ON THE ASYMPTOTIC DISTRIBUTION OF F-STATISTIC UNDER THE NULL-HYPOTHESIS IN A RANDMIZED P B I B DESIGN WITH M ASSOCIATE CLASSES UNDER THE NEYMAN MODEL

By

Sadao Ikeda, Junjiro Ogawa and Motoyasu Ogasawara
University of North Carolina; Nihon University; Nihon University

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SUMMARY

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DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA

Chapel Hill, N. C.

SUMMARY

This article gives a probability-theoretical justification to the previous work [5] on the asymptotic null-distribution of the F-statistic for testing a null-hypothesis in a randomized partially balanced incomplete block design with m associate classes under the Neyman model, which is the most general case among the series of our preceding works along the same line [1, 2, 3, 4, 5]. The theory of asymptotic equivalence of probability distributions developed by one of the authors [6, 7] seems to be effectively usable for the present study. The investigation in this article throws a light on the asymptotic nature of the power function of the F-statistic under consideration, which will be worked out in the forthcoming paper.

1. Statement of the Problem

Throughout this article the notation and terminology are the same as those of [5] unless otherwise stated.

We are concerned with a partially balanced incomplete block design with m associate classes, which has v treatment with the association, b blocks of size k each, r replications of each treatment, and the number of incidence of any pair of treatments λ_u if they are u-th associates.

Let the incidence matrices of treatments and blocks be Φ and ψ respectively, then the Neyman model assuming no interaction between the treatments and the units is given by

$$x = \gamma 1 + \Phi\tau + \psi\beta + \pi + e, \tag{1.1}$$

where $x' = (x_1, \ldots, x_n)$ is the observation vector, $\tau' = (\tau_1, \ldots, \tau_v)$ and $\beta' = (\beta_1, \ldots, \beta_b)$ are treatment-effects and block-effects being subjected to the restrictions

$$\sum_{i=1}^{v} \tau_i = 0 \quad \text{and} \quad \sum_{a=1}^{b} \beta_a = 0 \tag{1.2}$$

respectively, and $\pi' = (\pi_1, \ldots, \pi_n)$ stands for the unit-errors being subjected to the restriction

(1.3) $$\psi' \pi = 0.$$

Finally, $e' = (e_1, \ldots, e_n)$ is the technical-error being distributed as $N(o; \sigma^2 I_n)$.

Sums of squares due to treatments adjusted and errors are given by

(1.4) $$S_t^2 = x'\left(\sum_{u=1}^{m} V_u^{\#}\right)x$$

$$S_e^2 = x'\left(1-\frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#}\right)x$$

respectively, where

(1.5) $$V_u^{\#} = c_u\left(I - \frac{1}{k} B\right) T_u^{\#} \left(I - \frac{1}{k} B\right), \quad u = 1,2,\ldots, m$$

with

$$T_u^{\#} = \Phi A_u^{\#} \phi', \quad B = \psi \psi' \quad \text{and}$$

$$c_u = \frac{k}{rk - \rho u}, \quad u = 1,\ldots, m.$$

Here, $A_u^{\#}$, $u = 0,1,\ldots, m$, with $A_o^{\#} = G_o/v$ are $m + 1$ orthogonal idempotents of the association algebra, and ρ_u, $u = 1,\ldots, m$ and $\rho_o = rk$ are the characteristic roots of NN', N being the incidence matrix of the design, with respective multiplicity $\alpha_1, \ldots, \alpha_m, \alpha_o$, for which $\sum_{u=1}^{m} \alpha_u = v-1$.

We are interested in testing a partial null-hypothesis that some of the hypotheses $A_u^{\#}\tau = 0$, $u = 1,\ldots, m$, are true. We can take, without any loss of generality, the null-hypothesis

(1.6) $$H_{o(h)}: \ A_u^{\#}\tau = 0, \quad u = 1, \ldots, h$$

where h is a positive integer not greater than m. This hypothesis is called the 'partial' hypothesis, and is equivalent to the hypothesis $\Sigma_{u=1}^{h} A_u^{\#}\tau = 0$.

When $h = m$, this reduces to the 'total' null-hypothesis H_o: $\tau = 0$.

To test the null-hypothesis $H_o(h)$, we consider the partial sum of squares

$$S^2_{t(h)} = x' \left(\sum_{u=1}^{h} V_u^{\#} \right) x \tag{1.7}$$

instead of S^2_t given by (1.4). Then the null-distribution of the variate

$$\chi^2_{1(h)} = S^2_{t(h)}/\sigma^2 \tag{1.8}$$

before the randomization is the non-central chi-square distribution of the degrees of freedom $\bar{\alpha} = \alpha_1 + .. + \alpha_h$ with the non-centrality parameter

$$\begin{aligned} \bar{\kappa}_1 &= \pi' \Phi \left(\sum_{u=1}^{h} c_u A_u^{\#} \right) \Phi' \pi/\sigma^2 \\ &= \pi' \left(\sum_{u=1}^{h} V_u^{\#} \right) \pi/\sigma^2. \end{aligned} \tag{1.9}$$

Whence its probability element is given by

$$\exp -\left(\frac{\bar{\kappa}_1}{2}\right) \sum_{\mu=o}^{\infty} \frac{\left(\frac{\bar{\kappa}_1}{2}\right)^{\mu}}{\mu!} \frac{\left(\frac{\chi^2_{1(h)}}{2}\right)^{\frac{\bar{\alpha}}{2} + \mu - 1}}{\Gamma\left(\frac{\bar{\alpha}}{2} + \mu\right)} \cdot \exp\left(-\frac{\chi^2_{1(h)}}{2}\right) d \left(\frac{\chi^2_{1(h)}}{2}\right) \tag{1.10}$$

In the special case when $h = m$, the variate (1.8) reduces to

$$\chi^2_1 = S^2_t/\sigma^2, \tag{1.11}$$

and the null-distribution of this variate before the randomization is the non-central chi-square distribution of degrees of freedom $v-1$ with the non-centrality parameter

$$(1.12) \qquad \kappa_1 = \pi'(\sum_{u=1}^{m} V_u^{\#})\pi/\sigma^2,$$

whose probability clement is given by

$$(1.13) \qquad \exp\left(-\frac{\kappa_1}{2}\right) \sum_{\mu=0}^{\infty} \frac{\left(\frac{\kappa_1}{2}\right)^{\mu}}{\mu!} \cdot \frac{\left(\frac{\chi_1^2}{2}\right)^{\frac{v-1}{2}+\mu-1}}{\Gamma\left(\frac{v-1}{2}+\mu\right)}$$

$$\exp\left(-\frac{\chi_1^2}{2}\right) d\left(-\frac{\chi_1^2}{2}\right).$$

The null-distribution of the variate

$$(1.14) \qquad \chi_2^2 = S_e^2/\sigma^2$$

before the randomization is, independently of the hypotheses, the non-central chi-square distributions of degrees of freedom $n-b-v+1 = b(k-1)-(v-1)$ with the non-centrality parameter

$$(1.15) \qquad \kappa_1 = \pi'[I_n - \Phi(\sum_{u=1}^{m} c_u A_u^{\#})\Phi']\pi/\sigma^2$$

$$= \Delta/\sigma^2 - \bar{\kappa}_1 - \bar{\bar{\kappa}}_1 = \Delta/\sigma^2 - \kappa_1$$

where $\Delta = \pi'\pi$ and

$$(1.16) \qquad \bar{\bar{\kappa}}_1 = \pi'\left(\sum_{u=h+1}^{m} V_u^{\#}\right)\pi.$$

