SIMULTANEOUS CONFIDENCE BOUNDS ON A SET OF LINEAR FUNCTIONS OF LOCATION PARAMETERS FOR DEPENDENT AND INDEPENDENT NORMAL VARIATES

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1. Introduction and Summary. In practical situations, one is generally faced with multivariate problems in the form of testing the hypotheses or obtaining a set of simultaneous confidence bounds on certain parameters of interest. We shall consider here the variates under study to be normally distributed. A lot of work on the univariate and multivariate normal populations for the simultaneous confidence bounds on the location parameters has been done. (See references, not necessarily exhaustive.) Our aim in this paper is to give shorter confidence bounds on a given set of linear functions of location parameters when this set is previously chosen for study. For the univariate case, Dunn [6,8] using the Bonferroni inequality, obtained shorter confidence bounds when the number of linear functions is not too large. We may note that Nair [11], David [5], Dunn [6,7,8] and Siotani [22,24] have studied the closeness of the Bonferroni inequality while deriving the percentage points of certain statistics in univariate and multivariate normal cases. In this paper, we improve the Bonferroni inequality in all the situations considered by Siotani [22,23,24] and Dunn [6,7,8], and point out various uses of these results in obtaining simultaneous confidence bounds on linear functions of means (or location parameters) with confidence greater than or equal to $1-\alpha$ where α is the size of the test. Since our results are extensions of Dunn [6,8], Siotani [22,24] and Banerjee [2,3], their comments

on the shortness of the confidence bounds apply to our cases too. We may note that we require the percentage points of t, χ^2 or F distributions, in some cases maximum studentized t-statistic [12] and maximum Hotelling T^2 [22,23,24] when their tables are available. In all other cases, we use the sharper inequalities developed here, and for all practical purposes, they can be used when no better ones are available.

2. Some inequalities for multivariate normal distributions.

Lemma 1. Let D be a convex set symmetric about the origin. Then

$$(1) \qquad \int_D \exp[-\tfrac{1}{2}(\underset{\sim}{x}-y_o\underset{\sim}{b})'\Sigma^{-1}(\underset{\sim}{x}-y_o\underset{\sim}{b})]\, d\underset{\sim}{x}$$

is a monotonic decreasing function of $|y_o|$ if Σ: p × p is positive definite (p.d.), $\underset{\sim}{x}' = (x_1, \ldots, x_p)$, and $\underset{\sim}{b}' = (b_1, \ldots, b_p)$.

This follows from the following result proved by Anderson [1]:

Let D be a convex set symmetric about the origin and let $f(\underset{\sim}{x})$ be a function of $\underset{\sim}{x}' = (x_1, \ldots, x_p)$ such that (i) $f(\underset{\sim}{x}) \geq 0$, (ii) $f(\underset{\sim}{x}) = f(-\underset{\sim}{x})$, (iii) $\{\underset{\sim}{x} | f(\underset{\sim}{x}) \geq u\}$ is a convex set for every u, and (iv) $\int_D f(\underset{\sim}{x})\, d\underset{\sim}{x} < \infty$. Then

$$(2) \qquad \int_D f(\underset{\sim}{x}+y_o\underset{\sim}{b})\, d\underset{\sim}{x} \quad \geq \quad \int f(\underset{\sim}{x}+\underset{\sim}{b})\, d\underset{\sim}{x} \qquad \text{for } 0 \leq y_o \leq 1.$$

For lemma 1, $f(\underset{\sim}{x}) = \exp[-\frac{1}{2}\underset{\sim}{x}'\Sigma^{-1}\underset{\sim}{x}]$, and see that all the conditions for (2) are satisfied.

Lemma 2. If $g(\underset{\sim}{x})$ and $h(\underset{\sim}{x})$ are functions of real vector variable $\underset{\sim}{x}$ such that for any two points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, either $g(\underset{\sim}{x}_1) \geq g(\underset{\sim}{x}_2)$ and $h(\underset{\sim}{x}_1) \geq h(\underset{\sim}{x}_2)$ or $g(\underset{\sim}{x}_1) \leq g(\underset{\sim}{x}_2)$ and $h(\underset{\sim}{x}_1) \leq g(\underset{\sim}{x}_2)$. Then

$$(3) \qquad E[g(\underset{\sim}{x})\, h(\underset{\sim}{x})] \geq [E\, g(\underset{\sim}{x})]\, [E\, h(\underset{\sim}{x})],$$

where E stands for expectation of $\underset{\sim}{x}$.

Proof. Let y_1 and y_2 be any two independent and identical vector variates. Then the condition of lemma 2 gives us

$$[g(y_1) - g(y_2)]\,[h(y_1) - h(y_2)] \geq 0 \quad \text{for all } y_1 \text{ and } y_2.$$

Hence, we get

$$E\, g(y_1)\, h(y_1) + E\, g(y_2)\, h(y_2) \geq E\, g(y_2)\, h(y_1) + E\, g(y_1)\, h(y_2)$$

and this proved lemma 2.

Corollary 1. Let $f = f(x)$ be a function of random variables $\underset{\sim}{x}' = (x_1, \ldots x_p)$ and let E stand for the expectation with respect to $x_1, \ldots, x_p$. Then if $s + t = r$,

$$(4) \qquad E\, f^{2r} \geq (E\, f^{2s})\,(E\, f^{2t}) \text{ or } (E\, f)^2\,(E\, f^{2(r-1)}) \text{ and}$$

$$(5) \qquad E\, f^{r} \geq (E\, f^{s})\,(E\, f^{t}) \quad \text{if } f \geq 0 \text{ for all } \underset{\sim}{x}.$$

Proof of (4) can be derived from lemma 2 by using $f^{2s}(\underset{\sim}{x}) = g(\underset{\sim}{x})$ and $h(\underset{\sim}{x}) = f^{2t}(\underset{\sim}{x})$ and using $E(f^2) \geq (E\, f)^2$. (5) is the consequence of (4) by renaming f^2 by f. This can be derived from the result (15.4.6) given by Cramer [4, p. 176].

Corollary 2. Let $\underset{\sim}{x}_i : p \times 1$ $(i = 1, 2, \ldots, r)$ be independent and identical vector variates and be distributed independently of $y: q \times 1$. Then

$$(6) \qquad \Pr[h(\underset{\sim}{x}_i, y) \leq c_i,\ i = 1, 2, \ldots, r] \geq \prod_{i=1}^{r} \Pr[h(\underset{\sim}{x}_i, y) \leq c_i]$$

and

$$(7) \qquad \Pr[h(\underset{\sim}{x}_i, y) \geq c_i,\ i = 1, 2, \ldots, r] \geq \prod_{i=1}^{r} \Pr[h(\underset{\sim}{x}_i, y) \geq c_i].$$

Proof. Note than when y is fixed

$$\Pr[h(\underset{\sim}{x}_i, y) \leq c_i,\ i = 1, 2, \ldots, r \mid y] = \prod_{i=1}^{r} [f(y, c_i)]$$

where

$f(\underset{\sim}{y}, c_i) = \Pr[h(\underset{\sim}{x}_i, \underset{\sim}{y}) \leq c_i | \underset{\sim}{y}] \geq 0$ for all $\underset{\sim}{y}$. The functional form is the same for each $\underset{\sim}{x}_i$ being independent and identical vector variates. Hence the condition of lemma 2 is satisfied, i.e., if $f(\underset{\sim}{y}_1, c_i) \gtrless f(\underset{\sim}{y}_2, c_i)$ for $\underset{\sim}{y}_1 \neq \underset{\sim}{y}_2$, then $f(\underset{\sim}{y}_1, c_j) \gtrless f(\underset{\sim}{y}_2, c_j)$ for $i \neq j$. Hence using lemma 2, we get

$$E[\prod_{i=1}^{r} f(\underset{\sim}{y}, c_i)] \geq \prod_{i=1}^{r} E\, f(\underset{\sim}{y}, c_i) ,$$

and this proves corollary 2.

