underline{Q}' A_u^{\#} \underline{Q} = \underline{t}' A_u^{\#} \underline{Q} , \quad u = 1, 2, \ldots, m,$$

and

$$\sum_{u=1}^{m} A_u^{\#} = I_v - A_o^{\#} ,$$

it follows that

$$(5.6)\qquad \sum_{u=1}^{m} x'V_u^{\#}x = t'Q = s_t^2 \quad \text{; sum of squares due to treatments adjusted.}$$

Under this present model, we have

$$(5.7)\qquad x'V_u^{\#}x = \frac{k}{rk-\rho_u} QA_u^{\#}Q = e'V_u^{\#}e + 2\tau'A_u^{\#}(\Phi' - \frac{1}{k}N\psi)e + (r - \frac{\rho_u}{k})\tau A_u^{\#}\tau,$$

and therefore

$$(5.8)\qquad \chi_u^2 = x'V_u^{\#}x/\sigma^2$$

is distributed as the non-central chi-square distribution of degrees of freedom α_u and with the non-centrality parameter

$$(5.9)\qquad \delta_u = \frac{rk-\rho_u}{k\sigma^2} \tau A_u^{\#}\tau.$$

The sum of squares due to error s_e^2 is given by

$$(5.10)\qquad s_e^2 = x'(I - \frac{1}{k}B - \sum_{u=1}^{m} V_u^{\#})x = e'(I - \frac{1}{k}B - \sum_{u=1}^{m} V_u^{\#})e ,$$

and therefore

$$(5.11)\qquad \chi_e^2 = s_e^2/\sigma^2$$

is distributed as the central chi-square distribution of degrees of freedom W-b-v+1. The variates $\chi_1^2, \ldots, \chi_m^2, \chi_e^2$ are mutually independent in the stochastic sense. Hence under the null-hypothesis

$$(5.12)\qquad H_o^{(u)}:\ A_u^{\#}\tau = o,$$

the test statistic

$$(5.13)\qquad F_u = \frac{W-b-v+1}{\alpha_u} \frac{x'V_u^{\#}x}{s_e^2}$$

is distributed as the central F-distribution of degrees of freedom $(\alpha_u, W-b-v+1)$.

§6. Analysis of variance of PBIBDs of certain types. In this section, we present explicit expressions of the partition of the sum of squares due to treatments given by (5.6) and describe their statistical meanings for PBIBDS of certain types.

(1) PBIBD of group divisible type (see §3). In this case we have

$$A_0^{\#} = \frac{1}{mn}[\quad A_0 + A_1 \quad + A_2], \quad \alpha_0 = 1$$
$$A_1^{\#} = \frac{1}{mn}[(m-1)A_0 + (m-1)\ A_1 \ - A_2], \quad \alpha_1 = m-1$$
$$A_2^{\#} = \frac{1}{n}\ [(n-1)A_0 - \quad A_1 \quad] \quad \alpha_2 = m(n-1).$$

Now, we assume the following inner structure of the treatment-effects, which seems pertinent to the association under consideration.

$$(6.1) \qquad \tau_{(i-1)n+\alpha} = \theta_i + \pi_\alpha^i, \quad \alpha = 1, 2, \ldots, n; \ i = 1, 2, \ldots, m,$$

which are subjected to the restrictions

$$(6.2) \qquad \sum_{i=1}^{m} \theta_i = 0, \quad \sum_{\alpha=1}^{n} \pi_\alpha^i = 0, \quad i = 1, \ldots, m,$$

so that there are just mn-1 independent parameters. $\theta' = (\theta_1, \ldots, \theta_m)$ may be regarded as the group-effects and π_α^i may be regarded as the interaction between the i-th group and the $(i-1)n+\alpha$ -th treatment. The structural model (6.1) may be called the inner-parametric representation of the treatment-effects.

By direct calculations we obtain the following expressions:

$$
(6.3) \qquad A_1^* \tau = \left\| \begin{matrix}
(m-1)\theta_1 & -\theta_2 & -\theta_3 \cdots\cdots & -\theta_m \\
\cdots & \cdots & \cdots & \cdots \\
(m-1)\theta_1 & -\theta_2 & -\theta_3 & -\theta_m \\
-\theta_1 & +(m-1)\theta_2 & -\theta_3 \cdots\cdots & -\theta_m \\
\cdots & \cdots & \cdots & \cdots \\
-\theta_1 & +(m-1)\theta_2 & -\theta_3 \cdots\cdots & -\theta_m \\
\cdots & \cdots & \cdots & \cdots \\
-\theta_1 & -\theta_2 & -\theta_3 \cdots\cdots & +(m-1)\theta_m \\
\cdots & \cdots & \cdots & \cdots \\
-\theta_1 & -\theta_2 & -\theta_3 \cdots\cdots & +(m-1)\theta_m
\end{matrix} \right\| \begin{matrix} \}n \\ \\ \}n \\ \\ \}n \end{matrix} \Bigg\} m
$$