Hence the probability element of the variate χ_2^2 is given by

$$(1.17) \qquad \exp\left(-\frac{\kappa_2}{2}\right) \sum_{\nu=0}^{\infty} \frac{\left(\frac{\kappa_2}{2}\right)^{\nu}}{\nu!} \frac{\left(\frac{\chi_2^2}{2}\right)^{\frac{n-b-v+1}{2}+\nu-1}}{\Gamma\left(\frac{n-b-v+1}{2}+\nu\right)}$$

$$\exp\left(-\frac{\chi_2^2}{2}\right) d\left(\frac{\chi_2^2}{2}\right).$$

Since χ_1^2 and χ_2^2 are mutually independent in the stochastic sense, the null-distribution of the F statistic given by

(1.18) $$F = \frac{n-b-v+1}{\bar{\alpha}} \cdot \frac{S^2_{t(h)}}{S^2_e}$$

before the randomization is the non-central F-distribution, whose probability element is given by

(1.19) $$\exp\left(\frac{-\Delta}{2\sigma^2}\right) \sum_{\ell=0}^{\infty} \frac{\left(\frac{\Delta}{2\sigma^2}\right)^{\ell}}{\ell\,!} \sum_{\mu+\nu+\gamma=\ell} \frac{\ell\,!}{\mu!\,\nu!\,\gamma!}\ \bar{\theta}^{\mu}\bar{\bar{\theta}}^{\nu}(1-\bar{\theta}-\bar{\bar{\theta}})^{\gamma}$$

$$\cdot \frac{\Gamma\left(\frac{n-b-\bar{\bar{\alpha}}}{2}+\mu+\nu\right)}{\Gamma\left(\frac{\bar{\alpha}}{2}+\mu\right)\Gamma\left(\frac{n-b-v+1}{2}+\nu\right)} \left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\frac{\bar{\alpha}}{2}+\mu-1}$$

$$\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-\left(\frac{n-b-\bar{\bar{\alpha}}}{2}+\mu+\nu\right)} d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)$$

where we have put

(1.20) $$\bar{\bar{\alpha}} = \sum_{u=h+1}^{m} \alpha_u = v-1-\bar{\alpha},$$

$$\bar{\theta} = \pi'\left(\sum_{u=1}^{h} V^{\#}_u\right)\pi/\Delta \quad \text{and} \quad \bar{\bar{\theta}} = \pi'\left(\sum_{u=h+1}^{m} V^{\#}_u\right)\pi/\Delta.$$

When $h=m$, the null-distribution of the F-statistic

(1.21) $$F = \frac{n-b-v+1}{v-1}\ \frac{S^2_t}{S^2_e}$$

before the randomization is the non-central F-distribution, whose probability element is given by

$$\exp\left(-\frac{\Delta}{2\sigma^2}\right)\sum_{\ell=0}^{\infty}\frac{\left(\frac{\Delta}{2\sigma^2}\right)^{\ell}}{\ell!}\sum_{\mu+\nu=\ell}\frac{\ell!}{\mu!\,\nu!}\,\theta^{\mu}(1-\theta)^{\nu} \tag{1.22}$$

$$\frac{\Gamma\left(\frac{n-b}{2}+\ell\right)}{\Gamma\left(\frac{v-1}{2}+\mu\right)\Gamma\left(\frac{n-b-v+1}{2}+\nu\right)}\left(\frac{v-1}{n-b-v+1}F\right)^{\frac{v-1}{2}+\mu-1}$$

$$\left(1+\frac{v-1}{n-b-v+1}F\right)^{-\left(\frac{n-b}{2}+\ell\right)} d\left(\frac{v-1}{n-b-v+1}F\right)$$

where

$$\theta = \bar{\theta} + \bar{\bar{\theta}} = \pi'\left(\sum_{u=1}^{m} V_u^{\#}\right)\pi/\Delta \tag{1.23}$$

In the previous paper [5] the authors showed, by comparing the mean values and variances of both distributions in a very tedious way, that the permutation distribution of ($\bar{\theta},\bar{\bar{\theta}}$) due to randomization can be approximated by the Dirichlet distribution, whose probability element being given by

$$\frac{\Gamma\left(\frac{n-b}{2}\right)}{\Gamma\left(\frac{\bar{\alpha}}{2}\right)\Gamma\left(\frac{\bar{\bar{\alpha}}}{2}\right)\Gamma\left(\frac{n-b-v+1}{2}\right)}\;\bar{\theta}^{\frac{\bar{\alpha}}{2}-1}\,\bar{\bar{\theta}}^{\frac{\bar{\bar{\alpha}}}{2}-1}\,(1-\bar{\theta}-\bar{\bar{\theta}})^{\frac{n-b-v+1}{2}-1}\,d\bar{\theta}d\bar{\bar{\theta}}, \tag{1.24}$$

$$(0\le\bar{\theta},\bar{\bar{\theta}},\bar{\theta}+\bar{\bar{\theta}}\le 1)$$

for sufficiently large b, under the conditions

$$\Delta_p\left(\equiv\sum_{i=1}^{k}\pi_i^{(p)2}\right) = \Delta_o \quad\text{and}\quad \Gamma_p\left(\equiv\sum_{i=1}^{k}\pi_i^{(p)4}\right) = \Gamma_o,\; p=1,\dots,b \tag{1.25}$$

where $\pi_i^{(p)}$ stands for the unit error of the i-th unit of the p-th block. Then, they integrated out the factor $\bar{\theta}^{\mu}\bar{\bar{\theta}}^{\nu}(1-\bar{\theta}-\bar{\bar{\theta}})^{\gamma}$ in (1.19) with respect to (1.24) to show that the null-distribution of the F-statistic given by (1.18) after the randomization is approximated, for large b, by a familiar central

F-distribution with degrees of freedom $(\bar{\alpha}, n-b-v+1)$.

In the special case when $h=m$ (see [4], for $m=2$), it was shown, by the same method, that the permutation distribution of θ due to randomization is approximated by the Beta-distribution whose probability element being

$$(1.26) \qquad \frac{\Gamma(\frac{n-b}{2})}{\Gamma(\frac{v-1}{2})\Gamma(\frac{n-b-v+1}{2})} \theta^{\frac{v-1}{2}-1}(1-\theta)^{\frac{n-b-v+1}{2}-1} d\theta, \quad (0 \leq \theta \leq 1)$$

for large b, under the conditions (1.25). Then, the factor $\theta^{\mu}(1-\theta)^{\nu}$ in (1.22) was integrated out with respect to (1.26) to get the central F-distribution of degrees of freedom $(v-1, n-b-v+1)$ as an approximation to the null-distribution of the F-statistic given by (1.21) after the randomization.

