ON NONPARAMETRIC SIMULTANEOUS CONFIDENCE REGIONS AND TESTS FOR THE ONE CRITERION ANALYSIS OF VARIANCE PROBLEM*

PRANAB KUMAR SEN

University of North Carolina, Chapel Hill

Institute of Statistics Mimeo Series No. 463

March 1966

*Work supported with the Army Research Office, Durham, Grant DA-31-124-G432.

DEPARTMENT OF BIOSTATISTICS

UNIVERSITY OF NORTH CAROLINA

Chapel Hill, N. C.

ON NONPARAMETRIC SIMULTANEOUS CONFIDENCE REGIONS AND TESTS FOR THE ONE CRITERION ANALYSIS OF VARIANCE PROBLEM*

PRANAB KUMAR SEN

University of North Carolina, Chapel Hill
and University of Calcutta

Summary. For the one criterion analysis of variance problem, some nonparametric generalizations of the two well-known methods of Multiple comparisons by Tukey [26] and Scheffé [19], are proposed and studied here. The performance characteristics of the proposed methods are compared with those of the others, available in the literature.

1. Introduction. Let there be $c(\geq 2)$ independent samples, the ith sample comprising of n_i independent and identically distributed random variables (i·i·d·r·v·) distributed according to a continuous cumulative distribution function (cdf) $G_i(x)$, for $i = 1, \ldots, c$. In one way analysis of variance problem, it is assumed that

$$(1.1) \qquad G_i(x) = G(x - \theta_i),\ i = 1, \ldots, c,$$

$\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_c)$ being a real c-vector. The null hypothesis to be tested relates to

$$(1.2) \qquad H_0: \theta_1 = \ldots = \theta_c = 0.$$

Under the assumption of G being a normal cdf, the classical variance ratio (F-) test is known to possess some optimum properties as a comprehensive test for H_0 in (1.2). However, in many situations, we may not be merely satisfied with the rejection of (1.2), but may also desire to test more detailed hypotheses

*Work supported with the Army Research Office, Durham, Grant DA-31-124-G432.

concerning two or more of the components in $\underset{\sim}{\theta}$. An important class of parametric functions used for this purpose is the set of all contrasts among $\theta_1, \ldots, \theta_c$; a contrast ϕ being defined as

(1.3) $$\phi = \underset{\sim}{\ell} \cdot \underset{\sim}{\theta}, \ \underset{\sim}{\ell} = (\ell_1, \ldots, \ell_c), \ \underset{\sim}{\ell} \perp \underset{\sim}{I}_c = (1, \ldots, 1).$$

The class $\underset{\sim}{\phi}$ of all possible contrasts is generated by the space $\underset{\sim}{L}$ (of rank c-1) of all possible $\underset{\sim}{\ell}$ vectors, satisfying (1.3). Obviously, we can have always a set of c-1 linearly independent contrasts, say $\phi_1, \ldots, \phi_{c-1}$, which spans the class $\underset{\sim}{\Phi} = \{\phi: \ \phi = \underset{\sim}{\ell}\cdot\underset{\sim}{\theta}, \ \underset{\sim}{\ell} \in L, \ \underset{\sim}{\ell} \perp \underset{\sim}{I}_c\}$. It is also interesting to note that ϕ is translation invariate in the sense that for any real scaler δ,

(1.4) $$\phi = \underset{\sim}{\ell}(\underset{\sim}{\theta} + \delta \underset{\sim}{I}_c) = \underset{\sim}{\ell}\cdot\underset{\sim}{\theta}.$$

The problem of making further inferences about these contrasts, arising when the F-test rejects H_o in (2.2), has been considered by Nandi [15], Duncan [4], Tukey [26], Scheffé [19], Bose and Roy [2], Roy [18], Ghosh [8], Dwass [6], among many others. The two mostly used methods are due to Tukey [26] and Scheffé [19]. Essentially these methods are concerned with providing simultaneous confidence regions to all possible contrasts with a view to carry out any multiple comparison test having a preassigned level of significance. These procedures are all valid for normal cdf's only.

Now with the steady advancement of nonparametric techniques in this field of research, it has come to be recognized that the nonparametric competitors not only compare quite favourably with their parametric rivals but also possess some properties which may not be shared by the others (cf. Lehmann [12], [14]). Further, an interesting method of estimating shift parameters by rank order tests considered by Hodges and Lehmann [11] has made the opening of a new line of approach to the analysis of variance problems, on which further works are due to Lehmann ([13], [14]), Bhuchongkul and Puri [1] and Sen ([22], [23]). However, for the problem of simultaneous confidence regions or multiple comparison tests, these results may not be directly applicable.

The object of the present investigation is to provide some nonparametric generalizations of the well-known T and S methods of multiple comparisons (cf. [20, pp. 66-83]) through the use of the same class of rank order statistics as has been used in ([11], [13], [14], [21], [22], [23]). Some of the ideas of the present paper are scattered in rudimentary forms in the works of Dwass [7] and Steel ([24], [25]). However, no systematic approach to this problem (particularly the asymptotic theory) has yet been made elsewhere in the literature. The asymptotic properties of the proposed methods are studied and compared with those of the parametric S and T methods as well as the other ones by Steel ([24], [25]) and Dunn [5].

