# ON THE BLOCK STRUCTURE OF CERTAIN PBIB DESIGNS OF PARTIAL GEOMETRY TYPE

By

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1. Introduction. The association relation in an m-class partially balanced incomplete block (PBIB) design is a partition of the ordered pairs of distinct objects (or varieties, or treatments) of the design into m classes in accordance with certain well-known regularity conditions. If this partition is restricted to the pairs of the k objects of a single block, the restricted partition, which in this note will be called the association pattern of the block, is not specified by the definition of the design and in general appears to be quite irregular. The blocks of semi -regular group divisible designs were shown by Bose and Connor [2] to have association patterns of particularly simple form. A similar result was proved by Raghavarao [4] for certain designs of triangular type and Latin square type with two constraints ($L_2$). Theorem 1 of the present paper proves the same result for a more inclusive family of designs, designated here as SG designs. SG designs are related to partial geometries and are defined at the end of this section.

Majindar [3] proved several theorems on the intersection of blocks of BIB designs by means of enumeration methods which are not applicable in general to PBIB designs. Shah [5, 6] generalized these methods and theorems to the designs whose block association patterns had been analyzed in [ 2 ] and [4]. Theorems 2 to 5 of the present paper, on the intersection of blocks of SG designs, are a further extension.

Following Bose [1], we define a partial geometry with parameters ($\rho$, $\kappa$, t) as a finite collection of undefined objects called points and subsets called lines, subject to the following four axioms, where $\rho \geq 2$, $\kappa \geq 2$, $t \geq 1$ are integers.

A1. No pair of points is contained in more than one line.

A2. Each point is contained in $\rho$ lines.

A3. Each line contains $\kappa$ points.

A4. Given a point P and a line $\ell$ not containing P, exactly t points of $\ell$ are contained with P in lines.

The number of points is denoted by v. A partial geometry may be used to generate a two-class association relation for a set of v objects, which are identified with points. Two objects are defined to be first associates if some line of the geometry contains both of the corresponding points, and are defined to be second associates otherwise. It is shown in [1] that such an association relation satisfies the conditions for partial balance. A PBIB design will be said to be of partial geometry type if its association scheme can be generated in this way.

A group divisible association scheme with m groups of n objects can be regarded as a trivial case of a partial geometry, the groups comprising the lines. The above axioms are satisfied with $\rho = 1$, $\kappa = n$, $t = 0$. Conversely, we may deduce that a system which satisfies the axioms with $\rho = 1$ or $t = 0$ must correspond to an association scheme of group divisible type.

Association scheme parameters of partial geometry type are given in part by

(1.1) $$v = [(\rho - 1)(\kappa - 1) + t]/t,$$

(1.2) $$n_1 = \rho(\kappa - 1),$$

(1.3) $$n_2 = (\rho - 1)(\kappa - 1)(\kappa - t)/t.$$

Let a PBIB design of partial geometry type have parameters r, k, b, $\lambda_1$, $\lambda_2$, and let N be the v x b incidence matrix of the design. Then the symmetric matrix NN' has at most three distinct characteristic roots $\theta_0$, $\theta_1$, $\theta_2$ given by

$$\theta_0 = rk,$$

(1.4) $$\theta_1 = r + (\kappa - t - 1)\lambda_1 - (\kappa - t)\lambda_2,$$

$$\theta_2 = r - \rho\lambda_1 + (\rho - 1)\lambda_2.$$

Except in degenerate cases which do not arise in this investigation, these roots have multiplicities $\alpha_0 = 1$, $\alpha_1$, $\alpha_2$, where

(1.5) $\quad \alpha_1 = \rho\kappa(\rho - 1)(\kappa - 1)/t(\kappa + \rho - t - 1),$

(1.6) $\quad \alpha_2 = (\kappa - 1)(\kappa - t)\,[(\rho - 1)(\kappa - 1) + t]/t(\kappa + \rho - t - 1).$

It is easily verified from (1.1) that

(1.7) $\quad v(\kappa - t) = \kappa(v - \rho\kappa + \kappa + \rho - t - 1),$

and from (1.1) and (1.6) that

(1.8) $\quad v(\kappa - 1)(\kappa - t) = \kappa\alpha_2(\kappa + \rho - t - 1).$

The relations $r(k - 1) = n_1\lambda_1 + n_2\lambda_2$, $v = n_1 + n_2 + 1$, and (1.4) reduce to

$$r(k - 1) = (v - 1)\lambda_1 + (\lambda_2 - \lambda_1)n_2,$$

$$\theta_1 - r + \lambda_1 = (\lambda_1 - \lambda_2)(\kappa - t).$$

Eliminating $\lambda_2 - \lambda_1$ and employing (1.3) leads to

$$(\rho - 1)(\kappa - 1)\theta_1/t = v\lambda_1 - rk + [v(r - \lambda_1)/\kappa].$$