From the probability theoretical point of view, the above argument presents us two questions: (a) In what sense is the permutation distribution of $(\bar{\theta}, \bar{\bar{\theta}})$ or θ approximated by (1.25) or (1.26), and (b) in what sense is the null-distribution of the F-statistic given by (1.18) or (1.21) after the randomization approximated by the central F-distribution of degrees of freedom $(\bar{\alpha}, n-b-v+1)$ or $(v-1, n-b-v+1)$?
Furthermore, the condisions given by (1.25) under which our argument carried out seem too severe from the viewpoint of practical applications.

This article treats these problems and gives satisfactory answers to them, where the uniformity conditions (1.25) are weakened.

2. Asymptotic Permutation Distributions of $(\bar{\theta}, \bar{\bar{\theta}})$ and θ.

In this section we shall derive asymptotic distributions of $(\bar{\theta}, \bar{\bar{\theta}})$ and θ which are asymptotically equivalent to the permutation distributions of $(\bar{\theta}, \bar{\bar{\theta}})$ and θ in the sense of type (S) as $b \to \infty$, under certain conditions.

We shall take a special numbering of the whole set of units such that the i-th unit of the p-th block is numbered as $f = (p-1)k+i$, $i=1,2,\ldots,k$; $p=1,2,\ldots,b$. Then it is clear that $B=\psi\psi'$ becomes

$$(2.1) \qquad B = \left\|\begin{matrix} G_k & & & 0 \\ & G_k & & \\ & & \ddots & \\ 0 & & & G_k \end{matrix}\right\| ,$$

where G_k designate the $k \times k$ matrix whose elements are all unity. Let us take an arbitrary but fixed incidence matrix of the treatments Φ, and let

$$(2.2) \qquad U_\sigma = \left\|\begin{matrix} S_{\sigma_1} & & & 0 \\ & S_{\sigma_2} & & \\ & & \ddots & \\ 0 & & & S_{\sigma_b} \end{matrix}\right\|$$

be the permutation matrix corresponding to the set of permutations

$$\sigma_p = \begin{pmatrix} 1 & 2 & \cdots\cdots & k \\ \sigma_p(1) & \sigma_p(2) & & \sigma_p(k) \end{pmatrix}, \quad p=1,2,\ \ldots,\ b.$$

Then, for any incidence matrix of the treatments, Φ^*, there can be found a permutation matrix U_σ such that $U_\sigma \Phi^* = \Phi$ or $\Phi^* = U_\sigma^* \Phi$.

Now, since the $m+3$ matrices

$$(2.3) \qquad V_u\#,\quad u=1,2,\ldots,m,\quad (I_n - \frac{1}{k} B - \sum_{u=1}^{m} V_u\#),\quad \frac{1}{k} B - \frac{1}{n} G_n,\quad \frac{1}{n} G_n$$

with respective ranks

$$\alpha_u,\ u=1,2,\ldots,\ m,\ n-b-v+1,\ b-1,\qquad 1$$

are mutually orthogonal idempotent matrices [8], there exists an orthogonal matrix of order n

$$
P = \begin{Vmatrix} C \\ \cdots \\ D \end{Vmatrix} = \begin{Vmatrix}
c^{(1)}_{11} \cdots c^{(1)}_{1k} \cdots c^{(2)}_{11} \cdots c^{(2)}_{1k} \cdots\cdots c^{(p)}_{11} \cdots c^{(p)}_{1k} \cdots\cdots c^{(b)}_{11} \cdots c^{(b)}_{1k} \\
\vdots \\
c^{(1)}_{v-1\,1} \cdots c^{(1)}_{v-1\,k} \cdots c^{(2)}_{v-1\,1} \cdots c^{(2)}_{v-1\,k} \cdots\cdots c^{(p)}_{v-1\,1} \cdots c^{(p)}_{v-1\,1} \cdots\cdots c^{(b)}_{v-1\,1} \cdots c^{(b)}_{v-1\,k} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
d_{11} \quad d_{12} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots d_{1n} \\
\\
d_{n-v+1\,1} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots d_{n-v+1\,n}
\end{Vmatrix}
$$

satisfying the conditions

$$
P V^{\#}_u P' = \begin{Vmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \underbrace{1 \ \ddots \ 1}_{\alpha_u} & & \\ & & & & 0 & \\ 0 & & & & & \ddots \ 0 \end{Vmatrix}, \quad u=1,\ldots,\ m
$$

(2.5)
$$
P(I_n - \frac{1}{k} B - \sum_{u=1}^{m} V^{\#}_u)\ P' = \begin{Vmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & \underbrace{1 \ \ddots \ 1}_{n-b-v+1} & \\ 0 & & & & 0 \ \ddots \ 0 \end{Vmatrix},
$$

$$
P(\frac{1}{k} B - \frac{1}{n} G_n)\ P' = \begin{Vmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 \ \ddots \ 0 & & \\ 0 & & & \underbrace{1 \ \ddots \ 1}_{b-1} & \\ & & & & 0 \end{Vmatrix} \quad \text{and} \quad P(\frac{1}{n} G_n)P' =
$$

$$
\begin{Vmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & \ddots \ 1 \end{Vmatrix}
$$

simultaneously. The submatrix C consisting of the first $v-1$ rows of P is divided into two submatrices

$$C = \begin{Vmatrix} C_{\bar{\alpha}} \\ \cdots \\ C_{\bar{\bar{\alpha}}} \end{Vmatrix},$$

$C_{\bar{\alpha}}$ and $C_{\bar{\bar{\alpha}}}$, consisting of the first $\bar{\alpha}$ rows and of the last $\bar{\bar{\alpha}}$ rows of C, for which we have

$$(2.6) \qquad C_{\bar{\alpha}} \left(\sum_{u=1}^{h} V_u^{\#}\right) C'_{\bar{\alpha}} = I_{\bar{\alpha}} \quad \text{and} \quad C_{\bar{\bar{\alpha}}} \left(\sum_{u=h+1}^{m} V_u^{\#}\right) C'_{\bar{\bar{\alpha}}} = I_{\bar{\bar{\alpha}}} .$$

It is also noted that any row of the matrix C satisfies the following conditions

$$(2.7) \qquad \sum_{i=1}^{k} c_{si}^{(p)} = 0, \quad p=1,2,\ldots,b$$

Now, since π is orthogonal to the matrix B and hence to G_r, and the sum of m+3 matrices in (2.3) is equal to I_n, we have

$$(2.8) \qquad \Delta = \pi' U_\sigma \left(\sum_{u=1}^{h} V_u^{\#}\right) U_\sigma \pi + \pi' U_\sigma' \left(\sum_{u=h+1}^{m} V_u^{\#}\right) U_\sigma \pi + \pi U_\sigma' \left(I_n - \frac{1}{k} B - \sum_{u=1}^{m} V_u\right) U_\sigma \pi$$

$$= \pi' U_\sigma' P'P\left(\sum_{u=1}^{h} V_u^{\#}\right) P'P\, U_\sigma \pi + \pi U_\sigma' P'P\left(\sum_{u=h+1}^{m} V_u^{\#}\right) P'P\, U_\sigma \pi$$