2. **Nonparametric generalizations of T-method of multiple comparisons.** By analogy with the homoscedasticity condition implicit in the use of the T-method (cf. [28, p. 294]), we require here

$$n_1 = \dots = n_c = n. \tag{2.1}$$

Our proposed method is essentially based on Tukey's [26] principle as adapted for the one way classification. We shall consider first the method of paired comparisons, where we want to test for the difference of $\theta_i - \theta_j$ (referred to (1.1),) for all $i \neq j = 1, \dots, c$. Subsequently, we will consider the method of multiple comparisons, where we want to test for the significance of all possible contrasts.

2.1. The paired comparisons. Let us formulate first the class of rank order statistics which will be used throughout the paper. Let

$$\underset{\sim}{X}_i = (X_{i1}, \dots, X_{in}),\ i = 1, \dots, c;\quad \underset{\sim}{J}_n = (1, \dots, 1); \tag{2.2}$$

(2.3) $\underset{\sim}{E}_n = (E_{n,1}, \ldots, E_{n,2n}), \; E_{n,\alpha} = J_n(\frac{\alpha}{2n}), \; 1 \leq \alpha \leq 2n,$

where J_n is defined on the same line as in Chernoff and Savage [3, p. 972] and it satisfies all the four regularity conditions of theorem 1 of [3]. Throughout this paper, these regularity conditions will be implicit in the use of $\underset{\sim}{E}_n$. We write then

(2.4) $$\bar{E}_n = \frac{1}{2n} \sum_{\alpha=1}^{2n} E_{n,\alpha} \quad \text{and} \quad A_n^2 = \frac{1}{2n} \sum_{\alpha=1}^{2n} E_{n,\alpha}^2 - \bar{E}_n^2.$$

Also, we denote by $J(u) = \lim_{n \to \infty} J_n(u)$ for $0 < u < 1$ and write

(2.5) $$\mu = \int_0^1 J(u)\, du, \quad A^2 = \int_0^1 J^2(u)\, du - \mu^2.$$

Then, we assume further that

(2.6) $J(u)$ is $\uparrow$ in u: $0 < u < 1$;

(2.7) $|\bar{E}_n - \mu| = o(n^{-\frac{1}{2}}), \; |A_n - A| = o(1).$

Further, let $Z_{n,\alpha}^{(i,j)} = 1$, if the α-th smallest observation in the combined (i,j)-th sample is from the ith sample, and let $Z_{n,\alpha}^{(i,j)} = 0$, otherwise, for $\alpha = 1, \ldots, 2n$. Then, we shall be concerned with the following class of Chernoff-Savage [3] type of rank statistics

(2.8) $$h_n(\underset{\sim}{X}_i, \underset{\sim}{X}_j) = \frac{1}{n} \sum_{\alpha=1}^{2n} E_{n,\alpha} Z_{n,\alpha}^{(i,j)}, \text{ for } i \neq j = 1, \ldots, c.$$

In conjunction with (2.6), we shall assume that

(2.9) $h_n(\underset{\sim}{X}_i + a\underset{\sim}{I}_n, \underset{\sim}{X}_j)$ is $\uparrow$ in a for all $\underset{\sim}{X}_i, \underset{\sim}{X}_j$, $i \neq j = 1, \ldots, c$.

It may be noted that statistics of the type (2.8) include as particular cases, the Wilcoxon's [27] statistic, Normal score statistic, among others. Let us then consider the following statistic

$$(2.10) \qquad W_n = \max_{1 \le i, j \le c} [2n^{\frac{1}{2}} A_n^{-1} | h_n(\underset{\sim}{X}_i, \underset{\sim}{X}_j) - \bar{E}_n |],$$

which plays the basic role in our proposed method.

THEOREM 2.1 Under H_o in (1.2),

$$\lim_{n=\infty} P \{ W_n \le t \} = \chi_c(t),$$

where $\chi_c(t)$ is the cdf of the sample range in a sample of size c drawn from a standardized normal distribution.

PROOF. Let us define

$$(2.11) \qquad Y = B(X) = \int_{x_o}^{X} J'[G(x)] \, dG(\ell), \; G(x_o) = \tfrac{1}{2}.$$

Then, under (1.2), Y_{ij}, $j = 1, \ldots, A$, $i = 1, \ldots, c$ are $N(=nc)$ i·i·d·r·v. By theorem 1 of [3], it is known that Y has a finite absolute moment of order $2 + \eta$, for some $\eta > 0$, the variance of Y is A^2, defined in (2.5), and we denote its mean by ξ. Thus, on defining

$$(2.12) \qquad Z_{n,i} = n^{-\frac{1}{2}} A^{-1} \sum_{j=1}^{n} [Y_{ij} - \xi], \; i = 1, \ldots, c,$$

it follows that $Z_{n,1}, \ldots, Z_{n,c}$ are i·i·d·r·v· distributed asymptotically normally with zero mean and unit variance. Consequently, it is easily seen that for any given c,

$$(2.13) \qquad \lim_{n=\infty} P \left\{ \max_{1 \le i, j \le c} | Z_{n,i} - Z_{n,j} | \le t \right\} = \chi_c(t),$$

where $\chi_c(t)$ is defined in the statement of the theorem. Now, proceeding precisely on the same line as in the proof of theorem 1 of [3], and then using Poincares theorem on total probability, it is easily seen that

