On the Null-Distribution of the F-Statistic in a Randomized Partially Balanced Incomplete Block Design with Two Associate Classes under the Neyman Model

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Summary and Introduction

One of the authors, J. Ogawa, showed that for a complete block design [2], a Latin-square design [3], and for a balanced incomplete block design [4], the null-distribution of the F-statistic (i.e., the variance ratio) in their analysis of variance can be approximated by the familiar central F-distribution even under the Neyman model, i.e., an intra-block analysis model with both the unit errors and technical errors, if they are randomized. In the present article, the authors consider the null-distribution of the F-statistic in a randomized partially balanced incomplete block design with two associate classes under the Neyman model and they reached the same conclusion as mentioned above, though the calculations are much heavier than those for previous cases.

P. V. Rao [6] treated the same problem under the Fisher model for a special class of randomized partially balanced incomplete block designs with two associate classes with $\lambda_1 = 1$, $\lambda_2 = 0$.

Thus the result presented in the present paper seems to be the most general one among those thus far published along the line mentioned above.

The extension of the reasoning presented in this paper to a randomized partially balanced block design with m associate classes is not difficult and will be presented in a forth coming paper.

If one looks at those results obtained from the point of view of the usual normal theory, this may be regarded as the so-called "robustness" of the usual regression model with normal errors which ignores unit errors completely.

In §1 the spectral decomposition of the matrix NN', N being the incidence matrix of the design under consideration, is given and this is useful for discussions in later sections. The null-distribution of the F-statistic before the randomization under the Neyman model is presented in §2 and this turns out to be a non-central F-distribution whose non-centrality parameter depends upon the quantity θ which is a quadratic form of the unit errors. In §3 the exact mean and variance of the quantity θ with respect to the permutation distribution due to randomization are calculated. The calculations in this part are quite complicated. Then in §4, if the number of blocks is sufficiently large and a certain uniformity condition on the within-block variances of the unit errors is satisfied, the permutation distribution of θ is shown to be approximated by a suitable Beta-distribution. Finally in §5, it is shown that the null-distribution of the F-statistic after the randomization can be approximated by a familiar central F-distribution provided the two conditions mentioned above are satisfied.

§1. Spectral decomposition of the matrix NN'.

As for the definition of a partially balanced incomplete block design and notations being used in the present section, references should be made to [1] and [5].

Let the association matrices be $A_o = I_3$, A_1, A_2, and let their regular representations be

$$\rho_o = I_3, \rho_1 = \begin{Vmatrix} 0 & 1 & 0 \\ n_1 & p_{11}^1 & p_{11}^2 \\ 0 & p_{21}^1 & p_{21}^2 \end{Vmatrix}, \quad \rho_2 = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & p_{12}^1 & p_{12}^2 \\ n_2 & p_{22}^1 & p_{22}^2 \end{Vmatrix}$$

respectively, then it is known [5] that there exists a non-singular matrix

$$C = \begin{Vmatrix} 1 & 1 & 1 \\ c_{1o} & c_{11} & c_{12} \\ c_{2o} & c_{21} & c_{22} \end{Vmatrix},$$

such that

$$(1.1) \qquad C \rho_u C^{-1} = \begin{Vmatrix} n_u & 0 & 0 \\ 0 & z_{1u} & 0 \\ 0 & 0 & z_{2u} \end{Vmatrix}, \quad u = 0, 1, 2$$

simultaneously, where

$$z_{11} = (a+d)/2, \qquad z_{21} = (a-d)/2,$$

$$z_{12} = -(a+2+d)/2, \; z_{22} = -(a+2-d)/2 \; ,$$

with

$$(1.3) \qquad a = p^2_{21} - p^1_{12} - 1,$$

$$d = (p^2_{21} - p^1_{12} - 1)^2 + 4p^2_{21} = (p^1_{12} - p^2_{21} - 1)^2 + 4p^1_{12}.$$

One may put as follows:

$$(1.4) \qquad \begin{array}{lll} c_{1o} = 1, & c_{11} = \dfrac{z_{11}}{n_1}, & c_{12} = \dfrac{z_{12}}{n_2} \; , \\ c_{2o} = 1, & c_{21} = \dfrac{z_{21}}{n_1}, & c_{22} = \dfrac{z_{22}}{n_2} \; . \end{array}$$

Three orthogonal idempotents of the association algebra are given by

$$(1.5) \qquad \begin{cases} A^{\#}_o = \dfrac{1}{v} G_v \; , \\ A^{\#}_1 = (1 + \dfrac{z^2_{11}}{n_1} + \dfrac{z^2_{12}}{n_2})^{-1} \; (A_o + \dfrac{z_{11}}{n_1} A_1 + \dfrac{z_{12}}{n_2} A_2), \\ A^{\#}_2 = (1 + \dfrac{z^2_{21}}{n_1} + \dfrac{z^2_{22}}{n_2})^{-1} \; (A_o + \dfrac{z_{21}}{n_1} A_1 + \dfrac{z_{22}}{n_2} A_2), \end{cases}$$

with respective ranks $\alpha_o = 1$, α_1 and α_2.

Let the incidence matrix of the design by N, then it is known that

(1.6) $$NN'(=r A_o + \lambda_1 A_1 + \lambda_2 A_2) = \rho_o A_o^{\#} + \rho_1 A_1^{\#} + \rho_2 A_2^{\#} ,$$

where

(1.7) $$\rho_o = rk,\ \rho_1 = r + z_{11}\lambda_1 + z_{12}\lambda_2,\ \rho_2 = r + z_{21}\lambda_1 + z_{22}\lambda_2$$

and

(1.8) $$A_o^{\#} + A_1^{\#} + A_2^{\#} = I_v .$$

Thus (1.6) gives the spectral decomposition of the symmetric matrix NN' and ρ_o, ρ_1, ρ_2 are the characteristic roots of the matrix with multiplicities α_o, α_1, α_2 respectively. Column vectors of $A_u^{\#}$ are characteristic vectors corresponding to the characteristic root ρ_u of NN'.

§2. The null-distribution of F-statistic before the randomization under the Neyman model.

We shall be dealing with a partially balanced incomplete block design with two associate classes which has v treatments with such an association as explained in the previous section, b blocks of size k each, r replications of each treatment, and the numbers of incidence of any pair of treatments λ_1 or λ_2 according as they are 1st associates or 2nd associates. As for the notations being used in this section, references should be made to [1] and [5].

Let the incidence matrices of treatments and blocks be Φ and ψ respectively, then the Neyman model assuming no interaction between treatments and units is given by

(2.1) $$x = \gamma 1 + \Phi\tau + \psi\beta + \pi + e ,$$

where $x' = (x_1, \ldots, x_n)$ is the observation vector, $\tau' = (\tau_1, \ldots, \tau_v)$ and

$\beta' = (\beta_1, \ldots, \beta_b)$ are treatment-effects and block-effects subjected to the restrictions

$$\tau_1 + \ldots + \tau_v = 0 \text{ and } \beta_1 + \ldots + \beta_b = 0$$

respectively, and $\pi' = (\pi_1, \ldots, \pi_n)$ stands for the unit errors. Finally, $e' = (e_1, \ldots, e_n)$ is the technical error vector being distributed as $N(o', \sigma^2 I_n)$.

Now we are interested in testing the null-hypothesis

$$H_o: \quad \tau = 0 . \tag{2.2}$$

Likewise one may consider the testing the partial null-hypotheses

$$H_o^{(u)}: \quad A_u^{\#}\tau = 0, \qquad u = 1, 2.$$

Although the arguments for these hypotheses are similar to the one presented in this paper for H_o, we confine ourselves to the null-hypothesis H_o for the moment and testing of $H_o^{(u)}$ will be dealt with in the forth-coming paper mentioned in the introduction.

Sums of squares due to treatments adjusted and errors are given by

$$\begin{aligned} S_t^2 &= x'(V_1^{\#} + V_2^{\#})x , \\ S_e^2 &= x'(I - \frac{1}{k} B - V_1^{\#} - V_2^{\#})x \end{aligned} \tag{2.3}$$

respectively, where

$$V_u^{\#} = \frac{1}{r(rk-\rho_u)} (T_u^{\#} - \frac{1}{k} BT_u^{\#})(T_u^{\#} - \frac{1}{k} T_u^{\#}B)$$

$$= \frac{1}{rk-\rho_u} (I - \frac{1}{k} B)\, T_u^{\#}(I - \frac{1}{k} B) , \qquad u = 1, 2$$

with

$$T_u^{\#} = \Phi A_u^{\#}\Phi \quad \text{and} \quad B = \psi\psi' .$$

Under the null-hypothesis H_o, they can be expressed as follows:

$$S_t^2 = \pi'(I - \frac{1}{k}B)\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'(I-\frac{1}{k}B)\pi$$
$$+ 2\pi(I-\frac{1}{k}B)\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'(I-\frac{1}{k}B)e$$
$$+ e'(I-\frac{1}{k}B)\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'(I-\frac{1}{k}B)e,$$

$$S_e^2 = \pi'(I-\frac{1}{k}B)\,[I-\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'](I-\frac{1}{k}B)\pi$$
$$+ 2\pi'(I-\frac{1}{k}B)[I-\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi']\,(I-\frac{1}{k}B)e$$
$$+ e'(I-\frac{1}{k}B)[I-\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'](I-\frac{1}{k}B)e,$$

where

(2.6) $$c_1 = \frac{k}{rk-\rho_1}, \quad c_2 = \frac{k}{rk-\rho_2} \quad .$$

The null-distribution of the variate

$$\chi_1^2 = S_t^2/\sigma^2$$

before the randomization is the non-central chi-square distribution of degrees of freedom v-1 and with the non-centrality parameter

(2.7) $$\kappa_1 = \pi'\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'\pi/\sigma^2 \; .$$

Hence its probability element is

(2.8) $$\exp(-\frac{\kappa_1}{2}) \sum_{\mu=0}^{\infty} \frac{(\frac{\kappa_1}{2})^{\mu}}{\mu!} \; \frac{(\frac{\chi_1^2}{2})^{\frac{v-1}{2}+\mu-1}}{\Gamma(\frac{v-1}{2}+\mu)} \; \exp(-\frac{\chi_1^2}{2}).d(\frac{\chi_1^2}{2}) \; .$$