Thus,

(1.9) $\quad \lambda_1 = [r(k\kappa - v)/v(\kappa - 1)] + \kappa(\rho - 1)\,\theta_1/vt.$

Also, the condition $\theta_1 = 0$ is equivalent to

(1.10) $\quad rk - v\lambda_1 = v(r - \lambda_1)/\kappa$

and with (1.4) implies

(1.11) $\quad \lambda_2 - \lambda_1 = r(v - k)\kappa/v(\kappa - 1)(\kappa - t).$

If a design is of group divisible type with m groups of n objects, we set $\rho = 1$, $\kappa = n$, $t = 1$. Then expressions (1.2) and (1.4) for $n_1$, $\theta_0$, $\theta_1$, $\theta_2$ are valid and other parameters include

$$v = mn,$$

$$n_2 = (m - 1)n,$$

$$\alpha_1 = m - 1,$$

$$\alpha_2 = m(n - 1).$$

It is readily verified that (1.7) and (1.8) hold and that (1.9) can be replaced by

$$(1.12) \qquad \lambda_1 = [r(k - m)/m(n - 1)] + [(m - 1)\theta_1/m(n - 1)].$$

A group divisible design is semi-regular by definition if $rk - v\lambda_2 = 0$. It is easy to show that this is equivalent to $\theta_1 = 0$ and thus implies (1.11).

For convenient reference in this paper, we give this DEFINITION. An SG design is a two-class PBIB design which is of group divisible or partial geometry type and has the property $\theta_1 = 0$. Thus SG designs include semi-regular group divisible designs and certain designs of partial geometry type for which the matrix NN' is singular.

## 2. Block association patterns.

**THEOREM 1.** Let a two-class PBIB design be of group divisible type or of partial geometry type, $(\rho, \kappa, t)$. Then $\theta_1 = 0$ if and only if $k\kappa/v$ is an integer and each block of the design contains exactly $k\kappa/v$ objects from each line of the geometry

PROOF. Let $\ell$ be an arbitrary but fixed line, and let $y_i$ denote the number of objects of $\ell$ which occur in the i-th block of the design, $i = 1,\ldots,b$. Enumeration of the occurrences of objects of $\ell$ singly and in pairs within blocks of the design, we obtain

$$\sum_{i=1}^{b} y_i = r\kappa ,$$

$$\sum_{i=1}^{b} y_i(y_i - 1) = \kappa(\kappa - 1)\lambda_1.$$

Therefore,

$$(2.1) \qquad \sum_{i=1}^{b} (y_i - k\kappa/v)^2 = \kappa(\kappa - 1)\lambda_1 + r\kappa - (\kappa^2 rk/v).$$

Using (1.10), (1.9) and (1.12), the right hand side of (2.1) reduces to $\kappa^2(\kappa - 1)(p - 1)\theta_1/vt$ for a partial geometry and to $n(m - 1)\theta_1/m$ for a group divisible scheme. In either case, the sum in (2.1) is equal to zero if and only if $\theta_1 = 0$, proving the theorem.

Theorem 4 of [2] and Theorems 1.1 and 2.1 of [4] are special cases. The hypotheses which are assumed in [4] for the design parameters are the appropriate special cases of (1.9).

Designs with the following parameters cannot exist with partial geometries as association schemes because $\theta_1 = 0$ while $k\kappa/v$ is not an integer. This list of examples is confined to the range $r \leqq 15$, $k \leqq 15$ and to partial geometries not covered by [4].

| $v$ | $r$ | $k$ | $b$ | $\lambda_1$ | $\lambda_2$ | $n_1$ | $p^1_{11}$ | $\rho$ | $\kappa$ | $t$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 12 | 6 | 32 | 2 | 7 | 9 | 4 | 3 | 4 | 2 |
| 27 | 12 | 12 | 27 | 2 | 7 | 10 | 1 | 5 | 3 | 1 |
| 36 | 10 | 9 | 40 | 1 | 4 | 20 | 10 | 4 | 6 | 3 |
| 45 | 12 | 15 | 36 | 2 | 7 | 28 | 15 | 7 | 5 | 3 |
| 50 | 15 | 15 | 50 | 3 | 6 | 28 | 15 | 4 | 8 | 4 |