(2.14) $\quad [n^{\frac{1}{2}} A^{-1} \{ h_n(X_i, X_j) - \mu \} - \frac{1}{2}(Z_{n,i} - Z_{n,j})] = o_p(1),$

Simultaneously for all $i \neq j = 1, \ldots, c$. Thus, from (2.4), (2.5), (2.7), (2.10) and (2.13), we get that

(2.15) $\quad \{ W_n - \underset{1 \leq i, j \leq c}{\text{Max.}} | Z_{n,i} - Z_{n,j} | \} = o_p(1).$

Hence, the theorem follows from (2.13).

In small samples, the exact distribution of W_n(under H_o in (1.2),) may be quite involved and no suitable algebraic expression may be attached to it. However, there appears to be a permutation procedure of evaluating the exact null distribution of W_n. Let us write $\underset{\sim}{X}_N = (\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_c)$. Under (1.2), $\underset{\sim}{X}_N$ is composed on N i.i.d.r.v. and hence conditioned on the given $\underset{\sim}{X}_N$, all possible (N!) permutations of the variates among themselves are equally likely. This, conditioned on the given $\underset{\sim}{X}_N$, all possible $(N!/(n!)^c,)$ partitionings of these N variables into c subsets of equal sizes are equally likely, each having the (conditional) probability $[N!/(n!)^c]^{-1}$. Thus, if we consider the set of all these partitionings and for each one of them, we compute the value of W_n (with the aid of formulation (2.10),) we will arrive at the permutation distribution function of W_n. Since, G is assumed to be continuous (so that the possibility of ties may be ignored, in probability,) and as $h_n(\underset{\sim}{X}_i, \underset{\sim}{X}_j)(i \neq j = 1, \ldots, c)$ are all rank order statistics, it follows that the permutation distribution of W_n derived in this manner will agree with the exact null distribution of W_n. This procedure may be quite useful for small or moderately large values of n (particularly, if some modern computing facilities are available), while for large samples, we may use theorem 2.1 to approximate the true null distribution of W_n by $\chi_c(t)$, tables for which are abailable in Biometrika tables [16, pp. 165-171]. It may be noted that for Wilcoxon's [27] statistic, theorem 2.1 (in a slightly incorrect version) has been considered by Dwass [7].

Let now α: $0 < \alpha < 1$ be our preassigned level of significance. We denote by $W_{n,\alpha}$ and $R_{c,\alpha}$, the upper 100α % point of the exact null distribution of W_n and of $\chi_c(t)$ respectively, so that

$$(2.16) \qquad P\{W_n < W_{n,\alpha} | H_o\} = 1 - \alpha, \ \chi_c(R_{c,\alpha}) = 1 - \alpha,$$

and by theorem 2.1, $W_{n,\alpha} \rightarrow R_{c,\alpha}$.

The simplest type of paired comparison test may now be formulated as follows:

1. For all $1 \leq i < j \leq c$, compute the values of $h_n(\underset{\sim}{X}_i, \underset{\sim}{X}_j)$; the values of the remaining set will be obtained from the relation that

$$(2.17) \qquad h_n(\underset{\sim}{X}_i, \underset{\sim}{X}_j) + h_n(\underset{\sim}{X}_j, \underset{\sim}{X}_i) = 2\bar{E}_n \quad \text{for all } i, j = 1, \ldots, c.$$

2. Compute the value of $W_{n,\alpha}$ corresponding to the preassigned α.

3. Referred to (1.1), regard those $(\theta_i - \theta_j)$ to be significantly different from zero for which

$$(2.18) \qquad 2n^{\frac{1}{2}} A_n^{-1} | h_n(X_i, X_j) - \bar{E}_n | \geq W_{n,\alpha}.$$

It is easily seen that the test is an exact size $\alpha (0 < \alpha < 1)$ multiple comparison (similar) test.

Steel ([24], [25]) has used Wilcoxon's [27] statistic to derive some multiple comparison tests analogous to Duncan's [4] and Tukey's [26] methods. Here, we won't consider Duncan's method, while his generalization of Tukey's method may be regarded as a particular case of our paired comparison test in (2.18). Moreover, his study remains appreciably incomplete in the sense that he has not supplied any (asymptotic or exact) expression for $W_{n,\alpha}$ (even for his simplest case,) not to speak of any property of his proposed procedure. For $c = 3$, he has provided a table for the probability law of the minimum Wilcoxon-statistic (among the c(c-1) paired of samples) for sample sizes up to 6, and also some approximate values for $c \leq 10$, $n \leq 10$. Our results not only generalize his procedure to a wider class of rank order statistics but also suggests some asymptotically simplified form of W_n.

Now, often we are not merely satisfied with the detection of those pairs of (θ_i, θ_j) for which $\theta_i \neq \theta_j$, but also we want to attach a simultaneous confidence region to all possible $\theta_i - \theta_j$, $i \neq j = 1, \ldots, c$. For this, we shall use (2.18) and a method of deriving confidence intervals for shift parameters, considered by Lehmann [13] and Sen ([21], [22]). Let us write.