Theorem I. Let $\underset{\sim}{x}' = (x_1, \ldots, x_p)$ be distributed as multivariate normal with zero mean vector, $D_1 = D_1(x_1)$ and $D_2(x_2, \ldots, x_p)$ be convex sets symmetric about origin. Then

$$\Pr(D_1 D_2) \geq \Pr(D_1) \Pr(D_2) .$$

Proof. Let the covariance matrix of $(x_1, x_2, \ldots, x_p) = \underset{\sim}{x}'$ be $\underset{\sim}{\Sigma} = (\sigma_{ij})$, and let us partitioned it as

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_{11} & \underset{\sim}{\sigma}_1' \\ \underset{\sim}{\sigma}_1 & \underset{\sim}{\Sigma}_{22} \end{pmatrix} , \underset{\sim}{\Sigma}_{22}: (p-1) \times (p-1) .$$

Now, let us write

$$(8) \qquad x_i = a_i\, y_0 + b_i\, y_i \quad (i = 1, 2, \ldots, p)$$

where y_0, y_1, and $(y_2, \ldots, y_p)$ are independently distributed as normals with means zero. First of all, let us assume that $\underset{\sim}{\Sigma}: p \times p$ is p.d., and let $V(y_0) = V(y_1) = V(y_j) = 1$; $j = 2, \ldots, p$ and $\text{cov}(y_j, y_{j'}) = \delta_{jj'}$, $j \neq j' = 2, 3, \ldots, p$. Let $\underset{\sim}{\Delta} = (\delta_{jj'}): (p-1) \times (p-1)$, with $\delta_{jj} = 1$ $(j = 2, \ldots, p)$. Then, we can determine a_i and b_i $(i = 1, 2, \ldots, p)$ from

(9) $$\sigma_{ii} = a_i^2 + b_i^2,\ \sigma_{1j} = a_1 a_j \text{ and } \sigma_{jj'} = a_j a_{j'} + \delta_{jj'} b_j b_{j'}$$

for $i = 1, 2, \ldots, p$; $j, j' = 2, 3, \ldots, p$. The solution of (11) is

(10) $$a_j = \sigma_{1j}/a_1,\ b_1 = \sqrt{\sigma_{11} - a_1^2},\quad b_j = \sqrt{\sigma_{jj} - \sigma_{1j}^2/a_1^2} \quad \text{and}$$
$$\delta_{jj'} = (\sigma_{jj'} a_1^2 - \sigma_{1j}\sigma_{1j'})/\{(\sigma_{jj} a_1^2 - \sigma_{1j}^2)(\sigma_{j'j'} a_1^2 - \sigma_{1j'}^2)\}^{\frac{1}{2}}$$

for $j, j' = 2, 3, \ldots, p$ and for any a_1 such that $\underset{\sim}{\sigma}_1' \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\sigma}_1 < a_1^2 < 1$. The condition on a_1 ensures that $\underset{\sim}{\Delta}$ is p.d. Now using (10), it is easy to see that

(11) $$\Pr(D_1 D_2) = E[g(y_o)\ h(y_o)]$$

where

(12) $$g(y_o) = (2\pi b_1)^{-\frac{1}{2}} \int_{D_1} \exp[-\tfrac{1}{2}(x_1 - a_1 y_o)^2/b_1^2]dx_1$$

and

(13) $$h(y_o) = (2\pi)^{-\frac{1}{2}(p-1)} |\underset{\sim}{\Delta}_1|^{-\frac{1}{2}} \int_{D_2} \exp[-\tfrac{1}{2}(\underset{\sim}{z} - y_o \underset{\sim}{a})' \underset{\sim}{\Delta}_1^{-1} (\underset{\sim}{z} - y_o \underset{\sim}{a})]d\underset{\sim}{z},$$

in which $\underset{\sim}{z}' = (x_2, \ldots, x_p)$, $\underset{\sim}{a}' = (a_2, \ldots, a_p)$, $\underset{\sim}{\Delta}_1 = (\delta_{1,jj'})$ and $\delta_{1,jj'} = \delta_{jj'} b_j b_{j'}$ for $j, j' = 2, 3, \ldots, p$. Lemma 1 shows that $g(y_o)$ and $h(y_o)$ are monotonically decreasing functions of $|y_o|$, and so using lemma 2 in (13), we get

(14) $$\Pr(D_1 D_2) \geq [E\ g(y_o)]\ [E\ h(y_o)].$$

Now, it is easy to verify that

$$E[g(y_o)] = \Pr(D_1) \text{ and } E[h(y_o)] = \Pr(D_2).$$

This proves the theorem when $\underset{\sim}{\Sigma}$ is nonsingular. When $\underset{\sim}{\Sigma}$ is singular, let r be the rank of $\underset{\sim}{\Sigma}$ and without loss of generality, let us assume that $\underset{\sim}{\Sigma}_{11}: r \times r$

is non-singular,

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \begin{matrix} r \\ p-r \end{matrix}$$

$$\qquad r \qquad p-r$$

and $\underset{\sim}{\Sigma}_{22} = \underset{\sim}{\Sigma}'_{12} \underset{\sim}{\Sigma}_{11}^{-1} \underset{\sim}{\Sigma}_{12}$. We have the probability inequality

(15) $$\Pr[D_1(w_1)\ D_2(w_2, \ldots, w_p)] \geq \Pr[D_1(w_1)]\ \Pr[D_2(w_2, \ldots, w_p)]$$

where $w_1, w_2, \ldots, w_p$ are normally distributed with zero means and covariance matrix

$$\underset{\sim}{\Sigma}_q = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & q\, \underset{\sim}{I}_{p-r} + \underset{\sim}{\Sigma}_{22} \end{pmatrix} \quad \text{for very very small } q > 0.$$

Since (15) is true for all positive values of q, so taking the limit as $q \longrightarrow 0^+$, we get

$$\Pr(D_1 D_2) \geq \Pr(D_1)\ \Pr(D_2).$$

This proves the theorem I completely.

Corollary 3. Let $\underset{\sim}{x}_i$ $(i = 1, 2, \ldots, n)$ be n independent observations from the multivariate normal with zero means, $D_1(x_{11}, \ldots, x_{1n}) = D_1$ and D_2 $(x_{ij}, i = 2, 3, \ldots, p$ and $j = 1, 2, \ldots, n) = D_2$ be sectionwise convex regions symmetric about origin. Then

$$\Pr(D_1 D_2) \geq \Pr(D_1)\ \Pr(D_2).$$

Proof. Let $g(\underset{\sim}{x}_i)$ be the density function of $\underset{\sim}{x}_i$ $(i = 1, 2, \ldots, n)$ and $g_1(i) = g_1(x_{1i})$ and $g_2(x_{2i}, \ldots, x_{pi}) = g_2(i)$ be marginal density functions of x_{1i} and $(x_{2i}, \ldots, x_{pi}), i = 1, 2, \ldots, n$. Then

$$Pr(D_1D_2) = \int_{D_1D_2} \prod_{i=1}^{n} g(\underset{\sim}{x}_i)\, d\underset{\sim}{x}_1 \ldots, d\underset{\sim}{x}_n.$$

Now since D_1 is section-wise convex and symmetric in $x_{1i}(i = 1, 2, \ldots, n)$, it is convex and symmetric in x_{11} when others are fixed. Similarly using the argument for D_2, we use Theorem I for integration over $\underset{\sim}{x}_1$ and get

$$Pr(D_1D_2) \geq \int_{D_1D_2} g_1(1)g_2(1) \prod_{i=2}^{n} g(\underset{\sim}{x}_i)\, d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 \ldots d\underset{\sim}{x}_n .$$

Now using the same argument for $\underset{\sim}{x}_2$, then $\underset{\sim}{x}_3$, ..., we get finally

$$Pr(D_1D_2) \geq \int_{D_1D_2} \Big(\prod_{i=1}^{n} g_1(i)\Big)\Big(\prod_{i=1}^{n} g_2(i)\Big) d\underset{\sim}{x}_1 \ldots d\underset{\sim}{x}_n = Pr(D_1)Pr(D_2).$$

This proves the corollary 3.