$$+ \pi' U_\sigma' P' P \left(I_n - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#}\right) P'P U_\sigma \pi$$

$$= \pi' U_\sigma' P' \cdot \begin{Vmatrix} \underbrace{1 \ddots 1}_{\bar{\alpha}} & & 0 \\ & 0 \ddots 0 & \\ 0 & & \ddots 0 \end{Vmatrix} P U_\sigma \pi + \pi' U_\sigma' P' \begin{Vmatrix} 0 \ddots 0 & & 0 \\ & \underbrace{1 \ddots 1}_{\bar{\bar{\alpha}}} & \\ 0 & & 0 \ddots 0 \end{Vmatrix} P U_\sigma \pi$$

$$+ \pi' U_\sigma' P' \begin{Vmatrix} 0 \ddots 0 & & 0 \\ & 0 \ddots 0 & \\ 0 & & 1 \ddots 1 \end{Vmatrix} P U_\sigma \pi$$

and therefore

$$\bar{\theta} = (C_{\bar{\alpha}} U_\sigma \pi)' (C_{\bar{\alpha}} U_\sigma \pi) / \Delta,$$

$$\bar{\bar{\theta}} = (C_{\bar{\bar{\alpha}}} U_\sigma \pi)' (C_{\bar{\bar{\alpha}}} U_\sigma \pi) / \Delta, \tag{2.9}$$

$$\theta = \theta_1 + \theta_2 = (CU_\sigma \pi)'(C U_\sigma \pi)/\Delta.$$

We shall demonstrate the asymptotic normality of the vector variates $C_{\bar{\alpha}} U_\sigma \pi$, $C_{\bar{\bar{\alpha}}} U_\sigma \pi$ and $C U_\sigma \pi$ under certain conditions. For this, we prepare the following

<u>LEMMA 1</u> All the elements of the matrix C are of order $O(\frac{1}{\sqrt{b}})$ as $b \to \infty$, keeping v, k, $n_1, \ldots, n_m$ and p^i_{jk} fixed.

<u>Proof</u> Let $\mathbf{c}'$ be any row vector of the matrix C. Then, form (2.6) it follows that $\mathbf{c}$ must be a linear combination of v-1linearly independent column vectors of the matrix $(\Sigma_{u=1}^m V_u^{\#})$, ie.,

$$\mathbf{c} = \sum_{i=1}^{v-1} \lambda_i \mathbf{v}_i \tag{2.10}$$

where $(\mathbf{v}_1, \ldots, \mathbf{v}_{v-1})$ is a set of linearly independent column vectors of $\sum_{u=1}^{m} V_u^{\#}$.

Putting $\mathbf{v}_i' \mathbf{v}_j = w_{ij}$ $i,j=1,\ldots,v-1$, $\boldsymbol{\lambda}'=(\lambda_1, \ldots, \lambda_{v-1})$, and

$$W = r \, \| w_{ij} \|_{(i,j=1,\ldots,v-1)} \tag{2.11}$$

we have

$$\mathbf{c}'\mathbf{c} = \frac{1}{r} \boldsymbol{\lambda}' W \boldsymbol{\lambda} = 1. \tag{2.12}$$

Since the w_{ij}'s are elements of the matrix $\sum_{u=1}^{m} V_u^{\#}$, it follows from (1.5) that each element of the matrix W given by (2.11) is of order $O(1)$ under the limiting condition stated in the lemma. Furthermore since W is non-simgular and symmetric, its characteristic roots $\delta_1,\dots,\delta_{v1}$ are all positive and of order $O(1)$ as $b\to\infty$ and there exists an orthogonal matrix K of order $v1$, whose elements being of order $O(1)$ as $b\to\infty$, such that

$$KWK' = \left\| \begin{matrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_{v1} \end{matrix} \right\|$$

Hence, putting $\mu = K\lambda$, $\mu' = (\mu_1,\dots\mu_{v1})$, we obtain by (2.12)

$$\text{(2.13)} \qquad \mu'\mu = \lambda'\lambda \quad \text{and} \quad \sum_{i=1}^{v1} \delta_i \mu_i^2 = r,$$

from which it is easy to see that all components of λ must be of order $O(\sqrt{b})$ as $b\to\infty$, since r and b are of the same order of magnitude as $b\to\infty$.

Hence it follows from (2.10) that all the elements of the matrix C are of order $O(1/\sqrt{b})$ as $b\to\infty$, which proves the lemma.

Using this result, we can show the following

<u>LEMMA 2</u> Suppose that the following conditions are satisfied:

$$\text{(2.14)} \qquad \bar{\Delta} \equiv \frac{1}{b}\sum_{p=1}^{b} \Delta_p \to \Delta_o \quad \text{and} \quad \frac{1}{b}\sum_{p=1}^{b} |\Delta_p - \bar{\Delta}|^{1+\delta} \to 0 (b\to\infty)$$

for some $\Delta_o > 0$ and $\delta \geqq 1/2$. Then , the permutation distributions of the vector variates $C_{\underline{\alpha}} U_\sigma \pi$, $C_{\underline{\underline{\alpha}}} U_\sigma \pi$ and $C\, U_\sigma \pi$ converge to the multi-normal distributions $N(0\, \frac{\Delta_o}{k-1} I_{\underline{\alpha}})$, $N(0, \frac{\Delta_o}{k-1} I_{\underline{\underline{\alpha}}})$ and $N(0, \frac{\Delta_o}{k-1} I_{v1})$ respectively, in the sense of type (β) as $b\to\infty$.

PROOF We shall show that $CU_\sigma\pi$ converges to $N(0, \frac{\Delta_o}{k-1} I_{v1})$ in the sense of type $(\mathcal{S})$ as $b \to \infty$.

Let us put

$$(2.15)\qquad C\,U_\sigma\pi = \sum_{p=1}^{b} \xi^{(p)}, \quad \xi^{(p)} = \begin{bmatrix} \sum_{i=1}^{k} c_{1i}^{(p)} \pi_{\sigma(i)}^{(p)} \\ \sum_{i=1}^{k} c_{2i}^{(p)} \pi_{\sigma(i)}^{(p)} \\ \vdots \\ \sum_{i=1}^{k} c_{v1i}^{(p)} \pi_{\sigma(i)}^{(p)} \end{bmatrix},$$

Then, since

$$(2.16)\qquad E(\pi_{\sigma(i)}^{(p)}) = 0,\ E(\pi_{\sigma(i)}^{(p)2}) = \frac{1}{k}\Delta_p \ \text{ and } \ E(\pi_{\sigma(i)}^{(p)}\pi_{\sigma(j)}^{(p)}) = \frac{-1}{k(k-1)}\Delta_p (i \neq j)$$

and, $\xi^{(1)}, \ldots, \xi^{(b)}$ are mutually independent in the stochastic sense, we have

$$(2.17)\qquad E(CU_\sigma\pi) = 0$$

and by (2.7)

$$E((CU_\sigma\pi)(CU_\sigma\pi)') = \sum_{p=1}^{b} E(\xi^{(p)}\xi^{(p)\prime}) = \sum_{p=1}^{b} \left\| \frac{1}{k}\Delta_p \sum_{i=1}^{k} c_{ji}^{(p)} c_{j'i}^{(p)} - \frac{1}{k(k-1)} \sum_{i\neq j} c_{ji}^{(p)} c_{j'i}^{(p)} \right\|$$

$$(j, j' = 1, \ldots, v1)$$

$$(2.18)\qquad = \sum_{p=1}^{b} \frac{\Delta_p}{k-1} \left\| \sum_{i=1}^{k} c_{ji}^{(p)} c_{j'i}^{(p)} \right\| (j, j' = 1, \ldots, v1)$$

$$= \frac{\bar{\Delta}}{k-1} I_{v1} + \frac{1}{k-1} \sum_{p=1}^{b} (\Delta_p - \bar{\Delta}) \cdot \left\| \sum_{i=1}^{k} c_{ji}^{(p)} c_{j'i}^{(p)} \right\|$$