(2.19) $$\Delta_{ij} = \theta_i - \theta_j \text{ for } i, j = 1, \ldots, c.$$

From (1.1), we will have then

(2.20) $$G_i(x) = G(x-\theta_i) = G_j(x-\Delta_{ij}), \text{ for } i, j = 1, \ldots, c.$$

Let us then define

(2.20) $$\begin{cases} \mu_n^{(1)} = \bar{E}_n - \tfrac{1}{2}n^{-\frac{1}{2}} A_n W_{n,\alpha} \\ \mu_n^{(2)} = \bar{E}_n + \tfrac{1}{2}n^{-\frac{1}{2}} A_n W_{n,\alpha}. \end{cases}$$

Now, by (2.9) $h_n(\underset{\sim}{X}_i + a\underset{\sim}{I}_n, \underset{\sim}{X}_j)$ is monotonic in a. Hence, by the sliding principle, we arrive at the following two values:

(2.21) $$\begin{cases} \Delta_{ij.L} = \text{Inf } \{ a: \ h_n(\underset{\sim}{X}_i + a\underset{\sim}{I}_n, \underset{\sim}{X}_j) > \mu_n^{(1)} \} , \\ \Delta_{ij.U} = \text{Sup } \{ a: \ h_n(\underset{\sim}{X}_i + a\underset{\sim}{I}_n, \underset{\sim}{X}_j) < \mu_n^{(2)} \} ; \end{cases}$$

which defines an interval

(2.22) $$I_{ij} = \{ \Delta_{ij}: \Delta_{ij.L} \leq \Delta_{ij} \leq \Delta_{ij.U} \} , \ i \neq j = 1, \ldots, c.$$

Then, it follows from (2.18) though (2.22) that the probability is $1 - \alpha$ that

<u>the inequalities</u> $\Delta_{ij.L} \leq \Delta_{ij} \leq \Delta_{ij.U}$ <u>hold simultaneously for all</u> $i \neq j = 1, \ldots, c$.

We shall now consider certain asymptotic properties of the propose paired comparison procedure. To justify the approach theoretically and to avoid the limiting degeneracy (2.22), we shall now conceive of a sequence of c-tuplets of cdf's $\{G_{n,i}(x),\ i = 1, \ldots, c\}$, for which (1.1) holds and

$$n^{\frac{1}{2}} \underset{\sim}{\theta}_n \to \underset{\sim}{\lambda} = (\lambda_1, \ldots, \lambda_c) \text{ as } n \to \infty, \tag{2.23}$$

where θ_n is defined as in (1.1) and $\lambda_i,\ i = 1, \ldots, c$ are real and finite. We also define

$$\lambda_{ij} = \lambda_i - \lambda_j \text{ for } i, j = 1, \ldots, c. \tag{2.24}$$

We will be then interested in paried comparisons in λ_i's, instead of θ_i's. (It may be noted that as for the simultaneous confidence region $\{I_{ij}: 1 \leq i, j \leq c\}$, we may consider a somewhat more general formulation where as $n \to \infty$

$$n^{\frac{1}{2}} (\Delta_{ij} - \Delta_{ij}^0) \to \lambda_{ij}, \text{ defined in (2.24)}, \tag{2.25}$$

Δ_{ij}^0 being some (fixed) real quantity, not necessarily equal to zero. Since, the confidence intervals of the type (2.22) are all translation invariant (cf. [13], [22]), for the study of the asymptotic properties, it is immaterial whether we take $\Delta_{ij}^0 = 0$ or not, for all $i, j = 1, \ldots, c$.). The above formulation is analogous to Pitman's type of translation alternatives usually adopted to study the efficiency aspects of the nonparametric analysis of variance tests. Let us now consider the Hodges-Lehmann [11] point estimate $\hat{\Delta}_{ij}$ of Δ_{ij}, in (2.19). For this, we define

$$\begin{cases} \hat{\Delta}_{ij}^{(1)} = \text{Inf.} \ \{a: h_n (\underset{\sim}{X}_i + a\underset{\sim}{I}_n, \underset{\sim}{X}_j) > \mu_n\} \\ \hat{\Delta}_{ij}^{(2)} = \text{Sup.} \ \{a: h_n (\underset{\sim}{X}_i + a\underset{\sim}{I}_n, \underset{\sim}{X}_j) < \mu_n\} \end{cases} \tag{2.25}$$

Then, conventionally

(2.26) $\hat{\Delta}_{ij} = \frac{1}{2}(\hat{\Delta}_{ij}^{(1)} + \hat{\Delta}_{ij}^{(2)})$, $i, j = 1, \ldots, c$.

We also define

(2.27) $B(J,G) = \int_{-\infty}^{\infty} (d/dx)J[G(x)]\, dG(x)$.