The null-distribution of the variate

$$\chi_2^2 = S_e^2/\sigma^2$$

before the randomization is the non-central chi-square distribution of degrees of freedom n-v-b+1 and with the non-centrality parameter

(2.9) $$\kappa_2 = \pi'(I-\frac{1}{k}B)[I-\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'](I-\frac{1}{k}B)\pi/\sigma^2$$

$$= \Delta/\sigma^2 - \kappa_1 \quad , \qquad \Delta \equiv \pi'\pi \ .$$

Hence its probability element is given by

(2.10) $$\exp(-\frac{\kappa_2}{2}) \sum_{\nu=0}^{\infty} \frac{(\frac{\kappa_2}{2})^{\nu}}{\nu!} \frac{(\frac{\chi_2^2}{2})^{\frac{n-b-b+1}{2}+\nu-1}}{\Gamma(\frac{n-v-b+1}{2}+\nu)} \exp(-\frac{\chi_2^2}{2})\, d(\frac{\chi_2^2}{2}) \ .$$

Since χ_1^2 and χ_2^2 are stochastically independent of each other, the null-distribution of the F-statistic

(2.11) $$F = \frac{n-v-b+1}{v-1} \frac{s_t^2}{s_e^2}$$

before the randomization is, after a little algebra, given by

(2.12) $$\frac{\Gamma(\frac{n-b}{2})}{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-v-b+1}{2})} (\frac{v-1}{n-v-b+1}F)^{\frac{v-1}{2}-1} (1+\frac{v-1}{n-v-b+1}F)^{-\frac{n-b}{2}}$$

$$\exp(-\frac{\Delta}{2\sigma^2}) \sum_{\ell=0}^{\infty} \frac{(\frac{\Delta}{\sigma^2})^{\ell}}{\ell!} (1+\frac{v-1}{n-v-b+1}F)^{-\ell} \sum_{\mu+\nu=\ell} \frac{\ell!}{\mu!\nu!} \theta^{\mu} (1-\theta)^{\nu} (\frac{v-1}{n-v-b+1})^{\mu}$$

$$\frac{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-v-b+1}{2})\Gamma(\frac{n-b}{2}+\ell)}{\Gamma(\frac{n-b}{2})\Gamma(\frac{v-1}{2}+\mu)\Gamma(\frac{n-v-b+1}{2}+\nu)} \quad ,$$

where

(2.13) $$\theta = \Delta^{-1}.\pi'\Phi(c_1A_1^{\#}+c_2A_2^{\#})\Phi'\pi \quad .$$

The null-distribution of the F after the randomization should be obtained as

(2.14)

$$\frac{\Gamma(\frac{n-b}{2})}{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-v-b+1}{2})}\left(\frac{v-1}{n-v-b+1}F\right)^{\frac{v-1}{2}-1}\left(1+\frac{v-1}{n-v-b+1}F\right)^{-\frac{n-b}{2}} d\left(\frac{v-1}{n-v-b+1}F\right)$$

$$\exp\left(-\frac{\Delta}{2\sigma^2}\right)\sum_{\ell=0}^{\infty}\frac{\left(\frac{\Delta}{2\sigma^2}\right)^{\ell}}{\ell!}\left(1+\frac{v-1}{n-v-b+1}F\right)^{-\ell}\sum_{\mu+\nu=\ell}\frac{\ell!}{\mu!\,\nu!}\,\mathcal{E}(\theta^{\mu}(1-\theta)^{\nu})\left(\frac{v-1}{n-v-b+1}F\right)^{\mu}$$

$$\frac{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-v-b+1}{2})\Gamma(\frac{n-b}{2}+\ell)}{\Gamma(\frac{n-b}{2})\Gamma(\frac{v-1}{2}+\mu)\ \Gamma(\frac{n-v-b+1}{2}+\nu)} ,$$

where $\mathcal{E}(\theta^{\mu}(1-\theta)^{\nu})$ stands for the expectation with respect to the permutation distribution of θ due to randomization.

§3. The calculation of the mean and variance of the quantity θ with respect to the permutation distribution due to randomization.

Since

$$c_1A_1^{\#} + c_2A_2^{\#} = \mu_oA_o + \mu_1A_1 + \mu_2A_2 ,$$

with

(3.1)
$$\begin{cases} \mu_o = \dfrac{k}{(rk-\rho_1)(1+\frac{z_{11}^2}{n_1}+\frac{z_{12}^2}{n_2})} + \dfrac{k}{(rk-\rho_2)(1+\frac{z_{21}^2}{n_1}+\frac{z_{22}^2}{n_2})} , \\[2ex] \mu_1 = \dfrac{kz_{11}}{(rk-\rho_1)(1+\frac{z_{11}^2}{n_1}+\frac{z_{12}^2}{n_2})} + \dfrac{kz_{21}}{(rk-\rho_2)(1+\frac{z_{21}^2}{n_1}+\frac{z_{22}^2}{n_2})} , \\[2ex] \mu_2 = \dfrac{kz_{12}}{(rk-\rho_1)(1+\frac{z_{11}^2}{n_1}+\frac{z_{12}^2}{n_2})} + \dfrac{kz_{22}}{(rk-\rho_2)(1+\frac{z_{21}^2}{n_1}+\frac{z_{22}^2}{n_2})} , \end{cases}$$

let us put

$$(3.2) \qquad T^{\#} = \Phi(c_1 A_1^{\#} + c_2 A_2^{\#})\Phi' = \mu_o T_o + \mu_1 T_1 + \mu_2 T_2 \; .$$

3.1. Necessary notations.

Now we use the numbering of the whole units such that the i-th unit in the p-th block bears the number $f = (p-1)k + i$.

Let

$$(3.3) \qquad T_u = \left\| T_{pq}^{(u)} \right\| , \qquad T_{pq}^{(u)} = \left\| t_{ij}^{(u)pq} \right\|$$

$$p,q = 1,\dots,b \qquad\qquad i,j = 1,\dots,k.$$

where

$$t_{ij}^{(u)pq} = \begin{cases} 1, & \text{if the i-th unit in the p-th block and the j-th unit in the q-th block receive treatments which are u-th associates,} \\ 0, & \text{otherwise,} \end{cases}$$

and further let

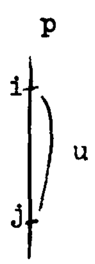
$$(3.4) \qquad T = \left\| T_{pq}^{\#} \right\| , \qquad T_{pq}^{\#} = \left\| t_{ij}^{\#pq} \right\| ,$$

where

$$(3.5) \qquad t_{ij}^{\#pq} = \mu_o t_{ij}^{(0)pq} + \mu_1 t_{ij}^{(1)pq} + \mu_2 t_{ij}^{(2)pq} \; .$$

Notations necessary for the calculations in this section are introduced here for the case $k \geq 4$. The cases in which $k = 2$ or 3 shall be examined later.

(i)

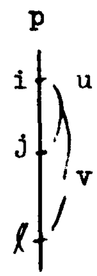


Number of the ordered pairs of treatments being of the u-th associated which occur in the p-th block is denoted by

$$\lambda^{(1)uu}_{pp} \equiv \sum_{i \neq j} t^{(u)pp}_{ij}, \quad u = 1, 2.$$

one can see that

$$\lambda^{(1)11}_{pp} + \lambda^{(1)22}_{pp} = k(k-1).$$

(ii) 

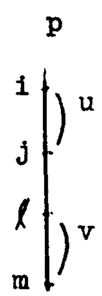
Number of the ordered triplets of treatments in the p-th block, in which the 1st and the second are u-th associates and the 1st and the 3rd are v-th associates is denoted by

$$\lambda^{(2)uv}_{pp} \equiv \sum_{i \neq j \neq \ell} t^{(u)pp}_{ij} t^{(v)pp}_{i\ell}, \quad u,v = 1, 2 .$$

One gets the following relations immediately:

$$\lambda^{(2)uv}_{pp} = \lambda^{(2)vu}_{pp},$$

$$\begin{cases} \lambda^{(2)11}_{pp} + \lambda^{(2)12}_{pp} = (k-2)\lambda^{(1)11}_{pp}, \\ \lambda^{(2)12}_{pp} + \lambda^{(2)22}_{pp} = (k-2)\lambda^{(1)22}_{pp}. \end{cases}$$

(iii) 

Number of two pairs of treatments in the p-th block, of which two treatments of the one pair are u-th associates and those of the other pair are v-th associates is denoted by

$$\lambda^{(3)uv}_{pp} \equiv \sum_{i \neq j \neq \ell \neq m} t^{(u)pp}_{ij} t^{(v)pp}_{\ell m}, \quad u, v, = 1, 2.$$

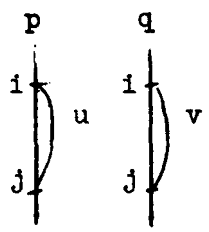
Trivial relations are

$$\lambda^{(3)uv}_{pp} = \lambda^{(3)vu}_{pp},$$

$$\begin{cases} \lambda^{(3)11}_{pp} + \lambda^{(3)12}_{pp} = (k-2)(k-3)\lambda^{(1)11}_{pp}, \\ \lambda^{(3)12}_{pp} + \lambda^{(3)22}_{pp} = (k-2)(k-3)\lambda^{(1)22}_{pp}, \end{cases}$$

Other similar 12 notations with their trivial relations are listed below.