and

$$
(6.4) \qquad A_2^* \tau = \left\| \begin{matrix}
(n-1)\pi_1^1 & -\pi_2^1 & \cdots\cdots & -\pi_n^1 \\
-\pi_1^1 & +(n-1)\pi_2^1 & \cdots\cdots & -\pi_n^1 \\
-\pi_1^1 & -\pi_2^1 & \cdots\cdots & +(n-1)\pi_n^1 \\
(n-1)\pi_1^2 & -\pi_2^2 & \cdots\cdots & -\pi_n^2 \\
-\pi_1^2 & +(n-1)\pi_2^2 & \cdots\cdots & -\pi_n^2 \\
-\pi_1^2 & -\pi_2^2 & \cdots\cdots & +(n-1)\pi_n^2 \\
\cdots & \cdots & \cdots & \cdots \\
(n-1)\pi_1^m & -\pi_2^m & \cdots\cdots & -\pi_n^m \\
-\pi_1^m & +(n-1)\pi_2^m & \cdots\cdots & -\pi_n^m \\
-\pi_1^m & -\pi_2^m & \cdots\cdots & +(n-1)\pi_n^m
\end{matrix} \right\| \begin{matrix} \}n \\ \\ \}n \\ \\ \}n \end{matrix} \Bigg\} m
$$

$A_1^* \tau$ represents the contrasts between group-effects, whereas $A_2^* \tau$ represents the contrasts between interactions.

Let

(6.5) $$\sum_{\alpha=1}^{n} Q_{(i-1)n+\alpha} = G_i, \quad i = 1, \ldots, m,$$

then since

$$x'V_1^{\#}x = \frac{k}{r(k-1)-(n-1)\lambda_1+\lambda_2} Q'A_1^{\#}Q = \frac{k}{mn\lambda_2} Q'A_1^{\#}Q ,$$

$$x'V_2^{\#}x = \frac{k}{r(k-1) + \lambda_1} Q'A_2^{\#}Q ,$$

and

$$A_1^{\#}Q = \frac{1}{n} \begin{bmatrix} G_1 \\ \vdots \\ G_1 \\ \vdots \\ G_m \\ \vdots \\ G_m \end{bmatrix} \begin{matrix} \left.\right\}n \\ \\ \left.\right\}n \end{matrix} , \quad A_2^{\#}Q = \begin{bmatrix} Q_1 - \frac{1}{n} G_1 \\ \vdots \\ Q_1 - \frac{1}{n} G_1 \\ \vdots \\ Q_{(m-1)n+1} - \frac{1}{n} G_m \\ \vdots \\ Q_{mn} - \frac{1}{n} G_m \end{bmatrix} \begin{matrix} \left.\right\}n \\ \\ \left.\right\}n \end{matrix} ,$$

it follows that

(6.6) $$x'V_1^{\#}x = \frac{k}{n\lambda_2 v} \sum_{i=1}^{m} G_i^2 ,$$

and

(6.7) $$x'V_2^{\#}x = \frac{k}{r(k-1) + \lambda_1} \sum_{i=1}^{m} \sum_{\alpha-1}^{n} (Q_{(i-1)n+\alpha} - \frac{1}{n} G_i)^2$$

$$= \frac{k}{r(k-1) + \lambda_1} \sum_{f=1}^{v} Q_f^2 - \frac{1}{n} \sum_{i=1}^{m} G_i^2 .$$

In terms of t, these turn out to be

$$(6.8)\qquad \begin{aligned} x'V_1^{\#}x &= \frac{\lambda_2 v}{k}\ t'A_1^{\#}t\ , \\ x'V_2^{\#}x &= \frac{r(k-1)+\lambda_1}{k}\ t'A_2^{\#}t\ , \end{aligned}$$

which were obtained by C. Y. Kramer and R. A. Bradley [5] by another method.

(2) PBIBD of triangular type (see §3). In this case, we have

$$A_0^{\#} = \frac{2}{n(n-1)}\left[A_0 + A_1 + A_2\right],\ \alpha_0 = 1,$$

$$A_1^{\#} = \frac{1}{n(n-1)}\left[(2n-4)A_0 + (n-4)A_1 - 4A_2\right],\ \alpha_1 = n-1,$$

$$A_2^{\#} = \frac{1}{(n-1)(n-2)}\left[(n-2)(n-3)A_0 - (n-3)A_1 + 2A_2\right],\ \alpha_2 = \frac{n(n-3)}{2}$$