Since

$$\left|\sum_{p=1}^{b}(\Delta_p - \bar{\Delta})\sum_{i=1}^{k} c_{si}^{(p)} c_{sj}^{(p)}\right|^a \leq \left(\sum_{p=1}^{b}|\Delta_p - \bar{\Delta}|^a\right)\left(\sum_{p=1}^{b}\left|\sum_{i=1}^{k} c_{si}^{(p)} c_{si}^{(p)}\right|^{a'}\right)^{\frac{a}{a'}}$$

$$\left(\frac{1}{a} + \frac{1}{a'} = 1\right)$$

and, by lemma 1,

$$\left(\sum_{p=1}^{b}\left|\sum_{i=1}^{k} c_{si}^{(p)} c_{si}^{(p)}\right|^{a'}\right)^{\frac{a}{a'}} \leqq \frac{M}{b}, \quad M = O(1) \quad (b \to \infty),$$

we obtain, putting $a=1+\delta$,

$$\left|\sum_{p=1}^{b}(\Delta_p - \bar{\Delta})\sum_{i=1}^{k} c_{si}^{(p)} c_{si}^{(p)}\right|^{1+\delta} \leqq \frac{1}{b}\sum_{p=1}^{b}\left|\Delta_p - \bar{\Delta}\right|^{1+\delta} \tag{2.19}$$

Hence, we have

$$E((CU_\sigma \pi)(CU_\sigma \pi)') = \sum_{p=1}^{b} E(\xi^{(p)}\xi^{(p)'}) \to \frac{\Delta_0}{k-1} I_{v-1}, \quad (b \to \infty). \tag{2.20}$$

Now, let $\varphi(\mathbf{t})$ be the characteristic function of $CU_\sigma \pi$. Then, by (2.17),

$$\log \varphi(\mathbf{t}) = -\frac{1}{2}\mathbf{t}' \sum_{p=1}^{b} E(\xi^{(p)}\xi^{(p)'})\, \mathbf{t} + \Theta \sum_{p=1}^{b} E(|\mathbf{t}' \xi^{(p)}|^3), \tag{2.21}$$

where $\Theta = O(1)$ as $b \to \infty$. Since

$$\begin{aligned} |\mathbf{t}' \xi^{(p)}| &\leq (\mathbf{t}'\mathbf{t})^{\frac{1}{2}} \left\{\sum_{s=1}^{v-1}\left(\sum_{i=1}^{k} c_{si}^{(p)} \pi_{\sigma(i)}^{(p)}\right)^2\right\}^{\frac{1}{2}} \\ &\leqq (\mathbf{t}'\mathbf{t})^{\frac{1}{2}} \left\{\Delta_p \sum_{s=1}^{v-1}\sum_{i=1}^{k} c_{si}^{(p)2}\right\}^{\frac{1}{2}}, \end{aligned}$$

we get

$$\sum_{p=1}^{b} E\,|\mathbf{t}' \xi^{(p)}|^3 \leqq (\mathbf{t}'\,\mathbf{t})^{\frac{3}{2}} \sum_{p=1}^{b} \Delta_p^{\frac{3}{2}}\left(\sum_{s=1}^{v-1}\sum_{i=1}^{k} c_{si}^{(p)2}\right)^{\frac{3}{2}} \tag{2.22}$$

From lemma 1 it follows that

$$\left(\sum_{s=1}^{v-1}\sum_{i=1}^{k} c_{si}^{(p)2}\right)^{\frac{3}{2}} = O\left(\frac{1}{b^{3/2}}\right) \quad (b \to \infty),$$

and hence

$$(2.23) \qquad \sum_{p=1}^{b} E(|\mathbf{t}' \xi^{(p)}|^3) \leq M\, (\mathbf{t}'\, \mathbf{t})^{\frac{3}{2}} \frac{1}{b^{2/3}} \sum_{p=1}^{b} \Delta_p^{\frac{3}{2}}$$

where $M = O(1)$ as $b \to \infty$. Since $\frac{1}{b} \sum_{p=1}^{b} \Delta_p^{\frac{3}{2}} \to \Delta_o^{\frac{3}{2}}$ $(b \to \infty)$, we have

$$(2.24) \qquad \sum_{p=1}^{b} E\,(|\mathbf{t}' \xi^{(p)}|^3) \to 0 \quad (b \to \infty).$$

Therefore, it follows from (2.20), (2.21) and (2.24) that

$$(2.25) \qquad \varphi(t) \to -\frac{1}{2} \frac{\Delta_o}{k-1} \mathbf{t}'\, \mathbf{t} \quad (b \to \infty)$$

which proves the lemma.

From this lemma, it is straightforward to prove the following lemma.

<u>LEMMA 3</u> Under the conditions (2.14),

(a) the permutation distribution of $b(k-1)\bar{\theta}$, $b(k-1)\bar{\bar{\theta}}$ and $b(k-1)\theta$ converge to $\chi^2_{\bar{\alpha}}$ $\chi^2_{\bar{\bar{\alpha}}}$ and χ^2_{v-1} in the sense of type $(\mathcal{S})$ as $b \to \infty$, where χ^2_f designates a random variable distributed according to the chi-square distribution with degrees of freedom f, and

(b) $\{b(k-1)\bar{\theta},\ b(k-1)\ \bar{\bar{\theta}}\}$ and hence $\{\bar{\theta},\ \bar{\bar{\theta}}\}$ are both asymptotically independent set of random variables in the sense of type $(\mathcal{S})$ as $b \to \infty$.

It should be noted that the limiting chi-square distribution in(a), $\chi^2_{\bar{\alpha}}$ and $\chi^2_{\bar{\bar{\alpha}}}$, can be assumed to be independent in the ctochastic sense.

Applying Theorem 4.1 of [7], we have

<u>THEOREM 1</u> Under the condition (2.14) of Lemma 2, (a) the two random vectors

$$(2.26) \qquad (\bar{\theta}, \bar{\bar{\theta}}) \text{ and } \left(\frac{1}{b(k-1)} \chi^2_{\bar{\alpha}},\ \frac{1}{b(k-1)} \chi^2_{\bar{\bar{\alpha}}}\right)$$

are asymptotically equivalent in the sense of type $(\mathcal{S})$ as $b \to \infty$, where we can assume that $\frac{1}{b(k-1)}\chi^2_{\alpha}$ and $\frac{1}{b(k-1)}\chi^2_{\bar{\alpha}}$ are mutually independently distributed for each fixed b, and (b) two random variables

(2.27) $$\theta \text{ and } \frac{1}{b(k-1)}\chi^2_{v-1}$$

is asymptotically equivalent in the sense of type (S) as $b \to \infty$.