(iv)

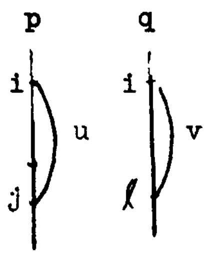


$$\lambda^{(4)uv}_{pq} \equiv \sum_{i\neq j} t^{(u)pp}_{ij} t^{(v)qq}_{ij}\ , \quad u,v = 1,\ 2.$$

$$\begin{cases} \lambda^{(4)11}_{pq} + \lambda^{(4)12}_{pq} = \lambda^{(1)11}_{pp}\ , \\ \lambda^{(4)21}_{pq} + \lambda^{(4)22}_{pq} = \lambda^{(1)22}_{11}\ , \end{cases}$$

$$\begin{cases} \lambda^{(4)11}_{pq} + \lambda^{(4)21}_{pq} = \lambda^{(1)11}_{qq}\ , \\ \lambda^{(4)12}_{pq} + \lambda^{(4)22}_{pq} = \lambda^{(1)22}_{qq}\ . \end{cases}$$

(v)

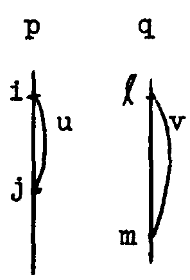


$$\lambda^{(5)uv}_{pq} \equiv \sum_{i\neq j\neq \ell} t^{(u)pp}_{ij} t^{(v)qq}_{i\ell}\ , \quad u,v = 1,2\ .$$

$$\begin{cases} \lambda^{(5)11}_{pq} + \lambda^{(5)12}_{pq} = (k-2)\lambda^{(1)11}_{pp}\ , \\ \lambda^{(5)21}_{pq} + \lambda^{(5)22}_{pq} = (k-2)\lambda^{(1)22}_{pp}\ , \end{cases}$$

$$\begin{cases} \lambda^{(5)11}_{pq} + \lambda^{(5)21}_{pq} = (k-2)\lambda^{(1)11}_{qq}\ , \\ \lambda^{(5)12}_{pq} + \lambda^{(5)22}_{pq} = (k-2)\lambda^{(1)22}_{qq}\ . \end{cases}$$

(vi)

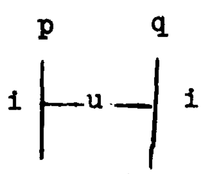


$$\lambda^{(6)uv}_{pq} \equiv \sum_{i\neq j\neq \ell\neq m} t^{(u)pp}_{ij} t^{(v)qq}_{\ell m}\ , \quad u,v = 1,\ 2.$$

$$\begin{cases} \lambda^{(6)11}_{pq} + \lambda^{(6)12}_{pq} = (k-2)(k-3)\lambda^{(1)11}_{pp}\ , \\ \lambda^{(6)21}_{pq} + \lambda^{(6)22}_{pq} = (k-2)(k-3)\lambda^{(1)22}_{pp}\ , \end{cases}$$

$$\begin{cases} \lambda^{(6)11}_{pq} + \lambda^{(6)21}_{pq} = (k-2)(k-3)\lambda^{(1)11}_{qq}\ , \\ \lambda^{(6)12}_{pq} + \lambda^{(6)22}_{pq} = (k-2)(k-3)\lambda^{(1)22}_{qq}\ . \end{cases}$$

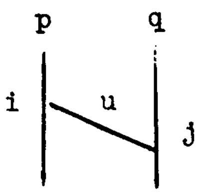
(vii)



$$\lambda^{(7)uu}_{pq} \equiv \sum_{i} t^{(u)pq}_{ii}\ , \quad u = 0,\ 1,\ 2.$$

$$\lambda^{(7)00}_{pq} + \lambda^{(7)11}_{pq} + \lambda^{(7)22}_{pq} = k.$$

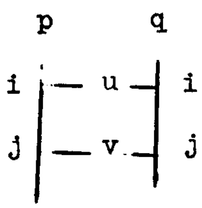
(viii)



$$\lambda^{(8)uu}_{pq} \equiv \sum_{i\neq j} t^{(u)pq}_{ij}\ , \quad u = 0,\ 1,\ 2.$$

$$\lambda^{(8)00}_{pq} + \lambda^{(8)11}_{pq} + \lambda^{(8)22}_{pq} = k(k-1).$$

(ix)

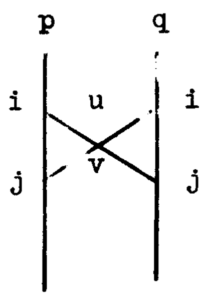


$$\lambda^{(9)uv}_{pq} \equiv \sum_{i\neq j} t^{(u)pq}_{ii}\ t^{(v)pq}_{jj}\ , \quad u,v = 0,\ 1,\ 2.$$

$$\lambda^{(9)uv}_{pq} = \lambda^{(9)vu}_{pq}\ .$$

$$\lambda^{(0)u0}_{pq} + \lambda^{(9)u1}_{pq} + \lambda^{(9)u2}_{pq} = (k-1)\lambda^{(7)uu}_{pq}\ , \quad u = 0,\ 1,\ 2.$$

(x)

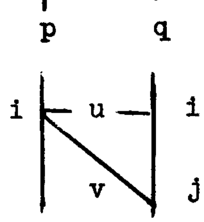


$$\lambda^{(10)uv}_{pq} \equiv \sum_{i\neq j} t^{(u)pq}_{ij}\ t^{(v)pq}_{ji}\ , \quad u,v = 0,\ 1,\ 2.$$

$$\lambda^{(10)uv}_{pq} = \lambda^{(10)vu}_{pq}\ .$$

$$\lambda^{(10)u0}_{pq} + \lambda^{(10)u1}_{pq} + \lambda^{(10)u2}_{pq} = \lambda^{(8)uu}_{pq}\ ,$$

(xi)



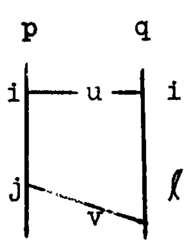
$$\lambda^{(11)uv}_{pq} \equiv \sum_{i\neq j} t^{(u)pq}_{ii}\ t^{(v)pq}_{ij}, \quad u,v = 0,\ 1,\ 2.$$

$$\lambda^{(11)00}_{pq} = 0\ .$$

$$\left\{\begin{aligned} \lambda^{(11)01}_{pq} + \lambda^{(11)02}_{pq} &= (k-1)\lambda^{(7)00}_{pq}\ , \\ \lambda^{(11)u0}_{pq} + \lambda^{(11)u1}_{pq} + \lambda^{(11)u2}_{pq} &= (k-1)\lambda^{(7)uu}_{pq}\ , \end{aligned}\right.$$

$$\left\{\begin{aligned} \lambda^{(11)10}_{pq} + \lambda^{(11)20}_{pq} &= \lambda^{(8)00}_{pq}\ , \\ \lambda^{(11)0u}_{pq} + \lambda^{(11)1u}_{pq} + \lambda^{(11)2u}_{pq} &= \lambda^{(8)uu}_{pq}\ , \quad u = 1,2, \end{aligned}\right.$$

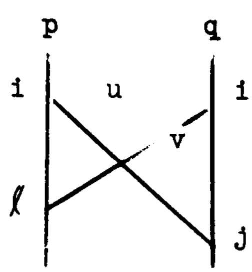
(xii)



$$\lambda^{(12)uv}_{pq} \equiv \sum_{i\neq j\neq \ell} t^{(u)pq}_{ii}\ t^{(v)pq}_{j\ell}\ , \quad u,v = 0,\ 1,\ 2.$$

$$\sum_{v=0}^{2} \lambda^{(12)uv}_{pq} = (k-1)(k-2)\lambda^{(7)uu}_{pq}\ , \quad u=0,\ 1,\ 2.$$

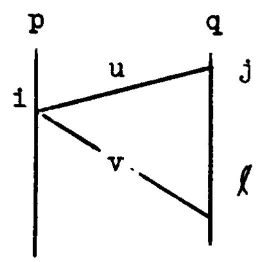
$$\sum_{u=0}^{2} \lambda^{(12)uv}_{pq} = (k-w)\lambda^{(8)vv}_{pq}, \quad v=0,\ 1,\ 2.$$

(xiii) 

$$\lambda^{(13)uv}_{pq} \equiv \sum_{i\neq j\neq \ell} t^{(u)pq}_{ij}\, t^{(v)pq}_{\ell i}\ , \quad u,\ v = 0,\ 1,\ 2.$$

$$\lambda^{(13)uv}_{pq} = \lambda^{(13)vu}_{pq} .$$

$$\sum_{v=0}^{2} \lambda^{(13)uv}_{pq} = (k-2)\lambda^{(8)uu}_{pq} ,$$

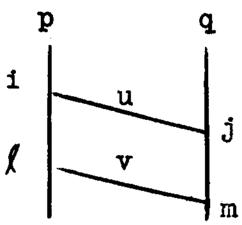
(xiv) 

$$\lambda^{(14)uv}_{pq} \equiv \sum_{i\neq k\neq \ell} t^{(u)pq}_{ij}\, t^{(v)pq}_{i\ell}\ , \quad u,\ v = 0,\ 1,\ 2.$$

$$\lambda^{(14)uv}_{pq} = \lambda^{(14)vu}_{pq} .$$

$$\lambda^{(14)00}_{pq} = 0$$

$$\sum_{v=0}^{2} \lambda^{(14)uv}_{pq} = (k-2)\lambda^{(8)uu}_{pq} .$$

(xv) 

$$\lambda^{(15)uv}_{pq} \equiv \sum_{i\neq j\neq \ell\neq m} t^{(u)pq}_{ij}\, t^{(v)pq}_{\ell m}\ , \quad u,\ v = 0,\ 1,\ 2.$$

$$\lambda^{(15)uv}_{pq} = \lambda^{(15)vu}_{pq} .$$

$$\sum_{v=0}^{2} \lambda^{(15)uv}_{pq} = (k-2)(k-3)\lambda^{(8)uu}_{pq} .$$

We list the relationships holding among those parameters of 15 classes.