Then by a simple and straight forward extension of theorem 1 of Sen [22], it follows that asymptotically I_{ij} in (2.22) reduces to

(2.28) $I_{ij} = \{\lambda_{ij} : |\lambda_{ij} - \hat{\lambda}_{ij}| \leq A\, R_{c,\alpha}/B(J,G)\}$,

where $\hat{\lambda}_{ij} = n^{\frac{1}{2}}(\hat{\Delta}_{ij} - \Delta_{ij}^{0})$ is the derived form of Hodges-Lehmann [11] estimate in (2.26), and A and $R_{c,\alpha}$ are defined in (2.5) and (2.16) respectively.

Let us now compare (2.38) with the confidence interval obtained by T-method, when G is assumed to be normal with a variance σ^2. If we define $\bar{\Delta}_{ij} = \bar{X}_i - \bar{X}_j$, as the difference of the ith and jth sample means, and if $R_{c,c(n-1),\alpha}$ is the upper 100α % point of the studentized range $R_{c,c(n-1)}$ (cf. Wilks [28, p. 294]), then we have the probability $1 - \alpha$ that the inequalities

(2.29) $\bar{\Delta}_{ij} - n^{-\frac{1}{2}} sR_{c,c(n-1),\alpha} \leq \Delta_{ij} \leq \bar{\Delta}_{ij} + n^{-\frac{1}{2}} sR_{c,c(n-1),\alpha}$

holds simultaneously for all $i, j = 1, \ldots, c$, where s^2 is the unbiased estimate of σ^2 carrying $c(n-1)$ degrees of freedom (d.f.). As it is well-known that

(2.30) $s^2 \xrightarrow{P} \sigma^2$, and $R_{c,c(n-1),\alpha} \to R_{c,\alpha}$ as $n \to \infty$;

we get from (2.29) that asymptotically the confidence interval in (2.29) reduces to

(2.31) $I_{ij}^{0} = \{\lambda_{ij} : |\lambda_{ij} - \bar{\lambda}_{ij}| \leq \sigma R_{c,\alpha}\}$, where $\bar{\lambda}_{ij} = n^{\frac{1}{2}}(\bar{\Delta}_{ij} - \Delta^0_{ij})$. Thus, if we take the ratio of the square of the width of the confidence intervals as a measure of the asymptotic efficiency, the asymptotic relative efficiency (A·R·E·) of our proposed method with respect to the T-method reduces to

$$(2.32) \qquad e(J,G) = \sigma^2 \, [B(J,G)]^2/A^2.$$

Thus, we arrive at the following theorem.

<u>THEOREM 2.2.</u> The A·R·E· of the proposed nonparametric generalization of the T-method of paired comparisons with respect to the T-method itself, is equal to the A·R·E· of the two sample rank order test (on which the proposed method is based,) with respect to the Student's t-test.

We shall discuss more about (2.32) later on.

It may be noted that the present author, in an earlier paper [22], has considered a method of estimating B(J,G) for any absolutely continuous G, satisfying the conditions of lemma 7.2 of Puri [17]. If we estimate B(J,G) separately from each (i,j)th samples (for $1 \leq i < j \leq c$), we may combine these together by an unweighted average, as the sample sizes are all equal. If we denote this estimate by $\hat{B}(J,G)$, then it follows from (2.28) that asymptotically, the confidence interval for $\lambda_i - \lambda_j$ may be written as

$$(2.33) \qquad \hat{\lambda}_{ij} - AR_{c,\alpha}/\hat{B}(J,G) \leq \hat{\lambda}_i - \lambda_j \leq \lambda_{ij} + AR_{c,\alpha}/\hat{B}(J,G).$$

2.2. <u>The multiple comparisons.</u> We start with a remark that the estimators $\hat{\Delta}_{ij}$ in (2.26) are incompatible in the sense that they may not satisfy the transitive relations viz., $\hat{\Delta}_{ij} + \hat{\Delta}_{jk} = \hat{\Delta}_{ik}$, which must be true for the corresponding parametric quantities. To remove this drawback, Lehmann ([12], [14]) has considered the following adjusted estimates. Let

$$(2.34) \qquad \hat{\Delta}_{i.} = \frac{1}{c} \sum_{j=1}^{c} \hat{\Delta}_{ij} \text{ for } i = 1, \ldots, c;$$

(2.35) $$Z_{ij} = \hat{\Delta}_{i.} - \hat{\Delta}_{j.} \text{ for } i \neq j = 1, \ldots, c.$$

It is easy to see that Z_{ij} satisfies the aforesaid transitive relation's. It is also well known (cf. [1, theorem 3.1]) that

(2.36) $$n^{\frac{1}{2}}(Z_{ij} - \hat{\Delta}_{ij}) = o_p(1), \text{ for all } i, j = 1, \ldots, c.$$

Our proposed method is based on certain properties of Z_{ij}'s.

<u>LEMMA 2.3.</u> $\underset{i}{\text{Range}} \{ Z_{ij} - \Delta_{ij} \} = \underset{j}{\text{Range}} \{ Z_{ij} - \Delta_{ij} \} = \underset{1 \leq k, \ell \leq c}{\text{Max}} | Z_{k\ell} - \Delta_{k\ell} |$.