<u>Corollary 4.</u> If $x_1, \ldots, x_p$ are distributed as multivariate normal with zero means, then

$$Pr(|x_i| \leq c_i,\ i = 1, 2, \ldots, p) \geq \prod_{i=1}^{p} Pr(|x_i| \leq c_i) .$$

This follows immediately from theorem I. This was conjuctured by Dunn [6].

<u>Corollary 5</u>. If $\underset{\sim}{x}_j' = (x_{1j}, \ldots, x_{pj})$, $j = 1, 2, \ldots, n$ are n independent observations from a multivariate normal with zero means and $\lambda_j \geq 0$ $(j=1,2,\ldots,n)$, then

$$Pr\Big(\sum_{j=1}^{n} \lambda\, x_{ij}^2 \leq c_i;\ i = 1, 2, \ldots, p\Big) \geq \prod_{i=1}^{p} Pr\Big(\prod_{j=1}^{n} \lambda_j x_{ij}^2 \leq c_i\Big).$$

This follows from corollary 3 or from theorem I.

<u>Theorem II</u>. Let $(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r)$ be distributed as normal with zero means and $\mathrm{cov}(\underset{\sim}{x}_i, \underset{\sim}{x}_j) = \beta_{ij} \underset{\sim}{\Sigma}$ where $\underset{\sim}{\Sigma}$: $p \times p$ is an unknown p.d. matrix and β_{ii} $(i = 1, 2, \ldots, r)$ are known; and let S: $p \times p$ be distributed as Wishart $(p, n, \underset{\sim}{\Sigma})$ and be independent of $(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r)$. Then

$$(16) \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i/\beta_{ii} \le c_i;\ i=1,2,\ldots,r) \ge \prod_{i=1}^{r} \{[B(\tfrac{p}{2}, \tfrac{n+1-p}{2})]^{-1} \int_0^{c_i} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n+1)} dy\}$$

and

$$(17) \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i/\beta_{ii} \le c_i;\ i=1,2,\ldots,r) \ge w \prod_{i=1}^{r} \{[B(\tfrac{1}{2}p, \tfrac{n+r-p}{2})]^{-1} \int_0^{c_i} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n+r)} dy\}$$

where

$$w = [\{\Gamma(\tfrac{n+r-p}{2})\}^r \prod_{i=1}^{p} \Gamma(\tfrac{n+r-i+1}{2})]/[\{\Gamma(\tfrac{n+r}{2})\}^r \prod_{i=1}^{p} \Gamma(\tfrac{n-i+1}{2})].$$

Proof. When $p = 1$ and $\beta_{ij} = \sqrt{\beta_{ii}\beta_{jj}}\, \alpha_i \alpha_j$ for $-1 \le \alpha_i \le 1$, (16) was proved by Dunnet and Sobel [9]. (16) is a sharper inequality than Bonferroni inequality whose closeness in some particular cases was considered by Siotani [22,23,24]. For the proof of (16) and (17), we may consider $\underset{\sim}{\Sigma} = \underset{\sim}{I}$, for the statistic under consideration is invariant under the transformations $\underset{\sim}{\Sigma}^{-\frac{1}{2}} \underset{\sim}{S} \underset{\sim}{\Sigma}^{-\frac{1}{2}} \longrightarrow \underset{\sim}{S}$ and $\underset{\sim}{\Sigma}^{-\frac{1}{2}} \underset{\sim}{x}_i \longrightarrow \underset{\sim}{x}_i$ $(i = 1, 2, \ldots, r)$. Now let us consider the conditional probability

$$\Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i/\beta_{ii} \le c_i;\ i=1,2,\ldots,r \mid \underset{\sim}{S}) = \Pr(\sum_{j=1}^{p} \lambda_j y_{ij}^2/\beta_{ii} \le c_i;\ i=1,2,\ldots,r \mid \underset{\sim}{S})$$

where λ_j are the ch. roots of $\underset{\sim}{S}^{-1}$, and $(y_{1j}, y_{2j}, \ldots, y_{rj})$, $(j = 1, 2, \ldots, p)$ are independently distributed as normal with zero means and covariance matrix (β_{ij}). Hence applying corollary 5, we get

$$(18) \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i/\beta_{ii} \le c_i;\ i=1,2,\ldots,r \mid \underset{\sim}{S}) \ge \prod_{i=1}^{r} \Pr[\underset{\sim}{z}_i' \underset{\sim}{S}^{-1} \underset{\sim}{z}_i \le c_i \mid \underset{\sim}{S}]$$

where $\underset{\sim}{z}_i$: $p \times 1$ $(i = 1, 2, \ldots, r)$ are independent normals with zero means and covariance matrix $\underset{\sim}{I}_p$.

Now, the right hand side of (18) can be written in two ways as follows:

(19) $\prod_{i=1}^{r} [f(\underset{\sim}{S}, c_i)]$ where $f(\underset{\sim}{S}, c_i) = \Pr(\underset{\sim}{z}'\underset{\sim}{S}^{-1}\underset{\sim}{z} \leq c_i | S) \geq 0$, for all S

p.d., and $\underset{\sim}{z}$ is $N(\underset{\sim}{0}, \underset{\sim}{I}_r)$, or

(20) $(2\pi)^{-\frac{1}{2}pr} |\underset{\sim}{S}|^{\frac{1}{2}r} \prod_{i=1}^{r} [g(\underset{\sim}{S}, c_i)]$ where $g(\underset{\sim}{S}, c_i) = \int_{\underset{\sim}{z}'\underset{\sim}{z} \leq c_i} \exp[-\frac{1}{2}\underset{\sim}{z}'\underset{\sim}{S}\underset{\sim}{z}]d\underset{\sim}{z}$

≥ 0 for all $\underset{\sim}{S}$ p.d.

Using (19) or (20) in (18), we get

(21) $\Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1}\underset{\sim}{x}_i/\beta_{ii} \leq c_i;\ i=1,2,\ldots,r) \geq [E \prod_{i=1}^{r} f(\underset{\sim}{S}, c_i)]$ or $[\omega_1 E \prod_{i=1}^{r} g(S_1, c_i)]$

where $\omega_1 = \prod_{i=1}^{p} \{\Gamma(\frac{n+r-i+1}{2})/\Gamma(\frac{n-i+1}{2})\}\pi^{-\frac{1}{2}pr}$, $\underset{\sim}{S}_1$ is Wishart (p, n+r, $\underset{\sim}{I}_p$) while $\underset{\sim}{S}$ is Wishart (p, n, $\underset{\sim}{I}_p$). Now using corollary 2, or lemma 2, we get the results as mentioned in (16) and (17) after some simplifications. We conjecture that (17) will be slightly better than (16), but involves more computations.

<u>Theorem III</u>. If $(x_1, \ldots, x_p)$ are distributed as multivariate normal with zero means, then

$$\Pr(|x_i| \geq c_i;\ i=1,2,\ldots,p) \geq \prod_{i=1}^{p} \Pr(|x_i| \geq c_i),$$

if the correlations $\rho_{ij} = \alpha_i\alpha_j$ for $i \neq j$; $i,j = 1, 2, \ldots, p$ and $-1 \leq \alpha_i \leq 1$.

<u>Proof</u>. Without loss of generality, we shall assume that $V(x_i) = 1$, $i=1,2,\ldots,p$.

Let $y_0, y_1, \ldots, y_p$ be independent normal variates with zero means and unit variances. Then

$$x_i = \alpha_i y_0 + \sqrt{1-\alpha_i^2}\, y_i \quad \text{for } i = 1, 2, \ldots, p \text{ and all } \alpha_i^2 < 1.$$

Then it is easy to see that

$$\Pr(|x_i| \geq c_i;\ i = 1, 2, \ldots, p) = E[\prod_{i=1}^{p} g_i(y_o)] \tag{22}$$

where

$$g_i(y_o) = \{2\pi(1-\alpha_i^2)\}^{-\frac{1}{2}} \int_{|x_i| \geq c_i} \exp[-\tfrac{1}{2}(x_i-\alpha_i y_o)^2/(1-\alpha_i^2)]dx_i \ . \tag{23}$$

We note that $g_i(y_o) = g_i(-y_o)$ and when $\alpha_i < 0$, we can change the sign of x_i and get the same function $g_i(y_o)$. Hence, we consider $\alpha_i \geq 0$. It is easy to see that $g_i(y_o)$, $i=1,2,\ldots,p$ are monotonically increasing functions of $|y_o|$. Then using lemma 2 in (22), we get theorem III when $|\alpha_i| < 1$. Now when some α_i's are equal to one, we can argue in the same way as done for the singular case of theorem I.