$$(\lambda^{(1)11}_{pp})^2 = 2\lambda^{(1)11}_{pp} + 4\lambda^{(2)11}_{pp} + \lambda^{(3)11}_{pp} ,$$

$$\lambda^{(1)22}_{pp} = k(k-1) - \lambda^{(1)11}_{pp} ,$$

$$\begin{cases} \lambda^{(2)12}_{pp} = (k-2)\lambda^{(1)11}_{pp} - \lambda^{(2)11}_{pp} , \\ \lambda^{(2)22}_{pp} = k(k-1)(k-2) - 2(k-2)\lambda^{(1)11}_{pp} + \lambda^{(2)11}_{pp} , \end{cases}$$

$$\lambda^{(3)12}_{pp} = (k-2)(k-3)\lambda^{(1)11}_{pp} - \lambda^{(3)11}_{pp} ,$$

$$\lambda^{(3)22}_{pp} = k(k-1)(k-2)(k-3) - 2(k-2)(k-3)\lambda^{(1)11}_{pp} + \lambda^{(3)11}_{pp}$$

$$\begin{cases} \lambda^{(4)22}_{pq} = k(k-1) - \lambda^{(1)11}_{pp} - \lambda^{(1)11}_{qq} + \lambda^{(4)11}_{pq}, \\ \lambda^{(5)22}_{pq} = k(k-1)(k-2)-(k-2)\lambda^{(1)11}_{pp}-(k-2)\lambda^{(1)11}_{qq}+\lambda^{(5)11}_{pq}, \end{cases}$$

$$\lambda^{(1)11}_{pp}\lambda^{(1)11}_{qq} = 2\lambda^{(4)11}_{pq}+4\lambda^{(5)11}_{pq}+\lambda^{(6)11}_{pq},$$

$$\begin{cases} \lambda^{(6)22}_{pq} = k(k-1)(k-2)(k-3)-(k-2)(k-3)\lambda^{(1)11}_{pp} \\ \qquad\qquad - (k-2)(k-3)\lambda^{(1)11}_{qq} + \lambda^{(6)11}_{pq}, \\ \lambda^{(7)22}_{pq} = k-\lambda^{(7)00}_{pq}-\lambda^{(7)11}_{pq}, \\ \lambda^{(8)22}_{pq} = k(k-1)-\lambda^{(8)00}_{pq}-\lambda^{(8)11}_{pq}, \\ \lambda^{(9)22}_{pq} = k(k-1)-2(k-1)\lambda^{(7)00}_{pq}-2(k-1)\lambda^{(7)11}_{pq} \\ \qquad\qquad +\lambda^{(9)00}_{pq}+2\lambda^{(9)01}_{pq}+\lambda^{(9)11}_{pq}, \\ \lambda^{(10)22}_{pq} = k(k-1)-2\lambda^{(8)00}_{pq}-2\lambda^{(8)11}_{pq}+ \\ \qquad\qquad + \lambda^{(10)00}_{pq}+2\lambda^{(10)01}_{pq}+\lambda^{(10)11}_{pq}, \end{cases}$$

$$\lambda^{(11)20}_{pq} = \lambda^{(8)00}_{pq}-\lambda^{(11)10}_{pq},$$

$$\begin{cases} \lambda^{(11)22}_{pq} = k(k-1)-(k-1)\lambda^{(7)00}_{pq}-(k-1)\lambda^{(7)11}_{pq} \\ \qquad\qquad - \lambda^{(8)00}_{pq}-\lambda^{(8)11}_{pq}+\lambda^{(11)01}_{pq}+\lambda^{(11)10}_{pq}+\lambda^{(11)11}_{pq}, \\ \lambda^{(12)22} = k(k-1)(k-2)-(k-1)(k-2)\lambda^{(7)00}_{pq}-(k-1)(k-2)\lambda^{(7)11}_{pq} \\ \qquad\qquad - (k-2)\lambda^{(8)00}_{pq}-(k-2)\lambda^{(8)11}_{pq} \\ \qquad\qquad + \lambda^{(12)00}_{pq}+\lambda^{(12)01}_{pq}+\lambda^{(12)10}_{pq}+\lambda^{(12)11}_{pq}, \end{cases}$$

$$\lambda^{(13)22}_{pq} = k(k-1)(k-2)-2(k-2)\lambda^{(8)00}_{pq}-2(k-2)\lambda^{(8)11}_{pq}$$
$$+ \lambda^{(13)00}_{pq}+2\lambda^{(13)01}_{pq}+\lambda^{(13)11}_{pq} ,$$

$$\lambda^{(14)22}_{pq} = k(k-1(k-2)-2(k-2)\lambda^{(8)00}_{pq}-2(k-2)\lambda^{(8)11}_{pq}$$
$$+ 2\lambda^{(14)01}_{pq}+\lambda^{(14)11}_{pq} ,$$

$$\lambda^{(15)22}_{pq} = k(k-1)(k-2)(k-3)-2(k-2)(k-3)\lambda^{(8)00}_{pq}$$
$$- 2(k-2)(k-3)\lambda^{(8)11}_{pq}+\lambda^{(15)00}_{pq}+2\lambda^{(15)01}_{pq}+\lambda^{(15)11}_{pq} ,$$

$$(\lambda^{(7)00}_{pq})^2 = \lambda^{(7)00}_{pq} + \lambda^{(9)00}_{pq} ,$$

$$\lambda^{(7)00}_{pq}\lambda^{(7)11}_{pq} = \lambda^{(9)01}_{pq} ,$$

$$(\lambda^{(7)11}_{pq})^2 = \lambda^{(7)11}_{pq} + \lambda^{(9)11}_{pq} ,$$

$$\lambda^{(7)00}_{pq}\lambda^{(8)00}_{pq} = \lambda^{(12)00}_{pq} ,$$

$$\lambda^{(7)00}_{pq}\lambda^{(8)11}_{pq} = \lambda^{(11)01}_{pq} + \lambda^{(11)01}_{qp} + \lambda^{(11)01}_{pq}$$

$$\lambda^{(7)11}_{pq}\lambda^{(8)00}_{pq} = \lambda^{(11)10}_{pq} + \lambda^{(11)10}_{qp} + \lambda^{(12)10}_{pq} ,$$

$$\lambda^{(7)11}_{pq}\lambda^{(8)11}_{pq} = \lambda^{(11)11}_{pq} + \lambda^{(11)11}_{qp} + \lambda^{(12)11}_{pq} ,$$

$$(\lambda^{(8)00}_{pq})^2=\lambda^{(8)00}_{pq}+\lambda^{(10)00}_{pq}+\lambda^{(14)00}_{pq}+\lambda^{(14)00}_{qp}+2\lambda^{(13)00}_{pq}+\lambda^{(15)00}_{pq} ,$$

$$(\lambda^{(8)11}_{pq})^2=\lambda^{(8)11}_{pq}+\lambda^{(10)11}_{pq}+\lambda^{(14)11}_{pq}+\lambda^{(14)11}_{qp}+2\lambda^{(13)11}_{pq}+\lambda^{(15)11}_{pq} ,$$

$$\lambda^{(8)00}_{pq}\lambda^{(8)11}_{pq}=\lambda^{(10)01}_{pq}=\lambda^{(14)01}_{pq}+\lambda^{(14)01}_{qp}+2\lambda^{(13)01}_{pq}+\lambda^{(15)01}_{pq} .$$

3.2 The mean value of θ.

Let us write

$$\theta = \Theta/\Delta,$$

where

(3.2.1) $$\Theta = \pi' T^{\#} \pi = \pi'(\mu_o T_o + \mu_1 T_1 + \mu_2 T_2)\pi \ .$$

Following Ogawa [4], we calculate $\mathcal{E}(\Theta)$ as follows:

(3.2.2) $$\mathcal{E}(\Theta) = \frac{1}{(k!)^b} \sum_{\sigma_1,\ldots,\sigma_b} \pi' U_\sigma' T^{\#} U_\sigma \pi$$

$$= \frac{1}{(k!)^b} \sum_{\sigma_1,\ldots,\sigma_b} \left[\sum_{p=1}^{b} \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)} + \sum_{p \neq q} \pi^{(p)'} S_{\sigma_p}' T_{pq}^{\#} S_{\sigma_q} \pi^{(q)} \right] .$$

Now, since

$$\pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)} = \sum_{i,j=1}^{k} t_{ij}^{\#pp} \pi_{\sigma(i)}^{(p)} \pi_{\sigma(j)}^{(p)}$$

$$= \sum_{i=1}^{k} t_{ii}^{\#pp} \pi_{\sigma(i)}^{(p)^2} + \sum_{i \neq k} t_{ij}^{\#pp} \pi_{\sigma(i)}^{(p)} \pi_{\sigma(j)}^{(p)} ,$$

and

$$\pi^{(p)'} S_{\sigma_p}' T_{pq}^{\#} S_{\sigma_q} \pi^{(q)} = \sum_{i=1}^{k} t_{ii}^{\#pq} \pi_{\sigma(i)}^{(p)} \pi_{\tau(i)}^{(q)} + \sum_{i \neq j} t_{ij}^{\#pq} \pi_{\sigma(i)}^{(p)} \pi_{\tau(j)}^{(q)},$$

where we have put $\sigma_p = \sigma$ and $\sigma_q = \tau$, and

$$\mathcal{E}(\pi_{\sigma(i)}^{(p)^2}) = \frac{1}{k} \Delta_p ,$$

$$\mathcal{E}(\pi_{\sigma(i)}^{(p)} \pi_{\sigma(j)}^{(p)}) = \frac{1}{k(k-1)} \Delta_p ,$$

$$\mathcal{E}(\pi_{\sigma(i)}^{(p)} \pi_{\tau(i)}^{(q)}) = \mathcal{E}(\pi_{\sigma(i)}^{(p)} \pi_{\tau(j)}^{(q)}) = 0 ,$$

we get

(3.2.3) $$\mathcal{E}(\Theta) = \frac{1}{k}\sum_{p=1}^{b} \Delta_p \left(\sum_{i=1}^{k} t_{ii}^{\#pp}\right) - \frac{1}{k(k-1)}\sum_{p=1}^{b} \Delta_p \left(\sum_{i\neq j} t_{ij}^{\#pp}\right),$$

where $\Delta_p = \pi^{(p)'}\pi^{(p)}$.

Now, it can be seen that

$$\sum_{i=1}^{k} t_{ii}^{\#pp} = \sum_{i=1}^{k} (\mu_o t_{ii}^{(0)pp} + \mu_1 t_{ii}^{(1)pp} + \mu_2 t_{ii}^{(2)pp}$$

$$= \mu_o \sum_{i=1}^{k} t_{ii}^{(0)pp} = k\mu_o ,$$

and

$$\sum_{i\neq j} t_{ij}^{\#pp} = \sum_{i\neq j} (\mu_o t_{ij}^{(0)pp} + \mu_1 t_{ij}^{(1)pp} + \mu_2 t_{ij}^{(2)pp})$$

$$= \mu_1 \sum_{i\neq j} t_{ij}^{(1)pp} + \mu_2 \sum_{i\neq j} t_{ij}^{(2)pp}$$

$$= \mu_1 \lambda_{pp}^{(1)11} + \mu_2 \lambda_{pp}^{(1)22}$$

$$= \mu_1 \lambda_{pp}^{(1)11} + \mu_2 [k(k-1) - \lambda_{pp}^{(1)11}]$$

$$= k(k-1)\mu_2 + (\mu_1-\mu_2)\lambda_{pp}^{(1)11} .$$

Hence we have

(3.2.4) $$\mathcal{E}(\Theta) = (\mu_o-\mu_2) - \frac{\mu_1-\mu_2}{k(k-1)} \sum_{p=1}^{b} \Delta_p \lambda_{pp}^{(1)11} ,$$

and consequently

(3.2.5) $$\mathcal{E}(\Theta) = \mu_o-\mu_2 - \frac{\mu_1-\mu_2}{k(k-1)} \frac{1}{\Delta} \sum_{p=1}^{b} \Delta_p \lambda_{pp}^{(1)11} .$$

As a special case, we shall examine the case of BIBD. In this case, since the spectral decomposition of the matrix NN' is

$$NN' = rk\frac{1}{v} G_v + (r-\lambda)(I_v - \frac{1}{v} G_v) \quad ,$$

and therefore

$$\rho_o = rk, \quad \rho_1 = \rho_2 = r-\lambda \ .$$

Whence it follows that

$$\mu_o = \frac{k(v-1)}{v^2\lambda} \ , \ \mu_1 = \mu_2 = - \frac{k}{v^2\lambda} \ .$$

Thus for a BIBD, (3.2.5) reduces

$$\mathcal{E}\,(\theta) = \frac{k}{v\lambda}$$

which confirms the earlier result given by Ogawa [4].