<u>PROOF.</u> Suppose for any fixed i, the range of $(Z_{ij} - \Delta_{ij})$ is attained by the pair of paired suffixes (i,k) and (i,ℓ). Then, using (2.35) we get that

(2.37) $$\underset{j}{\text{Range}} (z_{ij} - \Delta_{ij}) = \{ (Z_{ik} - \Delta_{ik}) - (Z_{i\ell} - \Delta_{i\ell}) \} = (Z_{\ell k} - \Delta_{\ell k}).$$

Since, the right hand side of (2.37) is independent of i, it holds for all $i = 1, \ldots, c$, and hence, is also the unrestricted maximum. Hence,

$$\underset{i}{\text{Range}} (Z_{ij} - \Delta_{ij}) = \underset{1 \leq k, \ell \leq c}{\text{Max.}} (Z_{k\ell} - \Delta_{k\ell}).$$

The other relation also holds similarly.

Hence the lemma.

<u>LEMMA 2.4.</u> If $\phi = \sum_1^c \ell_i \theta_i$ be any contrast in $\underset{\sim}{\theta}$ and if for some $\ell (=1, \ldots, c)$, we define $\hat{\phi}_\ell = \sum_1^c \ell_i Z_{i\ell}$, then

$$\underset{\ell}{\text{Sup}} | \hat{\phi}_\ell - \phi | \leq \frac{1}{2} \sum_1^c | \ell_i | \cdot \underset{1 \leq j, k \leq c}{\text{Max.}} | Z_{jk} - \Delta_{jk} | \ .$$

<u>PROOF.</u> We can rewrite ϕ as $\phi_\ell = \sum_1^c \ell_i \, \Delta_{i\ell}$, and hence,

$$|\hat{\phi}_\ell - \phi| = |\sum_1^c \ell_i \, [Z_{i\ell} - \Delta_{i\ell}]|$$
$$\leq \frac{1}{2}\sum_1^c |\ell_i| \underset{i}{\text{Range}} \, [Z_{i\ell} - \Delta_{i\ell}]$$
$$= \frac{1}{2}\sum_1^c |\ell_i| \underset{1\leq j,k\leq c}{\text{Max.}} | Z_{kj} - \Delta_{kj}| \text{, (by lemma 2.3).}$$

Since, the right hand side of (2.38) is independent of ℓ, and the inequality holds for all $\ell = 1, \ldots, c$, the lemma follows directly from (2.38).

If we now let

(2.39) $\ell_{ij} = \frac{1}{c}\ell_i$ for $j = 1, \ldots, c$, $i = 1, \ldots, c$,

then the contrast ϕ may also be expressed as $\sum_{i=1}^{c} \sum_{j=1}^{c} \ell_{ij} Z_{ij}$. Consequently, from lemma 2.4 we get that

(2.40) $$\left|\phi - \sum_{i=1}^{c} \sum_{j=1}^{c} \ell_{ij} Z_{ij}\right| \leq \left(\frac{1}{2}\sum_{i=1}^{c} |\ell_i|\right) \underset{1\leq j,k\leq c}{\text{Max.}} |Z_{jk} - \Delta_{jk}| \, .$$

Now corresponding to the c(c-1) estimates $\hat{\Delta}_{ij}$ in (2.26), we complete the values of Z_{ij} for $i \neq j = 1, \ldots, c$. Further, from the c(c-1) simultaneous confidence intervals I_{ij} in (2.22), we compute the value of

(2.41) $\underset{1\leq j,k\leq c}{\text{Max.}} | Z_{kj} - \Delta_{kj}|$ subject to $\Delta_{ij} \, \varepsilon I_{ij}$ for all $i \neq j = 1, \ldots, c$.

We denote this maximum by $H_{n,\alpha}$. So that from (2.22) and the probability statement made just after it, we get that

(2.42) $$P\left\{\underset{1\leq j,k\leq c}{\text{Max.}} | Z_{jk} - \Delta_{jk}| \leq H_{n,\alpha}\right\} = 1 - \alpha.$$

Consequently, from (2.40) and (2.42), we conclude that the <u>probability is</u> $1 - \alpha$ <u>that the inequalities</u>

$$(2.43)\qquad \sum_{i=1}^{c}\sum_{j=1}^{c}\ell_{ij}Z_{ij}-\tfrac{1}{2}H_{n,\alpha}\sum_{1}^{c}|\ell_i| \le \phi = \sum_{1}^{c}\ell_i\theta_i \le \sum_{i=1}^{c}\sum_{j=1}^{c}\ell_{ij}Z_{ij}+\tfrac{1}{2}H_{n,\alpha}\sum_{1}^{c}|\ell_i|$$

hold simultaneously for all $\phi \in \underset{\sim}{\Phi}$.

This may be regarded as a nonparametric generalization of the well known T-method of multiple comparisons. (2.43) may be used to attach a simultaneous confidence interval to any number of contrasts in $\underset{\sim}{\theta}$ or to test the significance of them.