3. Alternative Asymptotic Permutation Distributions of $(\bar{\theta}, \bar{\bar{\theta}})$ and θ.

In this section, we derive other types of distribution which are asymptotically equivalent to $(\bar{\theta}, \bar{\bar{\theta}})$ and θ in the sense of type (S) as $b \to \infty$.

In the first place, we consider the latter variable θ.

LEMMA 4 Two random variables

(3.1) $$\frac{1}{b(k-1)}\chi^2_{v-1} \quad \text{and} \quad Ⓗ_{v-1}$$

are asymptotically equivalent in the sense of type (B) ([7]) as $b \to \infty$, where $Ⓗ_{v-1}$ stands for a random variable whose probability element is given by (1.26).

PROOF Since type (B) and type (I) asymptotic equivalence are mutually equivalent in this case, Theorem 1.4.2 of [6] is applicable.

The probability density functions of the variables in (3.1) are given by

(3.2) $$F(y) = \frac{[b(k-1)]^{\frac{v-1}{2}}}{2^{\frac{v-1}{2}}\Gamma(\frac{v-1}{2})} y^{\frac{v-1}{2}-1} e^{-\frac{b(k-1)}{2}y}, \quad (0 < y < \infty)$$

and

(3.3) $$G(y) = \frac{\Gamma\left(\frac{b(k-1)}{2}\right)}{\Gamma(\frac{v-1}{2})\,\Gamma(\frac{n-b-v+1}{2})} y^{\frac{v-1}{2}-1}(1-y)^{\frac{n-b-v+1}{2}},$$

$$(0 < y < 1)$$

respectively. Hence, the Kullback-Leibler mean information in the Theorem 1.4.2 of [6] is given in the present case by

(3.4) $$I(g : f) = \int_0^1 g(y) \log \frac{g(y)}{f(y)} dy$$

This is calculated as follows:

(3.5) $$\log \frac{g(y)}{f(y)} = \log \Gamma\left(\frac{b(k-1)}{2}\right) - \log \Gamma\left(\frac{n-b-v+1}{2}\right) + \frac{v-1}{2}$$

$$\times \log 2 - \frac{v-1}{2} \log (b(k-1)) + \left(\frac{n-b-v+1}{2} - 1\right) \log (1-y) + \frac{b(k-1)}{2} y$$

By using the Stirling formula, we have

(3.6) $$\left| \log \Gamma\left(\frac{b(k-1)}{2}\right) - \left(\frac{b(k-1)}{2} - \frac{1}{2}\right) \log \frac{b(k-1)}{2} - \frac{b(k-1)}{2} + \log \sqrt{2\pi} \right| \to 0$$

$$\left| \log \Gamma\left(\frac{n-b-v+1}{2}\right) - \left(\frac{n-b-v+1}{2} - \frac{1}{2}\right) \log \frac{n-b-v+1}{2} - \frac{n-b-v+1}{2} + \log \sqrt{2\pi} \right| \to 0$$

as $b \to \infty$. Furthermore,

(3.7) $$\int_0^1 g(y) \frac{b(k-1)}{2} y dy = \frac{v-1}{2}$$

and again using the Stirling formula, we have

$$\left(\frac{n-b-v+1}{2} - 1\right) \int_0^1 g(y) \log (1-y) dy = -\left(\frac{n-b-v+1}{2} - 1\right)$$

$$\times \left[\frac{d}{dy} (\log \Gamma (p+q) - \log \Gamma(p)) \right]_{\substack{\left(p=\frac{n-b-v+1}{2}\right) \\ \left(q=\frac{v-1}{2}\right)}}$$

$$(3.8) \qquad = \lim_{b\to\infty} \left[-\frac{n-b-v+1}{2} \left\{ \log\left(1+\frac{v-1}{n-b-v+1}\right) + \frac{v-1}{b(k-1)(n-b-v+1)} \right\} \right]$$

$$= -\frac{v-1}{2}$$

It follows from (3.4) (3.5) (3.6) (3.7) and (3.8) that

$$(3.9) \qquad I\,(g:\ f) \to 0, \quad (b\to\infty),$$

which proves the lemma.

Next, we shall show the following

<u>LEMMA 5</u> Two variates

$$(3.10) \qquad \left(\frac{1}{b(k-1)}\chi^2_{\bar\alpha},\ \frac{1}{b(k-1)}\chi^{*2}_{\bar{\bar\alpha}}\right) \text{ and } (\Theta_{\bar\alpha},\ \Theta_{\bar{\bar\alpha}})$$

are asymptotically equivalent in the sense of type (B) as $b\to\infty$, where $(\Theta_{\bar\alpha}, \Theta_{\bar{\bar\alpha}})$ is a random vector whose probability element is identical to that of(1.24)

<u>PROOF</u> Analogously to the the proof of the preceding lemma, it suffices to prove that the Kullback-Leibler mean information of both variates of (3.10) tends to zero as $b\to\infty$.

The probability density function of the first variate of (3.10) is given by

$$(3.11) \qquad f(x,\ y) = \frac{[b(k-1)]^{\frac{v-1}{2}}}{2^{\frac{v-1}{2}}\Gamma(\frac{\bar\alpha}{2})\,\Gamma(\frac{\bar{\bar\alpha}}{2})}\ x^{\frac{\bar\alpha}{2}-1}\ y^{\frac{\bar{\bar\alpha}}{2}-1}\ e^{-\frac{b(k-1)}{2}(x+y)},$$

$$(0 < x,\ y < \infty)$$

and that of the second variate of (3.10) is

$$(3.12) \qquad g(x,\ y) = \frac{\Gamma\left(\frac{b(k-1)}{2}\right)}{\Gamma(\frac{\bar\alpha}{2})\,\Gamma(\frac{\bar{\bar\alpha}}{2})\,\Gamma(\frac{n-b-v+1}{2})}\ x^{\frac{\bar\alpha}{2}-1}\ y^{\frac{\bar{\bar\alpha}}{2}-1}\ (1-x-y)^{\frac{n-b-v+1}{2}-1},$$

$$(0 < x, y, x+y<1)$$

Hence the Kullback-Leibler mean information is given by

$$(3.13) \quad I(g:\ f) = \int_0^1 \int_0^1 g(x, y) \log \frac{g(x, y)}{f(x, y)}\, dxdy$$

Thus, from (3.11) and (3.12) we have

$$(3.14) \quad I(g:\ f) = E_{(x,\ y)} \left[\log \left\{\frac{\Gamma(\frac{b(k-1)}{2})\ 2^{\frac{v-1}{2}}}{\Gamma(\frac{n-b-v+1}{2})(b(k-1))^{\frac{v-1}{2}}}\right\} + \left(\frac{n-b-v+1}{2} - 1\right) \log (1-(x+y)) - \frac{b(k-1)}{2}(x+y)\right],$$

where $E(x,y)$ stands for the expectation with respect to (3.12). Under (3.12) $x + y$ is distributed according to the Beta-distribution (3.3). Therefore $I(g:\ f)$ given by (3.14) has the same value as that in the preceding lemma, and hence

$$(3.15) \qquad I(g:\ f) \to 0,\ (b \to \infty)$$

which proves the lemma.