3.3. The variance of θ.

$$\Theta^2 = (\pi' U_\sigma' T^{\#} U_\sigma \pi)^2$$

$$= (\sum_{p=1}^{b} \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)} + \sum_{p\neq q} \pi^{(p)'} S_{\sigma_p}' T_{pq}^{\#} S_{\sigma_q} \pi^{(q)})^2$$

$$= (\sum_p \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)})^2$$

$$+ 2(\sum_p \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)})(\sum_{r\neq s} \pi^{(r)'} S_{\sigma_r}' T_{rs}^{\#} S_{\sigma_s} \pi^{(s)})$$

$$+ (\sum_{r\neq s} \pi^{(r)'} S_{\sigma_r}' T_{rs}^{\#} S_{\sigma_s} \pi^{(s)})^2.$$

This can be expanded as follows:

$$\Theta^2 = \sum_p \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)} \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)}$$

$$+ \sum_{p\neq q} \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p}' \pi^{(p)} \pi^{(q)'} S_{\sigma_p}' T_{qq}^{\#} S_{\sigma_q} \pi^{(q)}$$

$$+ 2[\sum_{p\neq q} \pi^{(p)'} S_{\sigma_p} T_{pp}^{\#} S_{\sigma_p} \pi^{(p)} \{\pi^{(p)'} S_{\sigma_p}' T_{pq}^{\#} S_{\sigma_q} \pi^{(q)} + \pi^{(q)'} S_{\sigma_q}' T_{qp}^{\#} S_{\sigma_p} \pi^{(p)}\}$$

$$+ \sum_{p\neq q\neq r} \pi^{(p)'} S_{\sigma_p}' T_{pp}^{\#} S_{\sigma_p} \pi^{(p)} \pi^{(q)'} S_{\sigma_q}' T_{qr}^{\#} S_{\sigma_r} \pi^{(r)}]$$

$$+ \sum_{p \neq q} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} [\pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} + \pi^{(q)'} S'_{\sigma_q} T^{\#}_{qp} S_{\sigma_p} \pi^{(p)}]$$

$$+ \sum_{p \neq q \neq r} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} [\pi^{(p)'} S'_{\sigma_p} T^{\#}_{pr} S_{\sigma_r} \pi^{(r)} + \pi^{(r)'} S'_{\sigma_r} T^{\#}_{rp} S_{\sigma_p} \pi^{(p)}]^{\#}$$

$$+ \sum_{p \neq q \neq r} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} [\pi^{(q)'} S'_{\sigma_q} T^{\#}_{qr} S_{\sigma_r} \pi^{(r)} + \pi^{(r)} S'_{\sigma_r} T^{\#}_{rq} S_{\sigma_q} \pi^{(q)}]^{\#}$$

$$+ \sum_{p \neq q \neq r \neq s} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} \pi^{(r)'} S'_{\sigma_r} T^{\#}_{rs} S_{\sigma_s} \pi^{(s)^{\#}}$$

Since the terms which are linear with respect to some S_σ such as those with $\#$ in the above vanish by taking their expectations, we have

$$\mathcal{E}(\Theta^2) = \mathcal{E}(A) + \mathcal{E}(B) + \mathcal{E}(C) + \mathcal{E}(C'),$$

where

$$(3.3.1) \qquad \begin{cases} A = \sum_{p} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pp} S_{\sigma_p} \pi^{(p)} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pp} S_{\sigma_p} \pi^{(p)} , \\ B = \sum_{p \neq q} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pp} S_{\sigma_p} \pi^{(p)} \pi^{(q)'} S'_{\sigma_q} T^{\#}_{qq} S_{\sigma_q} \pi^{(q)} , \\ C = \sum_{p \neq q} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} \pi^{(q)}, \\ C' = \sum_{p \neq q} \pi^{(p)'} S'_{\sigma_p} T^{\#}_{pq} S_{\sigma_q} \pi^{(q)} \pi^{(q)'} S'_{\sigma_q} T^{\#}_{qp} S_{\sigma_p} \pi^{(p)}. \end{cases}$$

Since $C = C'$, it follows that

$$(3.3.2) \qquad \mathcal{E}(\Theta^2) = \mathcal{E}(A) + \mathcal{E}(B) + 2\mathcal{E}(C).$$

Now we shall calculate $\mathcal{E}(A)$, $\mathcal{E}(B)$ and $\mathcal{E}(C)$ in this order. We present the process of the calculation of $\mathcal{E}(A)$ in detail.

Since

$$(\pi^{(p)'} S'_{\sigma_p} T^{\#}_{pp} S_{\sigma_p} \pi^{(p)})^2$$

$$= (\sum_i t^{\#pp}_{ii} \pi^{(p)}_{\sigma(i)} {}^2 + \sum_{i \neq j} t^{\#pp}_{ij} \pi^{(p)}_{\sigma(i)} \pi^{(p)}_{\sigma(j)})^2$$

$$= \sum_i t^{\#pp}_{ii}{}^2 \pi^{(p)}_{\sigma(i)}{}^4 + \sum_{i \neq j} t^{\#pp}_{ii} t^{\#pp}_{jj} \pi^{(p)}_{\sigma(i)}{}^2 \pi^{(p)}_{\sigma(j)}{}^2 + 4 \sum_{i \neq j} t^{\#pp}_{ii} t^{\#pp}_{ij} \pi^{(p)}_{\sigma(i)}{}^3 \pi^{(p)}_{\sigma(j)}$$

$$+ 2 \sum_{i \neq j \neq \ell} t^{\#pp}_{ii} t^{\#pp}_{j\ell} \pi^{(p)}_{\sigma(i)}{}^2 \pi^{(p)}_{\sigma(j)} \pi^{(p)}_{\sigma(\ell)} + 2 \sum_{i \neq j} t^{\#pp}_{ij} t^{\#pp}_{ij} \pi^{(p)}_{\sigma(i)}{}^2 \pi^{(p)}_{\sigma(j)}{}^2$$

$$+ 4 \sum_{i \neq j \neq \ell} t^{\#pp}_{ij} t^{\ pp}_{i\ell} \pi^{(p)}_{\sigma(i)}{}^2 \pi^{(p)}_{\sigma(j)} \pi^{(p)}_{\sigma(\ell)}$$

$$+ \sum_{i \neq j \neq \ell \neq m} t^{\#pp}_{ij} t^{\#pp}_{\ell m} \pi^{(p)}_{\sigma(i)} \pi^{(p)}_{\sigma(j)} \pi^{(p)}_{\sigma(\ell)} \pi^{(p)}_{\sigma(m)},$$

and

(A_1) $$\mathcal{E}\,(\pi^{(p)}_{\sigma(i)}{}^4) = \frac{1}{k}\,\Gamma_p \quad \text{where} \quad \Gamma_p = \sum_{i=1}^{k} \pi^{(p)}_i{}^4 ,$$

$$\sum_{i=1}^{k} t^{\#pp}_{ii}{}^2 = k\mu_o{}^2 ,$$

(A_2) $$\mathcal{E}\,(\pi^{(p)}_{\sigma(i)}{}^2 \, \pi^{(p)}_{\sigma(j)}{}^2) = \frac{1}{k(k-1)}\,(\Delta_p^2 - \Gamma_p) ,$$

$$\sum_{i \neq j} t^{\#pp}_{ii} t^{\#pp}_{jj} = \sum_{i \neq j} (\mu_o t^{(0)pp}_{ii} + \mu_1 t^{(1)pp}_{ii} + \mu_2 t^{(2)pp}_{ii})$$

$$(\mu_o t^{(0)pp}_{jj} + \mu_1 t^{(1)pp}_{jj} + \mu_2 t^{(2)pp}_{jj})$$

$$= \sum_{i \neq j} \mu_o^2 = k(k-1)\mu_o^2 ,$$

(A_3) $$\mathcal{E}\,(\pi^{(p)}_{\sigma(i)}{}^3 \pi^{(p)}_{\sigma(j)}) = -\frac{1}{k(k-1)}\ \Gamma_p ,$$

$$\sum_{i \neq j} t^{\#pp}_{ii} t^{\#pp}_{ij} = \sum_{i \neq j} (\mu_o t^{(0)pp}_{ii} + \mu_1 t^{(1)pp}_{ii} + \mu_2 t^{(2)pp}_{ii})$$

$$(\mu_o t^{(0)pp}_{ij} + \mu_1 t^{(1)pp}_{ij} + \mu_2 t^{(2)pp}_{ij})$$

$$= \mu_o \mu_1 \sum_{i \neq j} t^{(1)pp}_{ij} + \mu_o \mu_2 \sum_{i \neq j} t^{(2)pp}_{ij}$$

$$= \mu_o \mu_1 \lambda^{(1)11}_{pp} + \mu_o \mu_2 \lambda^{(1)22}_{pp} = \mu_o \sum_{s=0}^{2} \mu_s \lambda^{(1)ss}_{pp} ,$$