<u>Corollary</u> 6. Let $(x_{1j}, \ldots, x_{pj})$, $j = 1, 2, \ldots, n$ be independent observations from a normal distribution with zero means and correlations $\rho_{ii'} = \alpha_i\alpha_{i'}$ for $-1 \leq \alpha_i \leq 1$, $i \neq i'$. Then if $\lambda_j \geq 0$ $(j = 1, 2, \ldots, n)$, we have

$$\Pr(\sum_{j=1}^{n} \lambda_j\, x_{ij}^2 \geq c_i;\ i=1,2,\ldots,p) \geq \prod_{i=1}^{p} \Pr(\sum_{j=1}^{n} \lambda_j x_{ij}^2 \geq c_i).$$

Proof is similar to that given in corollary 3.

<u>Corollary 7</u>. Let $\underset{\sim}{x}_j' = (x_{1j}, \ldots, x_{pj})$, $j=1,2,\ldots,n_1$ and $\underset{\sim}{y}_{j'} = (y_{1j'},\ldots,y_{pj'})$, $j' = 1,2,\ldots,n_2$ be n_1 and n_2 independent observations on $\underset{\sim}{x}$ and $\underset{\sim}{y}$, respectively, which are independently distributed as multivariate normals with zero means. If the correlations of $\underset{\sim}{y}$ are $\rho_{ii'} = \alpha_i\alpha_{i'}$ for $-1 \leq \alpha_i \leq 1$, then

$$\Pr(\sum_{j=1}^{n_1} x_{ij}^2 \leq c_i \sum_{j'=1}^{n_2} y_{ij'}^2;\ i=1,2,\ldots,p) \geq \prod_{i=1}^{p} \Pr(\sum_{j=1}^{n_1} x_{ij}^2 \leq c_i \sum_{j'=1}^{n_2} y_{ij'}^2).$$

This follows with the help of corollary 5 and corollary 6. The following theorem will not be applied to the construction of confidence bounds, but for the completeness and as an application of theorem III, it is given here.

Theorem IV. Let $(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r)$ be normal variates with zero means and $\mathrm{Cov}(\underset{\sim}{x}_i, \underset{\sim}{x}_j) = \beta_{ij} \underset{\sim}{\Sigma}$ where $\underset{\sim}{\Sigma}$: $p \times p$ is p.d. and $\beta_{ij}/\sqrt{\beta_{ii}\beta_{jj}} = \alpha_i\alpha_j$ for $-1 \leq \alpha_i \leq 1$, and let $\underset{\sim}{S}$ be distributed as Wishart $(p, n, \underset{\sim}{\Sigma})$ and be independent of $(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_r)$. Then

$$(24) \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i \geq c_i \, \beta_{ii};\ i=1,2,\ldots,p) \geq \prod_{i=1}^{r} [\{B(\tfrac{p}{2}, \tfrac{n+i-p}{2})\}^{-1} \int_{c_i}^{\infty} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n+i)} dy]$$

and when $r = 2$,

$$(25) \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i \geq c_i \, \beta_{ii};\ i=1,2) \geq \prod_{i=1}^{2} [\{B(\tfrac{1}{2}p, \tfrac{n+2-p}{2})\} \int_{c_i}^{\infty} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}n+2)} dy]$$

Proof. As remarked in the proof of theorem II, we may consider $\underset{\sim}{\Sigma} = \underset{\sim}{I}$ and

$$\Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i \geq c_i \, \beta_{ii};\ i=1,2,\ldots,p \mid \underset{\sim}{S}) = \Pr(\sum_{j=1}^{p} \lambda_j y_{ij}^2 \geq c_i \beta_{ii};\ i=1,2,\ldots,p \mid \underset{\sim}{S})$$

where $\lambda_j \geq 0$ are the characteristic roots of $\underset{\sim}{S}$; and $(y_{1j}, \ldots, y_{rj})$, $(j=1,2,\ldots,p)$ are independently distributed as normals with zero means and covariances $\beta_{ij} = \alpha_i\alpha_j\sqrt{\beta_{ii}\beta_{jj}}$ for $i \neq j$ and $-1 \leq \alpha_i \leq 1$. Then using corollary 6, we get

$$(26) \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i \geq c_i \, \beta_{ii};\ i=1,2,\ldots,p) \geq E[\prod_{i=1}^{r} \Pr(\underset{\sim}{z}_i' \underset{\sim}{S}^{-1} \underset{\sim}{z}_i \geq c_i \beta_{ii} \mid \underset{\sim}{S})]$$

where E stands for expectation over $\underset{\sim}{S}$ and $\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_r$ are independent normals with zero means and with covariances $\beta_{ii} \underset{\sim}{I}_p$ $(i = 1,2,\ldots,r)$ and independent of $\underset{\sim}{S}$ which is $W(p, n, \underset{\sim}{I}_p)$. Denoting the right hand side of (26) by A, we get

(27) $$A = \delta_1 \int_D \cdots \int [\underset{\sim}{I}_r + \underset{\sim}{Y}'\underset{\sim}{Y}]^{-\frac{1}{2}(n+r)} d\underset{\sim}{Y}, \text{ where } \underset{\sim}{Y} = (\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_r),$$

$$D = \{\underset{\sim}{y}_i'\underset{\sim}{y}_i \geq c_i;\ i = 1,2,\ldots,r\} \text{ and } \delta_1 = \prod_{i=1}^{p} \{\Gamma(\tfrac{n+r-i+1}{2})/\Gamma(\tfrac{n-i+1}{2})\}\pi^{-\frac{1}{2}pr}.$$

We note that if $\underset{\sim}{Y} = (\underset{\sim}{Y}_1\ \underset{\sim}{y}_r)$, then

$$|\underset{\sim}{I}_r+\underset{\sim}{Y}'\underset{\sim}{Y}| = |\underset{\sim}{I}_{r-1}+\underset{\sim}{Y}_1'\underset{\sim}{Y}_1| \cdot [1+\underset{\sim}{y}_r'\underset{\sim}{y}_r - \underset{\sim}{y}_r'\underset{\sim}{Y}_1(\underset{\sim}{I}_{r-1}+\underset{\sim}{Y}_1'\underset{\sim}{Y}_1)^{-1}\underset{\sim}{Y}_1'\underset{\sim}{y}_r].$$

Let

$$\underset{\sim}{z}_r = (\underset{\sim}{I}_p+\underset{\sim}{Y}_1\underset{\sim}{Y}_1')^{-\frac{1}{2}}\underset{\sim}{y}_r \quad\text{and}\quad \underset{\sim}{I}_p-\underset{\sim}{Y}_1(\underset{\sim}{I}_{r-1}+\underset{\sim}{Y}_1'\underset{\sim}{Y}_1)^{-1}\underset{\sim}{Y}_1' = (\underset{\sim}{I}_p+\underset{\sim}{Y}_1\underset{\sim}{Y}_1')^{-1}.$$

Then A can be rewritten as

(28) $$A = \delta_1 \int_{D_1}\int_{D_2} |\underset{\sim}{I}_{r-1}+\underset{\sim}{Y}_1'\underset{\sim}{Y}_1|^{-\frac{1}{2}(n+r-1)}(1+\underset{\sim}{z}_r'\underset{\sim}{z}_r)^{-\frac{1}{2}(n+r)} d\underset{\sim}{Y}_1\, d\underset{\sim}{z}_r,$$

where

$$D_1: \{\underset{\sim}{y}_i'\underset{\sim}{y}_i \geq c_i;\ i = 1,2,\ldots,(r-1)\} \text{ and } D_2: \{\underset{\sim}{z}_r'\underset{\sim}{z}_r + (\underset{\sim}{z}_r'\underset{\sim}{Y}_1\underset{\sim}{Y}_1'\underset{\sim}{z}_r) \geq c_r\}.$$

