ON MAXIMAL AND MINIMAL SUB-FIELDS OF CERTAIN TYPES

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Institute of Statistics Mimeo Series No. 422

January 1965

This research was supported by the National Science Foundation Grant No. GP 16-60

DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.

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Let $(\underline{X}, \underline{A}, \underline{P})$ be a given statistical model. We concern ourselves with particular families of sub-fields of $\underline{A}$, and, for each such family we ask ourselves whether the family has maximal and minimal elements with respect to the natural partial order of inclusion relation. For example the family $\{\underline{S}\}$ of sub-fields that are sufficient for $\underline{A}$ has received a great deal of attention from theoretical statisticians. Clearly, $\underline{A} \in \{\underline{S}\}$ and is the maximum element of $\{\underline{S}\}$. It is known [4] that, in general, $\{\underline{S}\}$ has no minimal element. However, if we assume that $\underline{P}$ is dominated by a σ-finite measure then it can be shown [1] that $\{\underline{S}\}$ has an essentially minimum element $\underline{S}_o$, i.e., given any $\underline{S}_1 \in \{\underline{S}\}$ and any $A \in \underline{S}_o$, there exists $B \in \underline{S}_1$ such that

$$P(A \triangle B) \equiv 0 \quad \text{for all} \quad P \in \underline{P},$$

where $\triangle$ stands for the operation of symmetric difference.

Let $\underline{B}$ be a fixed sub-field of $\underline{A}$ and let $\{\underline{C}\}$ be the familiy of all sub-fields that are independent of $\underline{B}$, i.e., for any such $\underline{C}$ it is true that

$$P(BC) \equiv P(B)P(C) \quad \text{for all} \quad B \in \underline{B}, \quad C \in \underline{C}, \text{ and } \quad P \in \underline{P}.$$

The minimum element of $\{\underline{C}\}$ is the trivial sub-field consisting of the null-set and the whole space. We have the following

Theorem 1: For any sub-field $\underline{C}$ that is independent of $\underline{B}$, there exists at least one maximal subfield C^* such that $\underline{C} \subset \underline{C}^*$.

The proof of Theorem 1 is given in the next section.

For the next problem, let us suppose that the family $\underline{P}$ is indexed by two parameters θ and φ , i.e.,

$$\underline{P} = \{P_{\theta,\varphi}\} \ , \ (\theta, \varphi) \in \Theta \times \Phi \ .$$

Consider all sub-fields $\{\underline{D}\}$ that are generated by statistics whose probability distributions do not involve the paramter φ , i.e., each $\underline{D}$ is a sub-field such that, for every $D \in \underline{D}$, $P_{\theta,\varphi}(D)$ is a function of θ only.

The trivial sub-field is again the minimum element of $\{\underline{D}\}$; but, does this family have maximal elements? We have

Theorem 2: Given any sub-field $\underline{D}$ such that the restriction of $P_{\theta,\varphi}$ to $\underline{D}$ does not involve φ, there exists a maximal such sub-field $\underline{D}^*$ that contains $\underline{D}$.

The above Theorem is an immediate generalization of Theorem 1 in [2].

In the next section we give the proof of the above two theorems. In the final section we comment on some further problems of the same kind.

2. Proofs of Theorems 1 and 2

We need the following well-known lemmas.

Lemma 1 (Zorn's Lemma): If for a partially ordered set it is true that every linearly ordered sub-set has an upper (lower) bound, then given any element x of the set there exists a maximal (minimal) element x^* in the set such that x is less (greater) than x^*.

[The terms that are underlined are defined in terms of the partial order relation].

Lemma 2 (Extension of Measures): Given a measure μ defined on a field $\underline{F}$ of sets there exists one and only one extension μ^* of μ to the Borel-extension $\underline{F}^*$ of $\underline{F}$.

Corollary: If the two measures μ and ν agree on a field $\underline{F}$ of sets they necessarily agree on the Borel-extension $\underline{F}^*$ of F.