(A_4)
$$\mathcal{E}\ (\pi^{(p)^2}_{\sigma(i)} \pi^{(p)}_{\sigma(j)} \pi^{(p)}_{\sigma(\ell)}) = - \frac{1}{k(k-1)(k-2)} (\Delta^2_p - 2\Gamma_p),$$

$$\sum_{i \neq j \neq \ell} t^{\#pp}_{ii} t^{\#pp}_{j\ell} = \sum_{i \neq j \neq \ell} (\mu_o t^{(0)pp}_{ii} + \mu_1 t^{(1)pp}_{ii} + \mu_2 t^{(2)pp}_{ii})$$

$$(\mu_o t^{(0)pp}_{j\ell} + \mu_1 t^{(1)pp}_{j\ell} + \mu_2 t^{(2)pp}_{j\ell})$$

$$= (k-2)\mu_o \sum_{j \neq \ell} (\mu_o t^{(0)pp}_{j\ell} + \mu_1 t^{(1)pp}_{j\ell} + \mu_2 t^{(2)pp}_{j\ell})$$

$$= (k-2)\mu_o (\mu_1 \lambda^{(1)11}_{pp} + \mu_2 \lambda^{(1)22}_{pp}) = (k-2)\mu_o \sum_{s=0}^{2} \mu_s \lambda^{(1)ss}_{pp} .$$

(A_5)
$$\mathcal{E}\ (\pi^{(p)^2}_{\sigma(i)} \pi^{(p)^2}_{\sigma(j)}) = \frac{1}{k(k-1)} (\Delta^2_p - \Gamma_p),$$

$$\sum_{i \neq j} t^{\#pp^2}_{ij} = \sum_{i \neq j} (\mu_o t^{(0)pp} + \mu_1 t^{(1)pp} + \mu_2 t^{(2)pp})^2$$

$$= \sum_{i \neq j} (\mu_1^2 t^{(1)pp}_{ij} + \mu_2^2 t^{(2)pp}_{ij})$$

$$= \mu_1^2 \lambda^{(1)11}_{pp} + \mu_2^2 \lambda^{(1)22}_{pp} = \sum_{s=0}^{2} \mu_s^2 \lambda^{(1)ss}_{pp} .$$

(A_6)
$$\mathcal{E}\ (\pi^{(p)^2}_{\sigma(i)} \pi^{(p)}_{\sigma(j)} \pi^{(p)}_{\sigma(\ell)}) = - \frac{1}{k(k-1)(k-2)} (\Delta^2_p - 2\Gamma_p),$$

$$\sum_{i \neq j \neq \ell} t^{\#pp}_{ij} t^{\#pp}_{i\ell} = \sum_{i \neq j \neq \ell} (\mu_o t^{(0)pp}_{ij} + \mu_1 t^{(1)pp}_{ij} + \mu_2 t^{(2)pp}_{ij})$$

$$(\mu_o t^{(0)pp}_{i\ell} + \mu_1 t^{(1)pp}_{i\ell} + \mu_2 t^{(2)pp}_{i\ell})$$

$$= \sum_{s,t} \mu_s\mu_t \sum_{i\neq j\neq \ell} t^{(s)pp}_{ij} t^{(t)pp}_{i\ell} = \sum_{s,t=0}^{2} \mu_s\mu_t \lambda^{(2)st}_{pp} .$$

(A$_7$) $$\mathcal{E}(\pi^{(p)}_{\sigma(i)}\pi^{(p)}_{\sigma(j)}\pi^{(p)}_{\sigma(\ell)}\pi^{(p)}_{\sigma(m)}) = \frac{1}{k(k-1)(k-2)(k-3)} (\Delta_p^2 - 2\Gamma_p),$$

$$\sum_{i\neq j\neq \ell\neq m} t^{\#pp}_{ij} t^{\#pp}_{\ell m} = \sum_{s,t=0}^{2} \mu_s\mu_t \sum_{i\neq j\neq \ell\neq m} t^{(s)pp}_{ij} t^{(t)pp}_{\ell m}$$

$$= \sum_{s,t=0}^{2} \mu_s\mu_t \lambda^{(3)st}_{pp}.$$

Hence we obtain

(3.3.3)
$$\mathcal{E}(A) = \sum_p [\mu_o^2\Gamma_p + \mu_o^2(\Delta_p^2 - \Gamma_p) - \frac{4}{k(k-1)} \mu_o\Gamma_p \sum_s \mu_s \lambda^{(1)ss}_{pp}$$
$$- \frac{2}{k(k-1)} \mu_o (\Delta_p^2 - 2\Gamma_p) \sum_s \mu_s \lambda^{(1)ss}_{pp}$$
$$+ \frac{2}{k(k-1)} (\Delta_p^2 - \Gamma_p) \sum_s \mu_s^2 \lambda^{(1)ss}_{pp}$$
$$- \frac{4}{k(k-1)(k-2)} (\Delta_p^2 - 2\Gamma_p) \sum_{s,t} \mu_s\mu_t \lambda^{(2)st}_{pp}$$
$$+ \frac{3}{k(k-1)(k-2)(k-3)} (\Delta_p^2 - 2\Gamma_p) \sum_{s,t} \mu_s\mu_t \lambda^{(3)st}_{pp}]$$
$$= \sum_p [\frac{1}{k(k-1)} \Delta_p^2 \sum_s (\mu_o - \mu_s)^2 \lambda^{(1)ss}_{pp}$$
$$+ (\Delta_p^2 - 2\Gamma_p) \frac{1}{k(k-1)} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp} - \frac{4}{k(k-1)(k-2)} \sum_{s,t} \mu_s\mu_t \lambda^{(2)st}_{pp}$$
$$+ \frac{3}{k(k-1)(k-2)(k-3)} \sum_{s,t} \mu_s\mu_t \lambda^{(3)st}_{pp}].$$

In similar manner one can obtain

(3.3.4)
$$\mathcal{E}(B) = \sum_{p\neq q} \Delta_p\Delta_q [\frac{1}{k(k-1)} \sum_s (\mu_o - \mu_s)^2 \lambda^{(1)ss}_{pp} - \frac{1}{k(k-1)} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp}$$
$$+ \frac{2}{k^2(k-1)^2} \sum_{s,t} \mu_s\mu_t \lambda^{(4)st}_{pq}$$

$$+ \frac{4}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t \lambda^{(5)st}_{pq}$$

$$+ \frac{1}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t \lambda^{(6)st}_{pq}]$$

$$= \mathcal{E}^2(\Theta) - \sum_p \Delta_p^2 [\frac{1}{k(k-1)} \sum_s (\mu_o - \mu_s)^2 \lambda^{(1)ss}_{pp} - \frac{1}{k(k-1)} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp}$$

$$+ \frac{2}{k^2(k-1)^2} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp} + \frac{4}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t \lambda^{(2)st}_{pp}$$

$$+ \frac{1}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t \lambda^{(3)st}_{pp}]$$

and

$$\mathcal{E}(C) = \sum_{p \neq q} \Delta_p \Delta_q [\frac{1}{k^2} \sum_s \mu_s^2 (\lambda^{(7)ss}_{pq} + \lambda^{(8)ss}_{pq})$$

(3.3.5) $$+ \frac{1}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t (\lambda^{(9)st}_{pq} + \lambda^{(10)st}_{pq} + 2\lambda^{(12)st}_{pq} + 2\lambda^{(13)st}_{pq} + \lambda^{(15)st}_{pq})$$

$$- \frac{2}{k^2(k-1)} \sum_{s,t} \mu_s \mu_t (2\lambda^{(11)st}_{pq} + \lambda^{(14)st}_{pq})].$$

Therefore one obtains the variance of θ as

$$\text{Var } \theta = \Delta^{-2} \text{ Var } (\Theta)$$

$$= \frac{\mathcal{E}(A) + \mathcal{E}(B) + 2\mathcal{E}(C) - \mathcal{E}^2}{\Delta^2}$$

$$= \sum_p \frac{\Delta_p^2 - 2\Gamma_p}{\Delta^2} [\frac{1}{k(k-1)} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp} \frac{4}{k(k-1)(k-2)} \sum_{s,t} \mu_s \mu_t \lambda^{(2)st}_{pp}$$

(3.3.6) $$+ \frac{3}{k(k-1)(k-2)(k-3)} \sum_{s,t} \mu_s \mu_t \lambda^{(3)st}_{pp}]$$

$$+ \sum_p \frac{\Delta_p^2}{\Delta^2} [\frac{1}{k(k-1)} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp} - \frac{2}{k^2(k-1)^2} \sum_s \mu_s^2 \lambda^{(1)ss}_{pp}$$

$$- \frac{4}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t \lambda_{pp}^{(2)st} - \frac{1}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t \lambda_{pp}^{(3)st}]$$

$$+ \sum_{p \neq q} \frac{\Delta_p \Delta_q}{\Delta^2} [\frac{2}{k^2} \sum_s \mu_s^2 (\lambda_{pq}^{(7)ss}) + \lambda_{pq}^{(8)\ st})$$

$$+ \frac{2}{k^2(k-1)^2} \sum_{s,t} \mu_s \mu_t (\lambda_{pq}^{(9)st} + \lambda_{pq}^{(10)st} + 2\lambda_{pq}^{(12)st} + 2\lambda_{pq}^{(13)st} + \lambda_{pq}^{(15)st})$$

$$- \frac{4}{k^2(k-1)} \sum_{s,t} \mu_s \mu_t (2\lambda_{pq}^{(11)st} + \lambda_{pq}^{(14)st})].$$

For a BIBD, (3.3.6) reduces

$$\text{Var } \theta = 2(v\lambda)^{-2} \sum_{p \neq q} \Delta_p \Delta_q [\lambda_{pq} + \frac{1}{(k-1)^2} (\lambda_{pq}^2 - \lambda_{pq})]$$

where

$$\lambda_{pq} = \lambda_{pq}^{(7)00} + \lambda_{pq}^{(8)00} = \text{number of treatments common to the}$$

p-th and q-th blocks,

and this confirms the Ogawa result once again [4].

§4. An approximation to the permutation distribution of θ by a certain Beta-distribution when the number b of the blocks is sufficiently large and certain uniformity conditions are imposed on the unit errors.