Since $D_2'\{z_r'z_r \geq c_r\}$ implies D_2, we get

(29) $$A \geq \delta_1 \int_{D_1}\int_{D_2'} |\underset{\sim}{I}_{r-1}+\underset{\sim}{Y}_1'\underset{\sim}{Y}_1|^{-\frac{1}{2}(r+r-1)}(1+\underset{\sim}{z}_r'\underset{\sim}{z}_r)^{-\frac{1}{2}(n+r)} d\underset{\sim}{z}_r\, d\underset{\sim}{Y}_1.$$

This procedure is continued and we get (24) after some simplifications. Now proving (25), we assume $p \geq 2$ and then after some simplifications, we can write A as

(30) $$A = \Gamma(\tfrac{n+2}{2})\Gamma(\tfrac{n+1}{2})\{\pi\Gamma(\tfrac{1}{2}p)\Gamma(\tfrac{p-1}{2})\Gamma(\tfrac{n-1}{2})\Gamma(\tfrac{n}{2})\}^{-1}\int_{-1}^{1}\int_{c_1}^{\infty}\int_{c_2}^{\infty} \frac{(y_1y_2)^{\frac{1}{2}p-1}(1-r^2)^{\frac{1}{2}(p-3)}\, dr\, dy_1\, dy_2}{\{(1+y_1)(1+y_2)-y_1y_2r^2\}^{\frac{1}{2}(n+2)}}$$

Now, we note that

$$\int_c^\infty y^m (1+y)^{-n}\, dy > m(n-1)^{-1} \int_c^\infty y^{m-1}(1+y)^{-n+1}\, dy \text{ if } n - 1 > m,$$

and

$$\{(1+y_1)(1+y_2)-y_1y_2r^2\}^{-\frac{1}{2}(n+2)} = \sum_{j=0}^{\infty} (r^2)^j[y_1y_2(1+y_1)^{-1}(1+y_2)^{-1}]^j$$
$$\Gamma(\tfrac{n+2}{2}+j)/[j\,!\,\Gamma(\tfrac{n+2}{2})\{(1+y_1)(1+y_2)\}^{\frac{1}{2}(n+2)}]$$

With the help of these results in (30), it can be verified that

$$A \geq \prod_{i=1}^{2} [\{B(\tfrac{1}{2}p,\ \tfrac{n+2-p}{2})\}^{-1} \int_{c_i}^{\infty} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n+2)}\, dy],$$

and this proves (25).

We give some passing remarks on the results of this section:

Remark 1. Corollaries 5, 6 and 7 are used to obtain one-sided confidence bounds on variances and ratios of variances.

Remark 2. For any non-singular correlation matrix in theorem III, it has been shown that $\Pr(x_i \geq c_i;\ i=1,2,\dots,p)$ is locally minimum at $\rho_{ij} = 0$ for $i \neq j$, $i,j = 1,2,\dots,p$. This gives some hope that theorem III may be true for any covariance matrix.

Remark 3. On account of the particular result in (25), we conjecture in place of (24), the following result for any r,

$$(25)' \qquad \Pr(\underset{\sim}{x}_i' \underset{\sim}{S}^{-1} \underset{\sim}{x}_i \geq c_i\beta_{ii};\ i=1,2,\dots,r) \geq \prod_{i=1}^{r} [B(\tfrac{1}{2}p, \tfrac{n+r-p}{2}) \int_{c_i}^{\infty} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n+r)} dy].$$

3. Simultaneous confidence bounds on a set of linear functions of location parameters.

(3.1) On means of k independent univariate normal variates with different variances.

Let x_{ij} $(j = 1, 2, \ldots, n_i;\ i = 1, 2, \ldots, k)$ be independently distributed as normals with means μ_i and variances σ_i^2 $(i = 1, 2, \ldots, k)$. We obtain the simultaneous confidence bounds on $\nu_j = \sum_{i=1}^{k} a_{ji}\mu_i$, $(j = 1,2,\ldots,p)$ in the similar form as that given by Banerjee [2,3] for $p = 1$. Let $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$, $s_i^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2$ and $y_j = \sum_{i=1}^{k} a_{ji}\,\bar{x}_i$, $(j = 1, 2, \ldots, p)$. Then s_i^2/σ_i^2, $(i = 1, 2, \ldots, k)$ and $(y_1, \ldots, y_p)$ are independently distributed and their respective distributions are χ^2 with (n_i-1) degrees of freedom (d.f.), $(i = 1, 2, \ldots, k)$, and multivariate normal with means $(\nu_1, \ldots, \nu_p)$ and covariances $\sigma_{jj'} = \sum_{i=1}^{k} a_{ji}a_{j'i}\,\sigma_i^2/n_i$, $j,j' = 1,2,\ldots, p$. Let E stand for the expectations over $s_1, s_2, \ldots, s_k$. Then by corollary 4, we get

$$\Pr[|y_j-\nu_j|^2 \leq \sum_{i=1}^{k} a_{ji}^2\, t_i^2\, s_i^2/n_i(n_i-1);\ j = 1, 2, \ldots, p] \tag{31}$$

$$\geq E[\prod_{j=1}^{p} \Pr\{|y_j-\nu_j|^2 \leq \sum_{j=1}^{k} a_{ji}^2\, t_i^2\, s_i^2/n_i(n_i-1) \text{ for fixed } s_1,\ldots,s_k\}].$$

where $t_1, \ldots, t_k$ are some fixed quantities to be determined. Now referring to Banerjee [2,3], we can write

$$\Pr[|y_j-\nu_j|^2 \leq \sum_{i=1}^{k} a_{ji}^2\, t_i^2\, s_i^2/n_i(n_i-1) \text{ for fixed } s_1, \ldots, s_k] \tag{32}$$

$$\geq \sum_{i=1}^{k} \omega_{i,j} \Pr[|x_j| \leq t_i s_i/\sigma_i\sqrt{n_i-1} \text{ for fixed } s_i].$$

where $x_1, \ldots, x_p$ are independent normals with zero means and unit variances, and

$$\omega_{i,j} = [a_{ji}^2\ \sigma_i^2/n_i]/[\sum_{i'=1}^{k} a_{ji'}^2\sigma_{i'}^2/n_{i'}].$$

Moreover, after some adjustments, with the help of lemma 2 or corollary 3, it is easy to show that for $1 \leq i_j$ $(j = 1,2,\ldots,p) \leq p$ (some $i_1, \ldots, i_p$ may be equal),

(33) $$\Pr[|x_j| \leq t_{i_j} s_{i_j}/\sigma_{i_j}\sqrt{n_{i_j}-1};\ j = 1, 2, \ldots, p] \geq \delta_{i_1}\delta_{i_2}\cdots\delta_{i_p}$$

where, $(j = 1, 2, \ldots, p)$,

(34) $$\delta_j = [B(\tfrac{1}{2},\tfrac{1}{2}n_j)]^{-1}\int_{-c_j}^{c_j} dt/(1+t^2)^{\frac{1}{2}n_j} \text{ and } c_j = t_j/\sqrt{n_j-1}\ .$$

Using (32) and (33) in (31), it is easy to show that

(35) $$\Pr[|y_j-\nu_j|^2 \leq \sum_{i=1}^{k} a_{ji}^2 t_i^2 s_i^2/n_i(n_i-1);\ j = 1, 2, \ldots, p] \geq \prod_{i=1}^{p}(\sum_{j=1}^{k}\omega_{i,j}\delta_j).$$

Now, choosing $t_1, \ldots, t_k$ from

(36) $$\delta_j = (1-\alpha)^{\frac{1}{p}} = \Pr[F_{1,n_j-1} \leq t_j^2],$$ $F_{m,n}$ is a F-statistic with m and n d.f. and noting $\sum_{j=1}^{p}\omega_{i,j} = 1$, we get

(37) $$\Pr[|y_j-\nu_i|^2 \leq \sum_{i=1}^{k} a_{ji}^2 t_i^2 s_i^2/n_i(n_i-1);\ j=1,2,\ldots,p] \geq (1-\alpha)\ .$$

Hence, simultaneous confidence bounds on ν_j $(j=1,2,\ldots,p)$ with confidence coefficient greater than or equal to $(1-\alpha)$ are

(38) $$y_j - \{\sum_{i=1}^{k} a_{ji}^2 t_i^2 s_i^2/n_i(n_i-1)\}^{\frac{1}{2}} \leq \nu_j \leq y_j + \{\sum_{i=1}^{k} a_{ji}^2 t_i^2 s_i^2/n_i(n_i-1)\}^{\frac{1}{2}}$$

where the t_j^2, s are to be determined from (36).