## 3. Intersection of blocks.

$B_1$ will denote an arbitrary but fixed block of a design, and $B_i$, $i = 2,\ldots,b$ will denote the other blocks. $x_i$ denotes the number of objects common to $B_1$ and $B_i$. We define

$$L = \sum_{i=2}^{b} x_i,$$

$$Q = \sum_{i=2}^{b} x_i^2 .$$

LEMMA 1. For some integers $x$ and $n$, let there exist blocks $B_{i_1},\ldots,B_{i_n}$

such that $x_{i_j} = x$, $j = 1,\dots,n$. Then, provided $x_i \neq x$ for some $i = 2,\dots,b$,

$$(3.1) \qquad n \leqq b - 1 - \{[(b-1)x - L]^2/[Q - 2xL + (b-1)x^2]\}.$$

If $n \geqq 1$,

$$(3.2) \qquad [L/(b-1)] - D \leqq x \leqq [L/(b-1)] + D,$$

where $$D^2 = [(b-1)Q - L^2](b - 1 - n)/n(b-1)^2.$$

PROOF. The $b - 1 - n$ integers $x_i$ other than $x_{i_j}$, $j = 1,\dots,n$, have mean $(L-nx)/(b-1-n)$. Computing the sum of squares of deviations from the mean, we obtain

$$Q - nx^2 - [(L - nx)^2/(b - 1 - n)] \geqq 0,$$

leading to

$$(3.3) \qquad n(b-1)x^2 - 2nLx + nQ + L^2 - (b-1)Q \leqq 0.$$

In particular, $Q - [L^2/(b-1)] \geqq 0$. Now the coefficient of $n$ in (3.3) can be expressed $Q - [L^2/(b-1)] + \{[L - (b-1)x]^2/(b-1)\}$ and is positive unless $x_i = x$, $i = 2,\dots,b$. Under the respective hypotheses, (3.3) may therefore be solved for $n$ to give (3.1) and solved for $x$ to give (3.2).

LEMMA 2. For an SG design and for any choice of block $B_1$,

$$(3.4) \qquad L = k(r-1),$$

$$(3.5) \qquad Q = k^2[(v-k)(b-r) - \alpha_2(v - rk)]/v\alpha_2.$$

PROOF. Enumeration of occurrences in other blocks of objects of $B_1$ gives (3.4). Let $B_1$ contain $z$ ordered pairs of first associates. Enumeration of occurrences within other blocks of pairs of objects of $B_1$ leads to

$$Q = z(\lambda_1 - 1) + [k(k-1) - z](\lambda_2 - 1) + k(r-1),$$

$$(3.6) \qquad Q = (\lambda_2 - \lambda_1)\,[k(k-1) - z] + (\lambda_1 - 1)k(k-1) + k(r-1).$$

A given object of $B_1$ is contained in $\rho$ lines of the partial geometry, and by Theorem 1, $B_1$ contains exactly $k\kappa/v$ objects of each of these lines. Thus $B_1$ contains exactly $\rho(k\kappa - v)/v$ first associates of the given object, and

$z = k\rho(k\ \kappa - v)/v$, for any choice of block $B_1$. If the design is of partial geometry type, this is used in (3.6), along with (1.11) and (1.9), and after simplification we obtain

$$(Q/rk) + (k/r) - (k^2/v) =$$

$$(v - k)\ \{\kappa[k(v - \rho\kappa) + v(\rho - 1)] + v\kappa(\kappa - t) - kv(\kappa - t)\}/v^2(\kappa - 1)(\kappa - t),$$

Application of (1.7) and (1.8) gives the right hand side the form

$$(v - k)\{\kappa[k(v - \rho\kappa) + v(\rho - 1)] + v\kappa(\kappa - t) - k\ \kappa(v - \rho\kappa + \kappa + \rho - t - 1)\}$$
$$/v\kappa\,\alpha_2(\kappa + \rho - t - 1)$$

and straightforward simplification gives

$$(Q/rk) + (k/r) - (k^2/v) = (v - k)^2/v\alpha_2.$$

This is equivalent to (3.5). If the design is of group divisible type, $z = k(k - m)/m$ and with the aid of (1.11) and (1.12), (3.6) again reduces to (3.5).