Lemma 3: If the family $\{\underline{B}_\alpha\}$ of sub-fields of $\underline{A}$ be linearly ordered with respect to the inclusion relation then

$$\underline{B} = \cup \underline{B}_\alpha$$

is a field of sub-sets.

We omit the proofs of the lemmas.

Now let $\underline{B}$ be a given sub-field of $\underline{A}$ and let $\underline{E}$ be the class of all sets $E \in \underline{A}$ such that E is independent of $\underline{B}$, i.e.

$$P(EB) \equiv P(E)P(B) \text{ for all } B \in \underline{B}, \ E \in \underline{E} \text{ and } P \in \underline{P}.$$

It is easy to check that $\underline{E}$ contains the null-set and the whole space and further that $\underline{E}$ is closed for complementation and countable disjoint unions. In case $\underline{E}$ is a σ-field there is nothing to prove in Theorem 1, as $\underline{E}$ is then the maximal sub-field for which we are searching. However, $\underline{E}$ is usually not a σ-field (see example 1).

Let $\{\underline{C}_\alpha\}$ be a family of sub-fields in $\underline{E}$ and be linearly ordered with respect to the inclusion relation and let

$$\underline{C}_o = \cup \underline{C}_\alpha .$$

Now, from Lemma 3, $\underline{C}_o$ is a field of sub-sets of $\underline{A}$ and, since $C_o \subset \underline{E}$, every member of $\underline{C}_o$ is independent of $\underline{B}$. Choose and fix $B \in \underline{B}$ and $P \in \underline{P}$.

Consider the two measures P(AB) and P(A)P(B) defined for all sets $A \in \underline{A}$. These two measures agree over the field $\underline{C}_o$ and hence, from the corollary to Lemma 2, they agree over the Borel-extension $\underline{C}_o^*$ of $\underline{C}_o$.

Remembering that B and P were arbitrary members of $\underline{B}$ and $\underline{P}$ respectively, we now have

$$P(AB) \equiv P(A)P(B) \text{ for all } A \in \underline{C}^* , \quad B \in \underline{B} \text{ and } P \in \underline{P} .$$

Thus, $\underline{C}_o^*$ includes every $\underline{C}_\alpha$ and is independent of $\underline{B}$. The conditions of Lemma 1 are satisfied and hence the proof of Theorem 1 is complete.

We now turn our attention to Theorem 2. Let $\underline{F}$ be the class of all sets $F \in \underline{A}$ such that

$$P_{\theta,\varphi}(F) \text{ is a function of } \theta \text{ only.}$$

Again, it is easy to check that $\underline{F}$ contains the null-set and the whole space and is closed for complementation and countable disjoint unions. In example 2 we shall see that $\underline{F}$ is usually not a sub-field of $\underline{A}$.

Let $\{\mathfrak{D}_\alpha\}$ be a family of sub-fields in $\underline{F}$ and be linearly ordered with respect to the inclusion relation and let

$$\mathfrak{D}_o = \bigcup \mathfrak{D}_\alpha .$$

As before $\mathfrak{D}_o$ is a field of sub-sets of $\underline{X}$. Let $\mathfrak{D}_o^*$ be the Borel-extension of $\mathfrak{D}_o$.

We define the measure Q_θ on $\mathfrak{D}_o$ as the restriction of $P_{\theta\varphi}$ on $\mathfrak{D}_o$. [Since $\mathfrak{D}_o \subset \underline{F}$, the measure Q_θ on $\mathfrak{D}_o$ must be independent of φ.]

From Lemma 2, for each $\theta \in \Theta$, the extension Q_θ^* of Q_θ from $\mathfrak{D}_o$ to $\mathfrak{D}_o^*$ is unique. From the corollary to Lemma 2, the two measures Q_θ^* and $P_{\theta\varphi}$ must agree on $\mathfrak{D}_o^*$.

In other words, the restriction of $P_{\theta,\varphi}$ to the sub-field $\mathfrak{D}_o^*$ is independent of φ. Also, $\mathfrak{D}_o^*$ includes every $\mathfrak{D}_\alpha$. The conditions of Lemma 1 are satisfied and hence the proof of Theorem 2 is complete.