3.2 On means of k independent univariate normal distributions with equal variances. Let x_{ij} ($j = 1, 2, \ldots, n_i$; $i = 1, 2, \ldots, k$) be independently distributed as normals with means μ_i ($i = 1, 2, \ldots, k$) and variance σ^2 which is unknown. Let $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$, $s^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_i)^2$ and $y_j = \sum_{i=1}^{k} a_{ji}\bar{x}_i$ ($j = 1, 2, \ldots, p$). Then $(y_1, \ldots, y_p)$ is distributed as normal with means $\nu_j = \sum_{i=1}^{k} a_{ji}\mu_i$ ($j = 1, 2, \ldots, p$), and is distributed independently of s^2/σ^2, which is distributed as χ^2 with $\sum_{i=1}^{k} n_i-k$ or $(n-k)$ d.f. Now with the help of corollary 4, we have

$$\Pr[|y_j-\nu_j| \leq c_i \sqrt{\sum_{i=1}^{k} (a_{ji}^2/n_i)}\ s;\ i = 1,2,\ldots,p] \geq (1-\alpha) \tag{39}$$

where

$$(1-\alpha) = E[\prod_{j=1}^{p} \Pr\{|y_j-\nu_j| \leq c_i \sqrt{(\sum_{i=1}^{k} a_{ji}^2/n_i)}\ s \text{ for fixed } s\}] .$$

or

$$(1-\alpha) = \Gamma(\tfrac{p+n-k}{2})\{\pi^{\frac{1}{2}p}\Gamma(\tfrac{n-k}{2})\}^{-1} \int_D [1 + \sum_{j=1}^{p} t_j^2]^{-\frac{1}{2}(p+n-k)} dt_1 \ldots dt_p , \tag{40}$$

D being equal to $\{-c_j \leq t_j \leq c_j, j = 1, 2, \ldots, p\}$.

When $c_1 = c_2 = \ldots = c_p$ and $c_j\sqrt{n-k} = c$(say) $j = 1,2,\ldots,p$, then the tables for c for different values of α are given by Pillai and Ramachandran [12] and Dunn [7]. When the Tables are not available, we may use the inequality given in theorem II for the special case when p = 1.

Hence (39) gives us simultaneous confidence bounds on ν_j ($j = 1,2,\ldots,p$) with confidence coefficient greater than or equal to $(1-\alpha)$ as

$$y_j = \{c_i s \sqrt{\sum_{i=1}^{k} (a_{ji}^2/n_i)}\} \leq \nu_j \leq y_j + \{c_i s \sqrt{\sum_{i=1}^{k} (a_{ji}^2/n_i)}\} \tag{41}$$

for $j = 1,2,\ldots,p$, where $c_1, c_2, \ldots, c_p$ are to be determined from (40) along with the remarks just below (40).

3.3 MANOVA model for growth curve problems. The MANOVA model as defined and studied by Potthoff and Roy [13] and Khatri [10] is as follows:

Let $\underset{\sim}{X}$: $p \times n$ be a random matrix such that

$$F(\underset{\sim}{X}) = \underset{\sim}{B}\, \underset{\sim}{\xi}\, \underset{\sim}{A}$$

and the columns of $\underset{\sim}{X}$ are independent normals with common covariance matrix $\underset{\sim}{\Sigma}$: $p \times p$. The matrices $\underset{\sim}{B}$: $p \times q$ and $\underset{\sim}{A}$: $m \times n$ are assumed to be known and $\underset{\sim}{\xi}$: $q \times m$ is an unknown matrix of location parameters. Since we are only concerned with the estimable linear functions of $\underset{\sim}{\xi}$, we shall assume, without loss of generality, the ranks of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ to be m and q respectively, (see Khatri [10]). Hence, $(\underset{\sim}{B}'\underset{\sim}{B})^{-1}$ and $(\underset{\sim}{A}\underset{\sim}{A}')^{-1}$ exist and

$$E[\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{B}'\, \underset{\sim}{X}\, \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}] = \underset{\sim}{\xi} \quad .$$

We derive here simultaneous confidence bounds for the following two types of problems:

Problem 1: Let $\underset{\sim}{\xi} = (\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_m)$ and $\underset{\sim}{\eta}_j = \sum_{i=1}^{m} e_{ji}\, \underset{\sim}{\xi}_i$, $j = 1,2,\ldots,k$ where the matrix $\underset{\sim}{E} = (e_{ji})$: $k \times m$ is known. Here we are interested in deriving simultaneous confidence bounds on $\underset{\sim}{a}'\underset{\sim}{\eta}_j$ $(j=1,2,\ldots,k)$ for all non-null vectors $\underset{\sim}{a}$: $p \times 1$. Here, we assume that $\underset{\sim}{\Sigma}$ is unknown.

Problem 2: Let $\underset{\sim}{\xi}' = (\underset{\sim}{\epsilon}_1, \ldots, \underset{\sim}{\epsilon}_q)$ and $\underset{\sim}{\varphi}_j = \sum_{i=1}^{q} f_{ji}\underset{\sim}{\epsilon}_i$ $(j = 1,2,\ldots \ell)$ where the matrix $\underset{\sim}{F} = (f_{ji})$: $\ell \times q$ is known. Here, we are interested in deriving simultaneous confidence bounds on $\underset{\sim}{b}'\underset{\sim}{\varphi}_j$ $(j = 1,2,\ldots,\ell)$ for all non-null vectors $\underset{\sim}{b}$: $m \times 1$ when (1) $\underset{\sim}{V}$ is unknown and (2) v_{jj} $(j = 1,2,\ldots,\ell)$ are known. This problem is solved under the condition $v_{jj'} = \sqrt{v_{jj}v_{j'j'}}\, \alpha_j\alpha_{j'}$ for $-1 \leq \alpha_j \leq 1$, $j \neq j' = 1, 2, \ldots, \ell$ where $\underset{\sim}{V}$ is unknown and $(v_{jj'}) = \underset{\sim}{V} = \underset{\sim}{F}(\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{B}'\underset{\sim}{\Sigma}\underset{\sim}{B}(\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{F}'$

Solution of Problem 1: Let

$$\underset{\sim}{Y} = (\underset{\sim}{B}'\underset{\sim}{B})^{-1} \underset{\sim}{B}' \underset{\sim}{X} \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1} \underset{\sim}{E}' = (\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_k) ,$$

and

$$\underset{\sim}{S} = (\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{B}' \underset{\sim}{X} [\underset{\sim}{I}_n-\underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{A}]\underset{\sim}{X}' \underset{\sim}{B}(\underset{\sim}{B}'\underset{\sim}{B})^{-1} .$$