We assume the following inner structure of the treatment-effects:

$$n = 5,\quad v = \frac{n(n-1)}{2} = 10\ .$$

	$\theta_1+\theta_2+\pi_{12}$	$\theta_1+\theta_3+\pi_{13}$	$\theta_1+\theta_4+\pi_{14}$	$\theta_1+\theta_5+\pi_{15}$
$\theta_2+\theta_1+\pi_{21}$		$\theta_2+\theta_3+\pi_{23}$	$\theta_2+\theta_4+\pi_{24}$	$\theta_2+\theta_5+\pi_{25}$
$\theta_3+\theta_1+\pi_{31}$	$\theta_3+\theta_2+\pi_{32}$		$\theta_3+\theta_4+\pi_{34}$	$\theta_3+\theta_5+\pi_{35}$
$\theta_4+\theta_1+\pi_{41}$	$\theta_4+\theta_2+\pi_{42}$	$\theta_4+\theta_3+\pi_{43}$		$\theta_4+\theta_5+\pi_{45}$
$\theta_5+\theta_1+\pi_{51}$	$\theta_5+\theta_2+\pi_{52}$	$\theta_5+\theta_3+\pi_{53}$	$\theta_5+\theta_4+\pi_{54}$	

The inner parameters are subjected to the restrictions

$$(6.9)\qquad \sum_{i=1}^{n}\theta_i = 0,\ \pi_{ij} = \pi_{ji},\ \sum_{j(\neq i)}\pi_{ij} = 0,\ i = 1, 2, \ldots, n,$$

so that there are just v-1 independent parameters.

By direct calculations, we obtain the following expressions:

$$
(6.10)\quad A_1^{\#}\tau = \frac{1}{n}
\left[\begin{array}{l}
\left.\begin{array}{l}
(n-2)\theta_1 + (n-2)\theta_2 - 2\theta_3 - \cdots\cdots - 2\theta_{n-1} - 2\theta_n \\
(n-2)\theta_1 - 2\theta_2 + (n-2)\theta_3 - \cdots\cdots - 2\theta_{n-1} - 2\theta_n \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(n-2)\theta_1 - 2\theta_2 - 2\theta_3 - \cdots\cdots - 2\theta_{n-1} + (n-2)\theta_n
\end{array}\right\} n-1 \\
\left.\begin{array}{l}
-2\theta_1 + (n-2)\theta_2 + (n-2)\theta_3 - \cdots\cdots - 2\theta_{n-1} - 2\theta_n \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-2\theta_1 + (n-2)\theta_2 - 2\theta_3 - \cdots\cdots - 2\theta_{n-1} + (n-2)\theta_n
\end{array}\right\} n-2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-2\theta_1 - 2\theta_2 - 2\theta_3 - \cdots\cdots - 2\theta_{n-1} + (n-2)\theta_n \quad 1
\end{array}\right]
$$

and

(6.11) α-th component of $A_2\tau$

$$
= \frac{1}{(n-1)(n-2)} \Big[(n-1)(n-3) \text{ interaction } \pi_{ij}\,(i < j) \text{ of } \tau_\alpha
$$

$$
- (n-3)\,\Sigma \text{ interactions corresponding to 1st associates of } \tau_\alpha
$$

$$
+ 2\,\Sigma \text{ interactions corresponding to 2nd associates of } \tau_\alpha. \Big]
$$

Thus $A_1^{\#}\tau$ represents the contrasts between main effects θ and $A_2^{*}\tau$ represents the contrasts between interactions π_{ij}.

§7. Numerical illustration. There are n kinds of ingredients $I_1, I_2, \ldots, I_n$ which are known to be efficient in gaining weights of hogs if added in the feed stuff. We are interested to know whether there are interactions between any two of the ingredients when the mixtures of the two are added in the feed stuff.

We make $v = n(n-1)/2$ mixtures of the possible pairs (I_i, I_j), $i \neq j$. The main effects of the n original ingredients are denoted by θ_i, $i = 1, 2, \ldots, n$ and the interaction between I_i and I_j is denoted by π_{ij}. Then the inner-parametric representations of the mixtures are given by

$$\tau_\alpha = \theta_i + \theta_j + \pi_{ij}$$

if the α-th treatment is the mixture of I_i and I_j for $\alpha = 1, 2, \ldots, v = \frac{n(n-1)}{2}$. Hence in this situation, the association scheme of triangular type is naturally defined among the v treatments.

Suppose by taking ten litters of 4 hogs each as blocks, a PBIB of triangular type with parameters

$$n = 5,\ v = 10,\ b = 10,\ r = k = 4,\ \lambda_1 = 1,\ \lambda_2 = 2$$

is adopted yielding the following results. Observations are the gains of weights of hogs in pounds after feeding the mixtures of ingredients for 3 months. This experiment is a hypothetical one and the data are borrowed from R. C. Bose and T. Shimamoto [3] and therefere this example should be regarded as a purely illustrative one.