THEOREM 2. If a given block B of an SG design is disjoint from d other blocks, then

(3.7) $$d \leqq b - 1 - \{v\alpha_2(r - 1)^2/[(b - r)(v - k) - \alpha_2(v - rk)]\},$$

with equality only if

(3.8) $$k[(b - r)(v - k) - \alpha_2(v - rk)]/v\alpha_2(r - 1) \equiv \mu$$

is an integer and each of the remaining $b - d - 1$ blocks has exactly $\mu$ objects in common with B.

PROOF. (3.7) follows from (3.1) with $x = 0$ and Lemma 2. From the proof of Lemma 1, equality in (3.1) holds only if each of the $b - n - 1$ integers $x_i$ is equal to $L/(b-1-n)$, leading to (3.8).

The following relation for any PBIB design may be verified by straightforward simplification.

(3.9) $$(b - 1)[(b - r)(v - k) - \alpha_2(v - rk)] - v\alpha_2(r - 1)^2$$
$$= (b - \alpha_2 - 1)(b - r)(v - k).$$

THEOREM 3. In an SG design, the following three statements are equivalent.

(a) Some block B has the same number of objects in common with each of the remaining blocks.

(b) $b = \alpha_2 + 1$. (Equivalently, $b = v - \alpha_1$.)

(c) $x = k(r - 1)/(b - 1)$ is an integer and any two blocks have exactly x objects in common.

PROOF. Assume (a). Then by Theorem 2 for Block B, (3.7) holds with equality and with $d = 0$. Using (3.9), (b) follows. Next assume (b). Consider Theorem 2 for an arbitrary choice of block B. (b) and (3.9) imply that (3.7) holds with equality and with $d=0$. Thus (3.8) holds. Using (b), (3.8) reduces to $\mu = k(r-1)/(b-1)$, proving (c). It is trivial that (c) implies (a).

THEOREM 4. If x is the number of objects common between two given blocks in an SG design, then

$$[k(r - 1)/(b - 1)] - D \leqq x \leqq [k(r - 1)/(b - 1)] + D, \tag{3.10}$$

where

$$D^2 = k^2(v - k)(b - r)(b - 2)(b - \alpha_2 - 1)/v\alpha_2(b - 1)^2.$$

PROOF. This theorem follows from (3.2) with $n = 1$, Lemma 2, and some simplification (in which (3.9) is useful).

A design is defined to be <u>resolvable</u> if it is possible to arrange the blocks into disjoint subsets, or replications, such that each object occurs in precisely one block of each replication. Necessarily, $v/k = b/r = u$, say, is an integer, representing the number of blocks in a replication. Then each block is disjoint from at least $u - 1$ other blocks. A design is defined to be <u>affine resolvable</u> if it is resolvable and each block has the same number of objects in common with each of the $b - u$ blocks not in its own replication.

THEOREM 5. In an SG design, let $v = ku$ where $u \geqq 2$ is an integer, and let some block B have $u - 1$ blocks disjoint from it. Then

(3.11) $$b \geqq v - \alpha_1 + r - 1,$$

and the following two statements are equivalent.

(d) $b = v - \alpha_1 + r - 1.$

(e) $k/u$ is an integer and B has exactly $k/u$ objects in common with each non-disjoint block.

PROOF. Let $v = ku$, $b = ru$. Applying Theorem 2 with $d = u - 1$, (3.7) reduces to $b \geqq \alpha_2 + r$, which is equivalent to (3.11). If $\alpha_2 = b - r$ and $v = ku$, (3.8) reduces to $k/u = \mu$, and Theorem 2 implies the equivalence of (d) and (e).

COROLLARY 5.1. Let an SG design be resolvable. Then (3.11) holds and the following three statements are equivalent.

(f) Some block B has the same number of objects in common with each block not in its replication.

(g) $b = v - \alpha_1 + r - 1.$

(h) The design is affine resolvable.

PROOF. Theorem 5 is applicable for an arbitrary choice of block B, implying (3.11). Assume (f) for a particular block B. Then the k objects of block B are distributed in equal numbers over the u blocks of each replication not containing B, whence $k/u$ is an integer and (e) holds, implying (g). Assume (g). Then (e) holds for any choice of block B, implying (h). It is trivial that (h) implies (f).

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