Then, it is easy to show that $\underset{\sim}{S}$ and $\underset{\sim}{Y}$ are independently distributed and their respective distributions are

$$\text{Wishart } [\underset{\sim}{S};\ v,\ n-m,\ \underset{\sim}{\Sigma}_1 = (\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{B}\ \underset{\sim}{\Sigma}\ \underset{\sim}{B}'(\underset{\sim}{B}'\underset{\sim}{B})^{-1}]$$

and normal with mean $(\underset{\sim}{\eta}_1, \ldots,\underset{\sim}{\eta}_k) = \underset{\sim}{\eta}$ and $\text{cov}(\underset{\sim}{Y}_j,\underset{\sim}{Y}_{j'}) = \beta_{jj'} \underset{\sim}{\Sigma}_1$ where $(\beta_{jj'}) = \underset{\sim}{E} (\underset{\sim}{A}\underset{\sim}{A}')^{-1} \underset{\sim}{E}'$. Then, by theorem II, we have

(42) $$\Pr[(\underset{\sim}{y}_j-\underset{\sim}{\eta}_j)'\underset{\sim}{S}^{-1}(\underset{\sim}{y}_j-\underset{\sim}{\eta}_j)/\beta_{jj} \leq c_j;\ j = 1,2,\ldots,k] \geq 1 - \alpha$$

where c_j $(j = 1,2,\ldots,k)$ are to be determined from

(43) $$(1-\alpha) = \prod_{i=1}^{k} \{[B(\tfrac{1}{2}p,\ \tfrac{n-m-p+1}{2})]^{-1} \int_0^{c_i} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n-m+1)} dy\}$$

(c.f. equation (16))

or

(44) $$(1-\alpha) = [\prod_{i=1}^{p} \{ \Gamma(\tfrac{n-m+k-i+1}{2})/ \Gamma(\tfrac{n-m-i+1}{2})] \prod_{i=1}^{k} \{[\Gamma(\tfrac{1}{2}p)]^{-1}\int_0^{c_i} y^{\frac{1}{2}p-1} (1+y)^{-\frac{1}{2}(n-m+k)} dy\}.$$

(c.f. equation (17)).

Hence (42) gives us the simultaneous confidence bounds on $\underset{\sim}{a}'\underset{\sim}{\eta}_j$ $(j=1,2,\ldots,k)$ with confidence greater than or equal to $(1-\alpha)$ as

(45) $$\underset{\sim}{a}'\underset{\sim}{Y}_j - \{\beta_{jj}c_j(\underset{\sim}{a}'\underset{\sim}{S}\underset{\sim}{a})\}^{\frac{1}{2}} \leq \underset{\sim}{a}'\underset{\sim}{\eta}_j \leq \underset{\sim}{a}'\underset{\sim}{Y}_j + \{\beta_{jj}c_j(\underset{\sim}{a}'\underset{\sim}{S}\underset{\sim}{a})\}^{\frac{1}{2}}$$

for all non-null vectors $\underset{\sim}{a}$: $p \times 1$ and c_j $(j = 1,2,\ldots,k)$ are to be determined from (43) or (44). One way to find $c_1, c_2, \ldots, c_k$ is to take $c_1 = c_2 = \ldots = c_k = c$ (say).

Solution of Problem 2: Suppose $\underset{\sim}{V}$ is unknown. Let

$$\underset{\sim}{Z} = \underset{\sim}{F}(\underset{\sim}{B}'\underset{\sim}{B})^{-1} \underset{\sim}{B}' \underset{\sim}{X} \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}, \; \underset{\sim}{Z}' = (\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_\ell),$$

and

$$(s_{1,ii'}) = \underset{\sim}{S}_1 = \underset{\sim}{F}(\underset{\sim}{B}'\underset{\sim}{B})^{-1} \underset{\sim}{B}' \underset{\sim}{X}[\underset{\sim}{I}_n - \underset{\sim}{A}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{A}]\underset{\sim}{X}'\underset{\sim}{B}(\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{F}' = \underset{\sim}{T}'\underset{\sim}{T}$$

where $\underset{\sim}{T}: \ell \times (n-m)$. Then it is easy to show that $\underset{\sim}{Z}$ and $\underset{\sim}{T}$ are independently distributed and their respective distributions are normals with means $(\varphi_1, \ldots, \varphi_\ell)$ and $\underset{\sim}{O}$, and $\text{cov}(z_j, z_{j'}) = v_{jj'}(AA')^{-1}$ and $\text{cov}(t_{ji}, t_{j'i'}) = v_{jj'}$ if $i = i'$; otherwise zero, $(j, j' = 1, 2, \ldots, \ell;\; i, i' = 1, 2, \ldots, n-m)$. Then using corollary 7, we have

(46) $$\Pr[(\underset{\sim}{z}_j - \underset{\sim}{\varphi}_j)'(\underset{\sim}{A}\underset{\sim}{A}')(\underset{\sim}{z}_j - \underset{\sim}{\varphi}_j) \leq c_j s_{1,jj};\; j=1,2,\ldots,\ell] \geq (1-\alpha) =$$

$$\prod_{j=1}^{k} \Pr[(\underset{\sim}{z}_j - \underset{\sim}{\varphi}_j)'(\underset{\sim}{A}\underset{\sim}{A}')(\underset{\sim}{z}_j - \underset{\sim}{\varphi}_j) \leq c_j s_{1,jj}] \text{ provided } v_{jj'} = \sqrt{v_{jj} v_{j'j'}}\, \alpha_j \alpha_{j'} \text{ with } -1 \leq \alpha_j \leq 1.$$

Hence (46) gives us simultaneous confidence bounds on $\underset{\sim}{b}'\underset{\sim}{\varphi}_j$ $(j = 1, 2, \ldots, \ell)$ with confidence coefficient greater than or equal to $(1-\alpha)$ as

(47) $$\underset{\sim}{b}'\underset{\sim}{z}_j - \{c_j s_{1,jj};\, \underset{\sim}{b}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1} \underset{\sim}{b}\}^{\frac{1}{2}} \leq \underset{\sim}{b}'\varphi_j \leq \underset{\sim}{b}'z_j + \{c_j s_{1,jj} \underset{\sim}{b}'(\underset{\sim}{A}\underset{\sim}{A}')^{-1}\underset{\sim}{b}\}^{\frac{1}{2}}$$

for all non-null vector $\underset{\sim}{b}: m \times 1$ and $c_1, \ldots, c_\ell$ are to be determined from

(48) $$1-\alpha = \prod_{j=1}^{\ell} [\{B(\tfrac{1}{2}m, \tfrac{1}{2}(n-m)\}^{-1} \int_0^{c_j} y^{\frac{1}{2}m-1} (1+y)^{-\frac{1}{2}m} dy].$$

One way to find $c_1, \ldots, c_\ell$ is to take $c_1 = c_2 = \ldots = c_\ell = c$ (say).

Now, suppose that v_{jj} ($j = 1,2,\ldots,\ell$) are known. Then using corollary 5, we get

(49) $$\Pr[(z_j-\varphi_j)'(AA')(z_j-\varphi_j) \leq c_j v_{jj};\ j=1,2,\ldots,\ell] \geq (1-\alpha) = \prod_{j=1}^{p} \Pr[(z_j-\varphi_j)'(AA')(z_j-\varphi_j) \leq c_j v_{jj}].$$

Hence (49) gives us simultaneous confidence bounds on $b'\varphi_j$ ($j=1,2,\ldots,\ell$) with confidence coefficient greater than or equal to $(1-\alpha)$ as

(50) $$b'z_j - \{c_j v_{jj} b'(AA')^{-1}b\}^{\frac{1}{2}} \leq b'\varphi_j \leq b'z_j + \{c_j v_{jj} b'(AA')^{-1}b\}^{\frac{1}{2}}$$

for all non-null vectors b: $m \times 1$; $c_1, \ldots, c_\ell$ are to be determined from

(51) $$(1-\alpha) = \prod_{j=1}^{\ell} [\{2^m \Gamma(\tfrac{1}{2}m)\}^{-1} \int_0^{c_j} y^{\frac{1}{2}m-1} \exp(-\tfrac{1}{2}y)dy].$$

One way to find $c_1, \ldots, c_\ell$ is to take $c_1 = c_2, \ldots, c_\ell = c$ (say).