3. Examples and Comments.

The following two examples demonstrate that the maximal element is usually not unique.

Example 1: Let $\underline{X}$ consist of the four points a,b,c and d and let $\underline{P}$ consist of just one probability distribution namely the uniform distribution over the four points. Consider the three sub-fields $\underline{B}$, $\underline{C}_1$ and $\underline{C}_2$ each consisting of four sub-sets of $\underline{X}$:

$\underline{B}$ consists of $\underline{X}$, {a,b} and their complements,

$\underline{C}_1$ .... $\underline{X}$, {a,c} ....,

and $\underline{C}_2$ .... $\underline{X}$, {a,d} ....

Here, each of $\underline{C}_1$ and $\underline{C}_2$ is independent of $\underline{B}$ and each of them is a maximal such sub-field. Incidentally, $\underline{C}_1$ and $\underline{C}_2$ are also independent of each other.

Example 2: Let $\underline{X}$ consist of the five points a,b,c,d, and e and let the probability distribution over the five points be

Points:	a	b	c	d	e
Probs:	$1-\theta$	$\theta\varphi$	$\theta\varphi$	$\theta(\frac{1}{2}-\varphi)$	$\theta(\frac{1}{2}-\varphi)$

where $0 < \theta < 1$ and $0 < \varphi < \frac{1}{2}$.

The family $\underline{F}$ of all sets whose probability does not involve φ consists of 12 sets, that is

$\underline{X}$, {a} , {b,d} , {b,e} , {c,d} , {c,e}

and their complements.

Note that $\underline{F}$ does not constitute a sub-field. There are two different maximal sub-fields in $\underline{F}$, namely,

$\mathfrak{A}_1$: consisting of $\underline{X}$, { a} , {b,d} , {c,e}
and their complements.

and $\mathcal{A}_2$: consisting of $\underline{X}$, $\{a\}$, $\{b,e\}$, $\{c,d\}$ and their complements.

Theorems 1 and 2 only establish the existence of maximal sub-fields in $\underline{E}$ and $\underline{F}$ respectively. It would be of some interest to develop general methods for proving the maximality of certain given sub-fields of $\underline{E}$ and $\underline{F}$. One such method, with very limited application, is given in Theorem 7 of [2].

Consider the problem where we have n independent observations $x_1, x_2, \ldots, x_n$ on a real random variable x with cumulative distribution function of the form

$$F\left(\frac{x-\varphi}{\theta}\right), \quad -\infty < \varphi < \infty, \quad 0 < \theta < \infty .$$

where the function F is known and θ and φ are the so-called scale and location parameters.

If y stand for the vector-valued statistic

$$(x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n)$$

then the distribution of y does not involve the location parameter φ. Is y a maximal such statistic? In the language of sub-fields, if $\underline{C}_y$ be the sub-field generated by y then is it true that $\underline{C}_y$ is maximal in the sense of Theorem 2? The author does not expect the answer to be 'yes' for all F.

4. Some further problems

The sub-field $\underline{B} \subset \underline{A}$ is said to be sufficient for the sub-field $\underline{C} \subset \underline{A}$ if for every $C \in \underline{C}$ there exists a $\underline{B}$-measurable function $f(x ; C)$ mapping $\underline{X}$ into the real line such that

$$P(BC) \equiv \int_B f(x ; C)\, dP(x) \quad \text{for all } P \in \underline{P} \text{ and } B \in \underline{B}.$$

In other words, $\underline{B}$ is sufficient for $\underline{C}$ if, for every $C \in \underline{C}$, there exists a choice for the conditional probability (function) of C given $\underline{B}$ that serves for all $P \in \underline{P}$.

Now, for a fixed $\underline{B}$, let us enquire about the family $\{\underline{C}\}$ of all sub-fiels $\underline{C}$ such that $\underline{B}$ is sufficient for $\underline{C}$. Clearly, the minimum element of $\{\underline{C}\}$ is the trivial sub-field consisting of only the null-set and the whole space. Do there exist maximal elements in $\{\underline{C}\}$?