From these results and Theorem 1 in the preceding section, we have the following

<u>THEOREM 2</u> Under the conditions (2.14) of Lemma 2, (a) the permutation distribution of $(\bar{\theta}, \bar{\bar{\theta}})$ are asymptotically equivalent to $(Ⓗ_{\bar{\alpha}}, Ⓗ_{\bar{\bar{\alpha}}})$, whose probability element being given by (1.24), in the sense of type (S) as $b \to \infty$, and (b) the permutation distribution of θ are asymptotically equivalent to $Ⓗ_{v1}$, whose probability element being (1.26), in the sense of type (S) as $b \to \infty$.

4. Asymptotic Null-distribution of the F-statistic after the Randomization

In the present section, we show that the null-distribution of the F-statistic given by (1.18) or (1.21) after the randomization can be asymptotically approximated in the sense of type ($\mathcal{S}$), by a familiar central F-distribution with degrees of freedom ($\bar{\alpha}$, n - b - v + 1) or (v-1, n-b-v+1), which is obtained by integrating out the conditional variates in (1.19) or (1.22) by the approximate distribution (1.24) or (1.26) of the permutation distribution of $(\bar{\theta}, \bar{\bar{\theta}})$ or θ .

First we shall prove the following

THEOREM 3 Let

$$(4.1) \qquad p_s(z|x)\, d\, z\, d\, F_s(x)$$

be a probability element of the two dimensional real random variable (Z^s, X^s), where $F_s(x)$ stands for the cumulative distribution function of X^s, s=1,2,...

Let Y^s be another probability distribution which is asymptotically equivalent to X^s in the sense of type ($\mathcal{S}$) as $s \to \infty$, and let the cumulative distribution function of Y^s be $G_s(y)$, s=1,2,...

Let us assume that the following conditions are satisfied:

(i) $p_s(z|x)$ is continuous with respect to (z, x) jointly, for every s.

(ii) For Y^s, there exist constants $c_s > 0$ and d_s, s=1,2,..., such that the variable

$$(4.2) \qquad \frac{1}{c_s}(Y^s - d_s), \quad s=1,2,\ldots$$

tends to a certain fixed distribution of the continuous type, Y, in the sense of type ($\mathcal{S}$) as $s \to \infty$. Then, two distributions whose probability density function being given by

$$(4.3) \qquad \int_{-\infty}^{\infty} p_s(z|x)\, dF_s(x) \text{ and } \int_{-\infty}^{\infty} p_s(z|x)\, d\, G_s(x)$$

are asymptotically equivalent in the sense of type ($\mathfrak{S}$) as $s \to \infty$.

PROOF To prove this, it is sufficient to show that

(4.4) $$\sup_{-\infty < a < \infty} \left| \int_{-\infty}^{a} dz \int_{-\infty}^{\infty} p_s(z|x)\, dF_s(x) - \int_{-\infty}^{a} dz \int_{-\infty}^{\infty} p_s(z|x)\, dG_s(x) \right| \to 0, \quad (s \to \infty).$$

Let us put

(4.5) $$\varphi_s(a|x) = \int_{-\infty}^{a} p_s(z|x)\, dz, \quad s=1,2,\dots,$$

and

(4.6) $$H_s(a) = \int_{-\infty}^{\infty} \varphi_s(a|x)\, dF_s(x) \text{ and } K_s(a) = \int_{-\infty}^{\infty} \varphi_s(a|x)\, dG_s(x),$$

$$s = 1,2,\dots$$

Then, (4.4) can be written as

(4.7) $$\sup_{-\infty < a < \infty} \left| H_s(a) - K_s(a) \right| \to 0, \quad (s \to \infty)$$

Now since

$$H_s(-\infty) = 0,\ H_s(+\infty) = 1 \text{ and } K_s(-\infty) = 0,\ K_s(+\infty) = 1,$$

for every s, there exists a point $a = a_s$ such that

$$\left| H_s(a_s) - K_s(a_s) \right| = \sup_{-\infty < a < \infty} \left| H_s(a) - K_s(a) \right|, \quad -\infty < a_s < \infty$$

for every s, and therefore (4.7) turns out to be

(4.8) $$\left| H_s(a_s) - K_s(a_s) \right| \to 0, \quad (s \to \infty).$$

Since the cumulative distribution function of the variable given by (4.2) is $G_s(c_s x + d_s)$, the condition (ii) of the theorem assures us that

(4.9) $$\sup_{-\infty < x < \infty} \left| G_s(c_s x + d_s) - G(x) \right| \to 0, \quad (s \to \infty)$$

where G(x) stands for the cumulative distribution function of Y, and therefore we have

(4.10) $$\sup_{-\infty < x < \infty} \left| F_s(c_s x + d_s) - G(x) \right| \to 0, \quad (s \to \infty)$$

Taking a transformation of the variable, we have

(4.11) $$\left| H_s(a_s) - K_s(a_s) \right| = \left| \int_{-\infty}^{\infty} \varphi_s (a_s | c_s x + d_s) d F_s(c_s x + d_s) \right.$$
$$\left. - \int_{-\infty}^{\infty} \varphi_s(a_s | c_s x + d_s) dG_s(c_s x + d_s) \right| .$$

The function $\varphi_s (a|x)$ given by (4.5) has the properties that $0 \leq \varphi_s(a|x) \leq 1$ for all (a,x), and $\varphi_s(a|x)$ is continuous in (a,x). Hence, we can always find a convergent subsequence $\{\varphi_s' (a_s' | c_s' x + d_s')\}$ $s' \to \infty$.) of $\{\varphi_s(a_s | c_s x + d_s)\}$ $(s=1,2,\ldots)$ and a function $\varphi_o(x)$ such that

(6.12) $$\varphi_s'(a_s' | c_s' x + d_s') \to \varphi_o(x) \quad (s' \to \infty)$$

Clearly, $\varphi_o(x)$ is continuous and $0 \leq \varphi_o(x) \leq 1$ for all x.

Now, for any given $e > o$, there exists an interval $I_M = (-M, M)$ such that

(4.13) $$\left| G(M) - G(-M) \right| < e,$$

and hence, by (4.9) and (4.10), we can find a integer s_e such that, if $s' \geq s_e$ then

(4.14) $$\begin{cases} \left| F_{s'}(c_{s'} M + d_{s'}) - F_{s'}(- c_{s'} M + d_{s'}) \right| \leq e, \\ \left| G_{s'}(c_{s'} M + d_{s'}) - G_{s'}(- c_{s'} M + d_{s'}) \right| \leq e, \end{cases}$$

Note that $\varphi_o(x)$ is uniformly continuous on the interval I_M, and the convergence (4.12) can be assumed to be uniform, i.e., there exists a positive integer s_e^*, such that $s' \geq s_e^*$ implies that

$$(4.15) \qquad \left| \varphi_{s'}(a_{s'} | c_{s'}x + d_{s'}) - \varphi_o(x) \right| < e$$

for all x belonging to I_M.