3.4 Some special problems. Let $x_1, \ldots, x_n$ be independent observations on x which is distributed as multivariate normal with mean μ and covariance matrix Σ: $p \times p$. Here let us first suppose that $\sigma_{ii} = \sigma^2$, ($i = 1,2,\ldots,p$) is unknown. Let $\bar{x}_i = \sum_{j=1}^{n} x_{ij}/n$ and $s^2 = \sum_{i=1}^{p} s_{ii}/p$, with $s_{ii} = \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2$. Then $(\bar{x}_1,\ldots,\bar{x}_p)$ is distributed as normal with mean vector $(\mu_1,\ldots,\mu_p)$ and is distributed independently of s^2. Using arguments similar to those of Dunn [6, pp. 1102-3], we get

(52) $$p \Pr[\ n|\bar{x}_i-\mu_i| \leq c_i s, i=1,2,\ldots,p] \geq \sum_{j=1}^{p} \Pr[\ n|\bar{x}_i-\mu_i| \leq c_i\ s_{jj}; i=1,2,\ldots,p];$$

But for $j \neq j'$,

$$\Pr[\ n|\bar{x}_i-\mu_i| \leq c_i\ s_{jj};\ i=1,2,\ldots,p] = \Pr[\ n|\bar{x}_i-\mu_i| \leq c_i\ s_{j'j'};\ i=1,2,\ldots,p].$$

Hence, we get

(53) $\Pr[\sqrt{n}\ |\bar{x}_i - \mu_i| \le c_i s\ i = 1,2,\ldots,p] \ge \Pr[\sqrt{n}\,|\bar{x}_i - \mu_i| \le c_i \sqrt{s_{11}};\ i=1,2,\ldots,p]$.

Now, using corollary 4 for fixed s_{11} and then integrating over s_{11}, we get

(54) $\Pr[|\bar{x}_i - \mu_i| \le c_i s/\sqrt{n};\ i= 1,2,\ldots,p] \ge (1-\alpha)$,

where

(55) $$(1-\alpha) = \int_D \Gamma\left(\frac{n+p-1}{2}\right)\{\pi^{\frac{1}{2}p}\Gamma\left(\frac{n-1}{2}\right)\}^{-1}[1+\sum_{i=1}^{p} t_i^2]^{-\frac{1}{2}(n+p-1)} dt_1\ldots dt_p ,$$

D being equal to $\{-c_j \le t_j \le c_j\ ;\ j=1,2,\ldots,p\}$.

When $c_i\sqrt{n-1} = c$ for $i=1,2,\ldots,p$, then tables of c for different values of α are available (see references [7] and [12]). When the tables are insufficient, we may use the inequality derived in theorem II. From (54), we get simultaneous confidence bounds on μ_i $(i=1,2,\ldots,p)$ with confidence coefficient at least $(1-\alpha)$ as

(56) $\bar{x}_i - (c_i s/\sqrt{n}) \le \mu_i \le \bar{x}_i + (c_i s/\sqrt{n})$, for $i=1,2,\ldots,p$

where $c_1, \ldots, c_p$ are to be determined from (55).

We can improve the result (56) if we know $(tr_q \rho)^{\frac{1}{2}}$ where ρ is the correlation matrix of $x_1, \ldots, x_p$, q is the rank of ρ and $tr_q\, \rho$ means sum of the principal minors of order q in ρ. This is done with the help of the following result (5.15, p.566 when $k=0$ and $p= (1-\mu)a_g$) of Ruben [20]:

Let $y_1, y_2, \ldots, y_q$ be independent normal variates with zero means and unit variances, and $\lambda_i > 0$ $(i=1,2,\ldots,q)$. Then,

(57) $$\Pr[\sum_{i=1}^{q} \lambda_i y_i^2 \le c(\lambda_1\lambda_2\ldots\lambda_q)^{1/q}] \le \Pr[\sum_{i=1}^{q} y_i^2 \le c] .$$

If $\lambda_1, \lambda_2, \ldots, \lambda_q$ are the nonzero ch. roots of Σ, then $tr_q \Sigma = \lambda_1 \lambda_2 \cdots \lambda_q$ and (51) is equivalent to

$$(58) \quad Pr[\sum_{i=1}^{p} x_i^2 \leq c(tr_q \Sigma)^{\frac{1}{2}} \text{ when } x \to N(0, \Sigma)] \leq Pr(\sum_{i=1}^{q} y_i^2 \leq c).$$

Hence, it is easy to see that

$$(59) \quad Pr[s^2 = \sum_{i=1}^{p} s_{ii}/p \leq c(tr_q \rho)^{\frac{1}{2}} \sigma^2] \leq Pr[\sum_{j=1}^{n-1} \sum_{i=1}^{q} y_{ij}^2 \leq cp],$$

where y_{ij}'s are independent normals with zero means and unit variances.

Now, when $\bar{x}_1, \ldots \bar{x}_p$ are fixed, then from (59), we get

$$(60) \quad Pr[s^2 \geq \max_i \{(\bar{x}_i - \mu_i)^2 n/c_i^2 \sigma^2\} (tr_q \rho)^{1/q} \sigma^2 \mid \bar{x}_1, \ldots, \bar{x}_p]$$

$$\geq Pr[\sum_{j=1}^{n-1} \sum_{i=1}^{q} y_{ij}^2 \geq p \max_i \{(\bar{x}_i - \mu_i)^2 n/c_i^2 \sigma^2\} \mid \bar{x}_1, \ldots, x_p].$$

Now (60) is equivalent to

$$(61) \quad Pr[|\bar{x}_i - \mu_i| \leq c_i s \sqrt{n} (tr_q \rho)^{1/q};\ i=1,2,\ldots,p]$$

$$\geq Pr[|\bar{x}_i - \mu_i| \leq c_i \chi \sigma / \sqrt{np} \quad ;\ i=1,2,\ldots,p]$$

where $\chi^2 = \sum_{j=1}^{n-1} \sum_{i=1}^{q} y_{ij}^2$ is a χ^2 variate with $(n-1)q$ d.f. Now, using corollary (4) in (61), we get

$$(62) \quad Pr[|\bar{x}_i - \mu_i| \leq c_i s/\sqrt{n} (tr_q \rho)^{1/q};\ i=1,2,\ldots,p] \geq (1-\alpha)$$

where

$$(63) \quad (1-\alpha) = \Gamma\left(\frac{q(n-1)+p}{2}\right)\left\{\pi^{\frac{1}{2}p} \Gamma\left(\frac{q(n-1)}{2}\right)\right\}^{-1} \int_D \{1+\sum_{i=1}^{p} t_i^2\}^{-\frac{1}{2}\{q(n-1)+p\}} dt_1 \ldots dt_p,$$

D being equal to $\{-c_i/\sqrt{p} \leq t_i \leq c_i/\sqrt{p}$ for $i=1,2,\ldots,p\}$.

When $c_i/\sqrt{p} = c/\sqrt{q(n-1)}$ $(i=1,2,\ldots,p)$, then tables [for c] are available. Compare (63) with (55). Hence (62) gives us simultaneous confidence bounds on μ_i $(i=1,2,\ldots,p)$ with confidence coefficient greater than or equal to $(1-\alpha)$ as

$$(64) \qquad \bar{x}_i - \{c_i\, s/\sqrt{n}(\mathrm{tr}_q\, \rho)^{1/q}\} \leq \mu_i \leq \bar{x}_i + \{c_i s/\sqrt{n}\, (\mathrm{tr}_q \rho)^{1/q}\}$$

for $i=1,2,\ldots,p$; where $c_1,\ldots,c_p$ are given by (61).

In the concluding, we remark that simultaneous confidence bounds obtained in section 3 will be shorter than the traditional ones when the number of linear functions is not too large and in some cases it may nearly be the shortest. All the results of section 3 can easily be extended to regression - like parameters (in testing independence of two sets), to testing the multi-collinearity of means (or to covariance analysis), and to the step down procedures, but they are not given here.

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Note:- The results of sections 2 and 3 can be shown to be true for complex multivariate Gaussian distributions with necessary modifications. These results will be presented in a later report.

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