In this section, we assume that the following uniformity conditions

(4.1) $$\Delta_p = \Delta_o \text{ and } \Gamma_p = \Gamma_o \text{ for } p = 1, 2, \ldots, b$$

are imposed on the unit errors. Under such conditions the mean of θ and the variance of θ, Var θ, can be presented as follows :

(4.2) $$\mathcal{E}(\theta) = \frac{1}{bk(k-1)} \sum_{s=1}^{2} (\mu_o - \mu_s) \sum_p \lambda_{pp}^{(1)ss}$$

$$\text{Var } \theta = (1-2 \frac{\Gamma_o}{\Delta_o^2})[\frac{1}{b^2k(k-1)} \sum_{s=0}^{2} \mu_s^2 \sum_p \lambda_{pp}^{(1)ss}$$

(4.3)

$$- \frac{4}{b^2 k(k-1)(k-2)} \sum_{s,t} \mu_s \mu_t \sum_p \lambda^{(2)st}_{pp}$$

$$+ \frac{3}{b^2 k(k-1)(k-2)(k-3)} \sum_{s,t} \mu_s \mu_t \sum_p \lambda^{(3)st}_{pp}]$$

$$+ [\frac{1}{b^2 k(k-1)} \sum_s \mu_s^2 \sum_p \lambda^{(1)ss}_{pp} - \frac{2}{b^2 k^2 (k-1)^2} \sum_s \mu_s^2 \sum_p \lambda^{(1)ss}_{pp}$$

$$- \frac{4}{b^2 k^2 (k-1)^2} \sum_{s,t} \mu_s \mu_t \sum_p \lambda^{(2)st}_{pp} - \frac{1}{b^2 k^2 (k-1)^2} \sum_{s,t} \mu_s \mu_t \sum_p \lambda^{(3)st}_{pp}]$$

$$+ \frac{2}{b^2 k^2} \sum_s \mu_s^2 \sum_{p \neq q} (\lambda^{(7)ss}_{pq} + \lambda^{(8)ss}_{pq})$$

$$+ \frac{2}{b^2 k^2 (k-1)^2} \sum_{s,t} \mu_s \mu_t \sum_{p \neq q} (\lambda^{(9)st}_{pq} + \lambda^{(10)st}_{pq} + 2\lambda^{(12)st}_{pq} + 2\lambda^{(13)st}_{pq} + \lambda^{(15)st}_{pq})$$

$$- \frac{4}{b^2 k^2 (k-1)} \sum_{s,t} \mu_s \mu_t \sum_{p \neq q} (2\lambda^{(11)st}_{pq} + \lambda^{(14)st}_{pq}) .$$

In order to express Var θ in terms of the parameters of the design under consideration, we show that

(4.4)

(i) $$\sum_p \lambda^{(1)ss}_{pp} = \lambda_s n_s v,$$

(ii) $$\sum_p (\lambda^{(7)ss}_{pp} + \lambda^{(8)ss}_{pp}) = (r^2 - \lambda_s) n_s v,$$

(iii) $$\sum_{p \neq q} (\lambda^{(9)st}_{pq} + \lambda^{(10)st}_{pq} + 2\lambda^{(12)st}_{pq} + 2\lambda^{(13)st}_{pq} + \lambda^{(15)st}_{pq})$$

$$= \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta \sum_{\gamma=0}^{2} p^\gamma_{\alpha\beta} p^\gamma_{st} n_\gamma v - \delta_{st} \lambda_s n_s v - \delta_{s0}(1-\delta_{t0})(k-2)\lambda_t n_t v$$

$$- \delta_{t0}(k-2)\lambda_s n_s v - 2\lambda^{(1)}_{st} - \lambda^{(2)}_{st},$$

(iv) $$\sum_{p \neq q} (2\lambda^{(11)st}_{pq} + \lambda^{(14)st}_{pq}) = r \sum_\alpha \lambda_\alpha p^\alpha_{st} n_\alpha v - \delta_{s0}(1-\delta_{t0})\lambda_t n_t v$$

$$- \delta_{t0}(1-\delta_{s0})\lambda_s n_s v - \lambda^{(1)}_{st},$$

where we have put

$$(4.5) \qquad \lambda^{(1)}_{st} = \sum_p \lambda^{(2)st}_{pp} \quad \text{and} \quad \lambda^{(2)}_{st} = \sum_p \lambda^{(3)st}_{pp} .$$

Indeed,

$$(4.4)\ (i) \qquad \sum_p \lambda^{(1)ss}_{pp} = \sum_p \sum_{i \neq j} t^{(s)pp}_{ij} = \sum_p \sum_{\alpha,\beta} n_{\alpha p}\, p^{\beta}_{\alpha s}\, n_{\beta p} = tr(N'A_sN)$$

$$= tr(NN'A_s) = tr(\sum_\alpha \lambda_\alpha \sum_\beta p^{\beta}_{\alpha s} A_\beta) = \lambda_s n_s v .$$

$$(4.4)\ (ii) \qquad \sum_{p \neq q} (\lambda^{(7)ss}_{pq} + \lambda^{(8)ss}_{pq}) = \sum_{p \neq q} (\sum_{i,j} t^{(s)pq}_{ij})$$

$$= \sum_{p,q} \sum_{\alpha,\beta} n_{\alpha p}\, p^{\beta}_{\alpha s}\, n_{\beta q} - \sum_p \sum_{\alpha,\beta} n_{\alpha p}\, p^{\beta}_{\alpha s}\, n_{\beta q}$$

$$= 1'N'A_sN1 - tr(N'A_sN) = r^2 n_s v - \lambda_s n_s v = (r^2 - \lambda_s) n_s v .$$

(4.4) (iii) It can be shown that

$$\sum_{p \neq q} (\lambda^{(7)00}_{pq} + \lambda^{(8)00}_{pq})^2 = \sum_{p \neq q} [(\lambda^{(7)00}_{pq} + \lambda^{(8)00}_{pq})$$

$$+ (\lambda^{(9)00}_{pq} + \lambda^{(10)00}_{pq} + 2\lambda^{(12)00}_{pq} + 2\lambda^{(13)00}_{pq} + \lambda^{(15)00}_{pq})].$$

Now, let us consider the following identity relation:

$$\sum_{p,q} (\sum_{\alpha,\beta} n_{\alpha p}\, p^{\beta}_{\alpha 0}\, n_{\beta q})^2 = tr(N'A_0N)^2 .$$

Since

$$\sum_{\alpha,\beta} n_{\alpha p}\, p^{\beta}_{\alpha 0}\, n_{\beta q} = \begin{cases} k & \text{if } p=q \\ \lambda^{(7)00}_{pq} + \lambda^{(8)00}_{pq} & \text{if } p \neq q \end{cases}$$

the left-hand side of the identity is shown to be

$$\sum_{p,q} (\sum_{\alpha,\beta} n_{\alpha p}\, p^{\beta}_{\alpha 0}\, n_{\beta q})^2 = bk^2 + \sum_{p \neq q} (\lambda^{(7)00}_{pq} + \lambda^{(8)00}_{pq}).$$

The right-hand side of the identity is

$$\operatorname{tr}(N'A_oN)^2 = \operatorname{tr}(N'N)^2 = \operatorname{tr}\Big(\sum_{\alpha,\beta,\gamma} \lambda_\alpha \lambda_\beta p^\gamma_{\alpha\beta} A_\gamma\Big) = \sum_\alpha \lambda^2_\alpha n_\alpha v \ .$$

Hence one obtains the relation

$$\sum_{p\neq q} \left(\lambda^{(9)00}_{pq} + \lambda^{(10)00}_{pq} + 2\lambda^{(12)00}_{pq} + 2\lambda^{(13)00}_{pq} + \lambda^{(15)00}_{pq}\right)$$

$$= \sum_\alpha \lambda^2_\alpha n_\alpha v - r(k-1)v \ .$$

It can be shown that

$$\sum_{p\neq q} \left(\lambda^{(7)00}_{pq} + \lambda^{(8)00}_{pq}\right)\left(\lambda^{(7)ss}_{pq} + \lambda^{(8)ss}_{pq}\right) = \sum_{p\neq q} \Big[2\left(2\lambda^{(11)0s}_{pq} + \lambda^{(14)0s}_{pq}\right)$$

$$+ \left(\lambda^{(9)0s}_{pq} + \lambda^{(10)0s}_{pq} + 2\lambda^{(12)0s}_{pq} + 2\lambda^{(13)0s}_{pq} + \lambda^{(15)0s}_{pq}\right)\Big].$$

Now,

$$\sum_{p,q} \sum_{\alpha,\beta,\gamma} n_{\alpha p}\, p^\beta_{\alpha 0}\, p^\gamma_{\alpha s}\, n_{\beta q} n_{\gamma q}$$

$$= \sum_p \sum_{\alpha,\gamma} n_{\alpha p}\, p^\gamma_{\alpha s}\, n_{\gamma q} + \sum_{p\neq q} \sum_{\alpha,\gamma} n_{\alpha p}\, p^\gamma_{\alpha s}\, n_{\alpha q}\, n_{\gamma q}$$

$$= \sum_p \lambda^{(1)ss}_{pp} + \sum_{p\neq q} \left(2\lambda^{(11)0s}_{pq} + \lambda^{(14)0s}_{pq}\right) ,$$

and on the other hand

$$\sum_{p,q} \sum_{\alpha,\beta,\gamma} n_{\alpha p}\, p^\beta_{\alpha 0}\, p^\gamma_{\alpha s}\, n_{\beta q}\, n_{\gamma q} = \sum_q \sum_{\alpha,\beta,\gamma} p^\beta_{\alpha 0}\, p^\gamma_{\alpha s}\, n_{\beta q} n_{\gamma q} \sum_p n_{\alpha p}$$

$$= r\sum_q \sum_{\alpha,\gamma} n_{\alpha q}\, p^\gamma_{\alpha s}\, n_{\gamma q} = r \operatorname{tr}(N'A_sN) = r\lambda_s n_s v$$

Thus one gets the relation

$$\sum_{p\neq q} \left(2\lambda^{(11)0s}_{pq} + \lambda^{(14)0s}_{pq}\right) = (r-1)\lambda_s n_s v \ .$$