Table 1

A Design of Triangular Type

Blocks / Treatments	1	2	3	4	5	6	7	8	9	10	Treatment Total
1 = (1,2)			2.31		2.86	1.65			2.58		9.40
2 = (1,3)		2.51					1.41	1.90	3.06		8.88
3 = (1,4)	2.89		2.29					1.95		2.04	9.16
4 = (1,5)				2.54		2.09	2.36			2.03	9.02
5 = (2,3)	2.28			2.81					2.20	2.07	9.36
6 = (2,4)		1.77	2.49	2.31			3.02				9.59
7 = (2,5)	2.72	2.29				1.57		2.60			9.18
8 = (3,4)				2.81	2.99	2.28		2.44			10.52
9 = (3,5)	2.54		2.44		2.23		2.12				9.33
10 = (4,5)		1.54			2.87				2.77	2.09	9.27
Block Totals	10.43	8.11	9.53	10.47	10.95	7.59	8.91	8.89	10.61	8.22	93.71

Table 2

<u>Association</u>

Treatment	1st Associates	2nd Associates
1	2, 3, 4, 5, 6, 7	8, 9, 10
2	1, 3, 4, 5, 8, 9	6, 7, 10
3	1, 2, 4, 6, 8, 10	5, 7, 9
4	1, 2, 3, 7, 9, 10	5, 6, 8
5	1, 2, 6, 7, 8, 9	3, 4, 10
6	1, 3, 5, 7, 8, 10	2, 4, 9
7	1, 4, 5, 6, 9, 10	2, 3, 8
8	2, 3, 5, 6, 9, 10	1, 4, 7
9	2, 4, 5, 7, 8, 10	1, 3, 6
10	3, 4, 6, 7, 8, 9	1, 2, 5

Now in this case, since

$$\rho_1 = z_{10}\lambda_o + z_{11}\lambda_1 + z_{12}\lambda_2 = 4 + 1.1 - 2.2 = 1,$$

$$\rho_2 = z_{20}\lambda_o + z_{21}\lambda_1 + z_{22}\lambda_2 = 4 - 2.1 + 1.2 = 4.$$

and

$$A_1^{\#} = \frac{1}{15}(6A_o + A_1 - 4A_2), \quad \alpha_1 = 4,$$

$$A_2^{\#} = \frac{1}{12}(6A_o - 2A_1 + 2A_2), \quad \alpha_2 = 5.$$

it follows that

$$A_1^{\#}Q = \frac{1}{15}(6Q + A_1Q = 4A_2Q),$$

$$A_2^{\#}Q = \frac{1}{6}(3Q - A_1Q + A_2Q) .$$

There is a relation

$$A_1^{\#}Q + A_2^{\#}Q = Q .$$

Finally the sums of squares due to main-effects and interactions are given by

$$x'V_1^{\#}x = \frac{k}{rk - \rho_1} Q'A_1^{\#}Q = \frac{4}{15} Q'A_1^{\#}Q ,$$

and

$$x'V_2^{\#}x = \frac{k}{rk - \rho_2} Q'A_2^{\#}Q = \frac{1}{3} Q'A_2^{\#}A ,$$

respectively satisfying the relation

$$\frac{4}{15} Q'A_1^{\#}Q + \frac{1}{3} Q'A_2^{\#}Q = t'Q.$$

Thus we get the following table of the analysis of variance (Table 4) by use of auxiliary Table 3.

Table 3

Adjusted Treatment Tables and Related Sums

	Q	A_1Q	A_2Q	A_1Q	A_2Q
1	-0.2700	0.0525	0.2175	-0.1625	-0.1075
2	-0.2500	-0.3075	0.5575	-0.2692	0.0192
3	-0.1075	0.8800	-0.7725	0.2217	-0.3292
4	0.2225	-1.0300	0.8075	-0.1950	0.4175
5	-0.5725	0.6600	-0.0875	-0.1617	-0.4108
6	0.3350	0.3175	-0.6525	0.3292	0.0058
7	0.4250	-1.1125	0.6875	-0.0875	0.5125
8	1.0450	-1.4225	0.3775	-0.2225	0.8225
9	-0.6250	0.6675	-0.0425	-0.1992	-0.4308
10	-0.2025	1.2950	-1.0925	0.2967	-0.4992
Total	0.0000				

Table 4

Analysis of Variance

Sources of Variations	Sum of Squares	d.f.	m.s.s.	Variance Ratio
Blocks	3.2284	9		
Treatment	0.7467	9	0.08295	
eliminating blocks				
main effect	0.1343	4	0.03357	0.230
interactions	0.6124	5	0.12248	0.841
Errors	3.0585	21	0.14550	
Total	7.0336	39		

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