Let $\underline{G}$ be the class of all sets $G \in \underline{A}$ such that $\underline{B}$ is sufficient for G in the sense mentioned above, namely, for every $G \in \underline{G}$ there exists a $\underline{B}$-measurable $f(x;G)$ such that

$$P(GB) \equiv \int_B f(x;G)dP(x) \quad \text{for all } P \in \underline{P} \text{ and } B \in \underline{B}.$$

The class $\underline{G}$ is similar to the classes $\underline{E}$ anf $\underline{F}$ considered before in that $\underline{G}$ contains the null-set and the whole space and is closed for complementation and countable disjoint unions. As before, $\underline{G}$ is usually not a sub-field. The rest of the arguments in Theorems 1 and 2 will apply if we could prove a result of the following type:

"If $\underline{B}$ is sufficient for each member of a field $\underline{C}$ of sets in $\underline{A}$, then $\underline{B}$ is sufficient for the Borel extension $\underline{C}^*$ of $\underline{C}$."

The above statement does not seem to be true in the generality stated above.

In the particular case where $\underline{B}$ is the trivial sub-field, the question posed above has a definite answer. For, in this case, $\underline{B}$ can be sufficient for G if and only if

$$P(G) \text{ is the same for all } P \in \underline{P},$$

and therefore, Theorem 2, or rather a particular case of it, namely, Theorem 1 in [2] applies.

Fraser in [3] introduced the notation of partial sufficiency in the following manner:

If $\underline{P} = \{P_{\theta,\varphi}\}$, $\theta \in \Theta$, $\varphi \in \Phi$, be a family of probability measures indexed by the two independent parameters θ and φ, then a sub-field $\underline{B} \subset \underline{A}$ will be called θ-sufficient for $\underline{A}$ (or simply θ-sufficient) if

i) the restriction of $P_{\theta,\varphi}$ to $\underline{B}$ does not depend on φ [i.e., $\underline{B}$ is a sub-field of the type considered in Theorem 2.],

and ii) given any $A \in \underline{A}$, there exists a choice (of the conditional probability (function) of A given $\underline{B}$ that does not depend on θ, i.e., for each $\theta \in \Theta$ there exists a $\underline{B}$-measurable function $f_\theta(x;A)$ such that

$$P_{\theta,\varphi}(AB) \equiv \int_B f_\theta(x;A)\, dP_{\theta,\varphi} \quad \text{for all } B \in \underline{B} \text{ and all } (\theta,\varphi).$$

Under what conditions does a θ-sufficient sub-field exist? Does there exist an essentially minimum such sub-field?

As a final problem on the existence of minimal sub-fields consider the following:

Given two sub-fields $\underline{B}$ and $\underline{C}$, let $\underline{B} \vee \underline{C}$ stand for the smallest sub-field that contains both $\underline{B}$ and $\underline{C}$.

Now, for a fixed $\underline{B} \subset \underline{A}$, let us consider the family $\{\underline{C}\}$ of all sub-fields $\underline{C} \subset \underline{A}$ such that

$$\underline{B} \vee \underline{C} = \underline{A}.$$

Every $\underline{C} \in \{\underline{C}\}$ may be called a complement of $\underline{B}$. The family $\{\underline{C}\}$ has $\underline{A}$ as its maximum element. Does $\{\underline{C}\}$ have minimal elements? The author expects the answer to be yes. It is easy to construct examples where there are several minimal complements to $\underline{B}$.

References

1. Bahadur, R.R. (1954), Sufficiency and Statistical decision functions, Ann. Math. Statist., 25, 425-462.

2. Basu, D. (1959), The family of ancillary statistics, Sankhya, 21, 247-256.

3. Fraser, D.A.S. (1956): Sufficient statistics with nuisance paramters, Ann. Math. Statist. 27, 838-842.

4. Pitcher, T.S., (1957), Sets of measures not admitting necessary and sufficient statistics or sub-fields, Ann. Math. Statist., 28, 267-268.