Since

$$\sum_{p,q}\Big(\sum_{\alpha,\beta} n_{\alpha p}\, a^{\beta}_{\alpha p o}\, n_{\beta q}\Big)\Big(\sum_{\gamma,\delta} n_{\gamma p}\, a^{\delta}_{\gamma s}\, n_{\delta q}\Big)$$

$$= \sum_{p}(\Sigma n_{\alpha p})\Big(\sum_{\gamma,\delta} n_{\gamma p}\, a^{\delta}_{\gamma s}\, n_{\delta q}\Big) + \sum_{p\neq q}\Big(\sum_{\alpha,\beta} n_{\alpha p}\, a^{\beta}_{\alpha 0}\, n_{\beta q}\Big)\Big(\sum_{\gamma,\delta} n_{\gamma p}\, a^{\delta}_{\gamma s}\, n_{\delta q}\Big)$$

$$= k\sum_{p}\lambda^{(1)ss}_{pp} + \sum_{p\neq q}(\lambda^{(7)00}_{pq}+\lambda^{(8)00}_{pq})(\lambda^{(7)ss}_{pq}+\lambda^{(8)ss}_{pq}) ,$$

and

$$\sum_{p,q}\Big(\sum_{\alpha,\beta} n_{\alpha p}\, a^{\beta}_{\alpha p o}\, n_{\beta q}\Big)\Big(\sum_{\gamma,\delta} n_{\gamma p}\, a^{\delta}_{\gamma s}\, n_{\delta q}\Big) = tr(N'A_o NN'A_s N)$$

$$= tr\,(NN'NN'A_s) = tr\Big(\sum_{\alpha,\beta}\lambda_\alpha\lambda_\beta \sum_{\gamma} p^{\gamma}_{\alpha\beta}A_\gamma A_s\Big) = tr\Big(\sum_{\alpha,\beta}\ \sum_{\gamma,\delta}\lambda_\alpha\lambda_\beta p^{\gamma}_{\alpha\beta}p^{\delta}_{\gamma s}A_\delta\Big)$$

$$= \sum_{\alpha,\beta,\gamma}\lambda_\alpha\lambda_\beta p^{\gamma}_{\alpha\beta}p^{0}_{\gamma s}v = \sum_{\alpha,\beta}\lambda_\alpha\lambda_\beta p^{s}_{\alpha\beta}n_s v = \sum_{\alpha,\beta=1}^{2}\lambda_\alpha\lambda_\beta p^{s}_{\alpha\beta}n_s v$$

$$+ 2r\lambda_s n_s v .$$

One obtains the relation

$$\sum_{p\neq q}(\lambda^{(9)0s}_{pq}+\lambda^{(10)0s}_{pq}+2\lambda^{(12)0s}_{pq}+2\lambda^{(13)0s}_{pq}+\lambda^{(15)0s}_{pq})$$

$$=\sum_{\alpha,\beta \geq 1}\lambda_\alpha\lambda_\beta p^{s}_{\alpha\beta}n_s v - (k-2)\lambda_s n_s v .$$

Similarly, the two way calculations of

$$\sum_{p,q}\ \sum_{\alpha,\beta,\gamma,\delta} n_{\alpha p}n_{\beta p}a^{\gamma}_{\alpha s}a^{\delta}_{\beta t}\, n_{\gamma q}n_{\delta q}$$

yields the relation

$$\sum_{p\neq q}(2\lambda^{(11)st}_{pq}+\lambda^{(14)st}_{pq}) = r\sum_{\alpha\geq 1}\lambda_\alpha p^{\alpha}_{st}n_\alpha v - \sum_{p}\lambda^{(2)st}_{pp} ,$$

and consequently

$$\sum_{p\neq q}(\lambda^{(9)st}_{pq}+\lambda^{(10)st}_{pq}+2\lambda^{(12)st}_{pq}+2\lambda^{(13)st}_{pq}+\lambda^{(15)st}_{pq})$$

$$= \sum_{\alpha,\beta\geq 1} \sum_{\gamma\geq 0} \lambda_\alpha\lambda_\beta p^\gamma_{\alpha\beta} p^\gamma_{st} n_\gamma v - \delta_{st}\lambda_s n_s v - 2\sum_p \lambda^{(2)st}_{pp} - \sum_p \lambda^{(3)st}_{pp} .$$

Thus one obtains

$$\text{Var }\theta = \left(\frac{2(2k-3)}{k(k-1)} - \frac{2\Gamma_o}{\Delta_o^2}\right)\left[\frac{v}{b^2k(k-1)} \sum_{s\geq 1} \mu_s^2\lambda_s n_s - \frac{4}{b^2k(k-1)(k-2)} \sum_{s,t\geq 1} \mu_s\mu_t\lambda^{(1)}_{st} + \frac{3}{b^2k(k-1)(k-2)(k-3)} \sum_{s,t\geq 1} \mu_s\mu_t\lambda^{(2)}_{st}\right] \tag{4.6}$$

$$- \frac{2v}{b^2k(k-1)^2} \sum_{s\geq 1} (\mu_o-\mu_s)^2\lambda_s n_s$$

$$+ \frac{2v}{b^2k^2(k-1)^2}\left[\sum_{s\geq 0} \mu_s^2 n_s r^2(k-1)^2 + \sum_{s,t,\gamma\geq 0} \sum_{\alpha,\beta\geq 1} \mu_s\mu_t\lambda_\alpha\lambda_\beta p^\gamma_{\alpha\beta}p^\gamma_{st}n_\gamma - 2r(k-1) \sum_{s,t\geq 0} \sum_{\alpha\geq 1} \mu_s\mu_t\lambda_\alpha p^\alpha_{st} n_\alpha\right] .$$

This can be rearranged as follows:

$$\text{Var }\theta = \frac{2(v-1)}{b^2(k-1)^2} - \frac{2(v-1)^2}{b^3(k-1)^2} + \frac{2v^2}{b^3k^2(k-1)^3} \sum_{s\neq t} (\mu_o-\mu_s)(\mu_s-\mu_t)\lambda_s\lambda_t n_s n_t$$

$$+ \left(\frac{2(2k-3)}{k(k-1)} - \frac{2\Gamma_o}{\Delta_o^2}\right)\left[\frac{v}{b^2k(k-1)} \sum_{s\geq 1} \mu_s^2 \lambda_s n_s - \frac{4}{b^2k(k-1)(k-2)} \sum_{s,t\geq 1} \mu_s\mu_t\lambda^{(1)}_{st} + \frac{3}{b^2k(k-1)(k-2)(k-3)} \sum_{s,t\geq 1} \mu_s\mu_t\lambda^{(2)}_{st}\right]. \tag{4.7}$$

Now for cases where $k = 2,3$, we have

$$2\Gamma_o = \Delta_o^2$$

and therefore the last tern in (4.7) vanishes for those cases.

If we consider the limiting process such as

$$b \longrightarrow \infty$$

fixing v, n_1, n_2, p^i_{jk} as constants, then, since vr=bk and $\sum_{i=1} n_i\lambda_i = r(k-1)$, r and at least one λ_i are of the same order as b, and

$$\mu_1\mu_2 = 0\,(\frac{1}{b^2}) .$$

It can be seen that

$$\lambda^{(1)}_{st} \leqq bk(k-1)(k-2)$$

$$\lambda^{(2)}_{st} \leqq bk(k-1)(k-2)(k-3)$$

and

$$\Gamma_o/\Delta^2_o \leqq 1 .$$

Thus we get

(4.8) $$\text{Var } \theta = \frac{2(v-1)}{b^2(k-1)^2}\,(1+0\,\frac{1}{b})) \quad \text{as } b \longrightarrow \infty .$$

For a BIBD this is

$$\text{Var } \theta = \frac{2(v-1)}{b^2(k-1)^2}\,(1 - \frac{v-1}{b(k-1)})$$

which confirms Ogawa's result [4], and for a case when λ_1=0, $\lambda_2 \neq 0$ (4.6) reduces

$$\text{Var } \theta = \frac{2(v-1)}{b^2(k-1)^2}\,(1 - \frac{v-1}{b(k-1)}) ,$$

which can be seen from (4.7). This confirms P. V. Rao's result [6].

§5. The approximate null-distribution of the F after the randomization.

We take the Beta-distribution

(5.1) $$\frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \theta^{\frac{v_1}{2} - 1} (1-\theta)^{\frac{v_2}{2} 1} d\theta$$

as an approximation to the permutation distribution of θ due to the randomization under the uniformity conditions (4.1). Then

(5.2) $$\frac{v_1}{v_1+v_2} = \mathcal{E}(\theta), \quad \frac{2v_1v_2}{(v_1+v_2)^2(v_1+v_2+2)} = \text{Var } \theta .$$

Hence we get

$$v_1 = \frac{2[\mathcal{E}(\theta)-\mathcal{E}^2(\theta)- \text{Var } \theta]}{\text{Var } \theta} \mathcal{E}(\theta)$$

$$v_2 = \frac{2[\mathcal{E}(\theta)-\mathcal{E}^2(\theta)- \text{Var } \theta]}{\text{Var } \theta} (1 - \mathcal{E}(\theta))$$

Notice that

$$\mathcal{E}(\theta) = \mu_o-\mu_2- \frac{\mu_1-\mu_2}{bk(k-1)} \sum_p \lambda^{(1)11}_{pp} = \frac{v-1}{b(k-1)}$$

and hence

$$1 - \mathcal{E}(\theta) = \frac{n-b-v+1}{b(k-1)} .$$

If we put

(5.4) $$\nu_1 = \varphi(v-1) \quad \text{and} \quad \nu_2 = \varphi(n-b-v+1),$$

then

(5.5) $$\varphi= \frac{1}{b(k-1)} \frac{2[\mathcal{E}(\theta)-\mathcal{E}^2(\theta)-\text{Var } \theta]}{\text{Var } \theta} = \frac{2}{b(k-1)} \frac{\frac{v-1}{b(k-1)} (1 - \frac{v-1}{b(k-1)})}{\frac{2(v-1)}{b^2(k-1)^2}(1+O\frac{1}{b}))} - 1$$

$$= 1 + O (\tfrac{1}{b}).$$

Therefore, for sufficiently large value of b, the permutation distribution of θ due to the randomization can be approximated by the Beta-distribution

$$(5.6) \qquad \frac{\Gamma(\frac{n-b}{2})}{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-b-v+1}{2})} \theta^{\frac{v-1}{2}-1} (1-\theta)^{\frac{n-b-v+1}{2}-1}$$

under the uniformity condition (4.1). Consequently the null-distribution of the statistic F after the randomization can be approximated by the familiar central F-distribution

$$(5.7) \qquad \frac{\Gamma(\frac{n-b}{2})}{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-b-v+1}{2})} \left(\frac{v-1}{n-b-v+1} F\right)^{\frac{v-1}{2}-1} \left(1 + \frac{v-1}{n-b-v+1}F\right)^{-\frac{n-b}{2}} d\left(\frac{v-1}{n-b-v+1} F\right),$$

which is obtained by averaging (2.14) with respect to θ having the density given by (5.6).

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