Now using (2.22), (2.28), (2.41) and (2.42), it readily follows that asymptotically

$$(2.44)\qquad n^{\frac{1}{2}} H_{n,\alpha} \to AR_{c,\alpha}/ B(J,G),$$

where A, $R_{c,\alpha}$ and B(J,G) are defined in (2.5), (2.16) and (2.27) respectively. Thus, if we define the derived estimates

$$(2.45)\qquad \hat{\lambda}^0_{ij} = n^{\frac{1}{2}} (Z_{ij} - \Delta^0_{ij}),\ i,\ j = 1, \ldots, c$$

(Δ^0_{ij} being defined in (2.25),) then from (2.43), (2.44) and (2.45), we get that (2.43) asymptotically reduces to

$$(2.46)\qquad -\tfrac{1}{2}\sum_{1}^{c}|\ell_i|AR_{c,\alpha}/B(J,G) \le \sum_{i=1}^{c}\sum_{j=1}^{c}\ell_{ij}(\lambda_{ij}-\hat{\lambda}^0_{ij}) \le \tfrac{1}{2}\sum_{1}^{c}|\ell_i|AR_{c,\alpha}/B(J,G).$$

If we now compare (2.46) with Tukey's [26] results, as adopted in the case of one way classified data with equal number of observations (cf. [28, p. 296]) we again get the same A. R. E. as obtained in (2.32). Hence, the following.

THEOREM 2.5. The conclusions of theorem 2.2 also hold for the multiple comparison tests considered here.

Now regarding (2.32), various known bounds are available in the literature. For example, if we use Wilcoxon's statistic, then for normal cdf, (2.32) is equal to 0.95, it has the minimum value of 0.864 for any continuous G (cf. [9]), while it can be arbitarily large for some specific G. Similarly, it is known that the use of normal scores results in a value of (2.32) which is always at least as large as one, while it may also be indefinitely large for some specific G. The relative value of the efficiencies of the Wilcoxon's test and normal score test depends on the particular G, and for various known G, a nice account of this is available with Hodges and Lehmann [10].

3. Nonparametric generalizations of S-method of multiple comparisons. The results of the preceding section have a somewhat limited scope of applicability in the sense that they deem the sample sizes to be all equal. The method to be considered now overcomes this drawback, and may be regarded as a nonparametric generalization of the well known S-method of multiple comparisons (cf. Schefee [19]). This method is essentially a confidence region procedure which is based primarily on the construction of a suitable simultaneous confidence region for the set of all possible contrasts among $\underset{\sim}{\theta}$.

In an earlier paper ([22]), the present author has ocnsidered such a simultaneous confidence region to the (c-1) parameters ($\theta_i - \theta_1$ for $i = 2, \ldots, c$) which may be used here. However, from computational standpoint this procedure appears to be a little involved in the sense that here we are faced with a set of (c-1) simultaneous equations in (c-1) unknowns, where each single equation is an involved function of all these (c-1) unknowns and is only asymptotically linear in them. Thus, the usual method of iteration, which has to be mostly adopted in such a case, becomes very tideous. To remove this difficulty, we shall consider the following approach which is computationally much more simple, and at the same time, asymptotically equivalent to the preceding one. This method will be very appropriate if one of the c populations may be regarded as control and the rest as treatment group. On the otherhand, if there is no such natural control group, our procedure is based on selecting one of the c groups as the standard or control group. However, as we shall see later on that asymptotically the procedure remains insensitive to the choice of any arbitary control group.

In this situation, the sample sizes $n_1, \ldots, n_c$ are not necessarily equal. Let us denote by

(3.1) $$N_{ij} = n_i + n_j \text{ for } i \neq j = 1, \ldots, c \text{ and } N = n_1 + \ldots + n_c.$$

Then, for the (i, j)th samples, we define $h_n(\underset{\sim}{X}_i, \underset{\sim}{X}_j)$ precisely in the same manner as in (2.8), with the only change that here $E_{N_{ij}}$ replaces $\underset{\sim}{E}_n$ in (2.3) and have N_{ij} instead of 2n) elements. The mean of the entries in $\underset{\sim}{E}_{N_{ij}}$ is denoted by $\bar{E}_{N_{ij}}$, and as in (2.7), we assume that

(3.2) $$|\bar{E}_{N_{ij}} - \mu| = o(N^{-\frac{1}{2}}),$$

where we define μ (and A^2) as in (2.5). Let us also define

(3.3) $$V_{N\cdot ij} = \frac{N_{ij}}{n_i n_j}\left(\sum_{\alpha=1}^{N_{ij}} [E_{N_{ij},\alpha} - \bar{E}_{N_{ij}}]\, Z_{N_{ij},\alpha}\right) = \frac{N_{ij}}{n_j}[h_N(\underset{\sim}{X}_i, \underset{\sim}{X}_j) - \bar{E}_{N_{ij}}],$$

for $i \neq j = 1, \ldots, c$.