Hence, if $s' \geq \max(s_e, s_e^*)$, then

$$(4.16) \qquad \left| \int_{-M}^{M} \varphi_{s'}(a_{s'} | c_{s'}x + d_{s'})\, dF_{s'}(c_{s'}x + d_{s'}) - \int_{-M}^{M} \varphi_o(x) dF_{s'}(c_{s'}x + d_{s'}) \right| < e$$

$$\left| \int_{-M}^{M} \varphi_{s'}(a_{s'} | c_{s'}x + d_{s'})\, dG_{s'}(c_{s'}x + d_{s'}) - \int_{-M}^{M} \varphi_o(x) dG_{s'}(c_{s'}x + d_{s'}) \right| < e$$

and, by (4.14)

$$(4.17) \qquad \begin{aligned} &\left| H_{s'}(a_{s'}) - \int_{-M}^{M} \varphi_{s'}(a_{s'} | c_{s'}x + d_{s'})\, dF_{s'}(c_{s'}x + d_{s'}) \right| \leq e \\ &\left| K_{s'}(a_{s'}) - \int_{-M}^{M} \varphi_{s'}(a_{s'} | c_{s'}x + d_{s'})\, dG_{s'}(c_{s'}x + d_{s'}) \right| \leq e. \end{aligned}$$

Thus,

$$(4.18) \qquad \left| \left| H_{s'}(a_s) - K_{s'}(a_s) \right| - \left| \int_{-M}^{M} \varphi_o(x)\, dF_{s'}(c_{s'}x + d_{s'}) - \int_{-M}^{M} \varphi_o(x) dG_{s'}(c_{s'}x + d_{s'}) \right| \right| \leq 4e.$$

Since, by (4.9) and (4.10),

$$(4.19) \qquad \left| \int_{-M}^{M} \varphi_o(x)\, dF_{s'}(c_{s'}x + d_{s'}) - \int_{-M}^{M} \varphi_o(x) dG(x) \right| \to 0, \quad (s \to \infty)$$

and

$$(4.20) \qquad \left| \int_{-M}^{M} \varphi_o(x)\, dG_{s'}(c_{s'}x + d_{s'}) - \int_{-M}^{M} \varphi_o(x) dG(x) \right| \to 0, \quad (s' \to \infty),$$

there exisits, from (4.18), a positive integer s_o such that $s' \geq s_o$ implies that

(4.21) $$\left| H_{s'}(a_{s'}) - K_{s'}(a_{s'}) \right| < 6e$$

This shows that (4.8) is true for the subsequence $\{s'\}$, from which it is easy to see that the theorem holds.

This result can easily be extended to bivariate case, which will be stated without proof in the following.

<u>THEOREM 4</u> Let

(4.22) $$p_s(z|x, y)\, dz\, d\, F_s(x, y)$$

be the probability element of a three-dimensional real random variable (z^s, X_1^s, X_2^s), where $F_s(x,y)$ designates the cumulative distribution function of (X_1^s, X_2^s), $s=1,2,\ldots$

Let (Y_1^s, Y_2^s) be another probability distribution which is asymptotically equivalent to (X_1^s, X_2^s) in the sense of type $(\mathcal{S})$ as $s \to \infty$, whose cumulative distribution is denoted by $G_s(x, y)$, $s=1,2,\ldots$

Suppose that the following conditions are satisfied:

(i) $p_s(z|x, y)$ is continuous with respect to (z, x, y) jointly, for every s.

(ii) For (Y_1^s, Y_2^s) there exist contants c_1^s, $c_2^s (> 0)$ and d_1^s, d_2^s, $s=1,2,\ldots$, such that the variable

(4.23) $$\left(\frac{1}{c_1^s}(Y_1^s - d_1^s),\ \frac{1}{c_2^s}(Y_2^s - d_2^s) \right)$$

tends to a certain fixed distribution (Y_1, Y_2) of the continuous type in the sense of type $(\mathcal{S})$ as $s \to \infty$.

Then, two distributions whose probability density functions being given by

(4.24) $$\int_{-\infty}^{\infty} p_s(z|x, y)\, d\, F_s(x, y) \quad \text{and} \quad \int_{-\infty}^{\infty} p_s(z|xy)\, dG_s(x,y)$$

are asymptotically equivalent in the sense of type ($\mathcal{S}$) as $s \to \infty$.

Applying these results to the problem stated in the section 1, we have the following

THEOREM 5 Let us consider the limiting process stated in Lemma 1 and the conditions (2.14) given in Lemma 2. Then, under the 'partial' null-hypothesis $H_{o(h)}$ given by (1.6), the distribution of the F-statistic given by (1.18) after the randomization is asymptotically equivalent to the F-distribution of degrees of freedom ($\bar{\alpha}$, n-b-v+1) in the sense of type ($\mathcal{S}$) as $b \to \infty$.

Likewise, in the special case when h=m, under the 'total' null-hypothesis H_o (τ =0), the distribution of the F-statistic given by (1.21) after the randomization is asymptotically equivalent to the F-distribution of degrees of freedom (v-1, n-b-v+1) in the sense of type ($\mathcal{S}$) as $b \to \infty$.

This gives a justification and a probability theoretical meaning to all the results hertherto obtained, concerning the asymptotic null-distribution of the F-statistic after the randomization [1,2,3,4,5].

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REFERENCES

[1] J. Ogawa, "The effect of randomization on the analysis of randomized block design", Ann. Inst. Stat. Math., Vol. 13, pp. 105-117 (1961)

[2] J. Ogawa, "On the randomization in Latin-square design under the Neyman model", Proc. Inst. Stat. Math., Vol. 10, pp. 1-16 (1962)

[3] J. Ogawa, "On the null-distribution of the F-statistic in a randomized balanced incomplete block design under the Neyman model. Ann. Math. Stat., Vol. 34, pp. 1558-1568 (1963)

[4] J. Ogawa, S. Ikeda and M. Ogasawara, "On the null-distribution of the F-statistic in a randomized partially balanced incomplete block design with two associate classes under the Neyman model", Inst. of Stat., Univ. of N. C., Mimeo Series No. 414 (1964)

[5] J, Ogawa, S. Ikeda and M. Ogasawara, "On the null-distribution of the F-statistic for testing a 'partial' null-hypothesis in a randomized partially balanced incomplete block design with m associate classes under the Neyman model" (35th session of the International Stat. Inst. 1965)

[6] S. Ikeda, "Asymptotic equivalence of probability distributions with applications to some problems of asymptotic independence", Ann. Inst. Stat. Math., Vol. 15, pp. 87-116 (1963)

[7] S. Ikeda, "On certain types of asymptotic equivalence of real probability distributions, I. Definitions and some of their properties," (submitted to the Inst. of Stat., University of N. C. Mimeo Series)

[8] J. Ogawa and G. Ishii, "On the analysis of partially balanced incomplete block designs in the regular case", Inst. of Stat., Univ. of N. C., Mimeo Series No. 412 (1964)