Conventionally, we let $V_{N\cdot ii} = 0$ for $i = 1, \ldots, c$ and we regard the first sample to constitute the control group. Then, we define

(3.4) $$S_N = \frac{1}{A^2} \sum_{i=2}^{c} \sum_{j=2}^{c} [n_i(\delta_{ij} N - n_j)/N] V_{N\cdot 1i} V_{N\cdot 1j},$$

where δ_{ij} is the usual Kronecker delta. If we write

(3.5) $$\bar{V}_{N\cdot 1\cdot} = \frac{1}{N} \sum_{i=1}^{c} n_i V_{N\cdot 1i} = \frac{1}{N} \sum_{i=2}^{c} n_i V_{N\cdot 1i},$$

then (3.4) may also be written as

$$(3.6) \qquad S_N = A^{-2} \sum_{i=1}^{c} n_i \, [V_{N.1i} - \bar{V}_{N.1.}]^2.$$

We may adopt a similar permutation approach (as in section 2) to find out the exact null distribution of S_N. On the otherhand, if N is large subject to

$$(3.7) \qquad n_i \,|\, N \longrightarrow \rho_i: \quad 0 < \rho_i < 1 \text{ for all } i = 1, \ldots, c,$$

then by an adaptation of the same proof as in lemma 1 of [1], we readily arrive at the conclusion that under H_o in (1.2), S_N, in (3.4), has asymptotically a chi-square distribution with (c-1) d.f. Further, from the same lemma, it follows that under the sequence of alternatives in (2.23)(with n replaced by N,) S_N has asymptotically a noncentral χ^2 distribution with (c-1) degrees of freedom and the noncentrality parameter

$$(3.8) \qquad \Delta_S = [B(J,G)]^2 \, A^{-2} \sum_{i=1}^{c} \rho_i (\lambda_i - \bar{\lambda})^2,$$

where $\bar{\lambda} = \sum_{1}^{c} \rho_i \, \lambda_i$, and B(J,G) is defined in (2.27).

Now from (2.20), we have

$$(3.9) \qquad G_1(x) = G_i(x - \Delta_{1i}) \text{ for } i = 1, \ldots, c.$$

So, if we define

$$(3.10) \qquad V_{N.ij}(a) = \frac{N_{ij}}{n_j} \, [h_N(\underset{\sim}{X}_i + a\underset{\sim}{I}_{n_i}, \underset{\sim}{X}_j) - \bar{E}_{N_{ij}}] \text{ for } i, j = 1, \ldots, c;$$

then it follows from (1.1), (3.5), (3.9) and (3.10) that for all $\underset{\sim}{\theta}$

$$(3.11) \qquad S_N(\theta) = A^{-2} \sum_{i=1}^{c} n_i \, [V_{N.1i}(\Delta_{1i}) - \bar{V}_{N.1.}(\underset{\sim}{\Delta})]^2$$

has the same distribution as of S_N in (3.6) under H_o in (1.2). Further from (2.9), (3.3) and (3.10) we may conclude that $V_{N.ij}(a)$ is $\uparrow$ in a for all $i \neq j = 1, \ldots, c$. So if we denote by $S_{N,\alpha}$ the upper 100α ·/· point of the null distribution of S_N (so that

$$(3.12) \qquad S_{N,\alpha} \xrightarrow{P} \chi^2_{c-1,\alpha} \text{ where } P\{\chi^2 \geq \chi^2_{c-1,\alpha}\} = \alpha,),$$

then from (3.11) we get that

$$(3.13) \qquad P\{S_N(\underset{\sim}{\theta}) < S_{N,\alpha} \mid \underset{\sim}{\theta}\} = 1 - \alpha.$$

Now, equating $h_N(\underset{\sim}{X}_1 + a\, \underset{\sim}{I}_{N_1}, \underset{\sim}{X}_i)$ to $\bar{E}_{N_{1i}}$ in the same way as in (2.25) and (2.26), we arrive at the Hodges-Lehmann [11] point estimate $\hat{\Delta}_{ij}$ of Δ_{1i} for $i = 2, \ldots, c_j$ the joint asymptotic normality of the standardized form of these estimates follow precisely as in the proof of theorem 3.1 of [1]. Let us now denote by

$$(3.14) \qquad \underset{\sim}{V} = (V_2, \ldots, V_c)$$

the running coordinate of the points $\underset{\sim}{V}_N(\Delta) = (V_{N.12}(\Delta_{12}), \ldots, V_{N.1c}(\Delta_{1c}))$, on the boundary of the ellipsoid in (3.13). For any particular $\underset{\sim}{V}$, if V_i is positive, we find out a value of Δ_{1i}, say Δ^*_{1i}, such that

$$(3.15) \qquad \Delta^*_{1i} = \text{Inf}\ \{\Delta_{1i}: \ V_{N.1i}(\Delta_{1i}) \geq V_i\} \ ;$$

on the otherhand, if V_i is negative, we define

$$(3.16) \qquad \Delta^*_{1i} = \text{Sup}\ \{\Delta_{1i}: \ V_{N.1i}(\Delta_{1i}) \leq V_i\} \ ,$$

for each $i = 2, \ldots, c$. In this manner, any point $\underset{\sim}{V}$ on the interior neighbourhood of the boundary of the allipsoid in (3.13) is mapped into a point

(3.17) $\quad \underset{\sim}{\Delta}^*_{1.} = (\Delta^*_{12}, \dots, \Delta^*_{1c}).$

Further, (3.13) is the equation of a closed convex set of points $\{S_N(\underset{\sim}{\theta})\}$, and as each $V_{N.1i}(a)$ is $\uparrow$ in a, i = 2, ..., c, it follows that the set of points

(3.18) $\quad C(\Delta_{1.}) = \{(\Delta_{12}, \dots, \Delta_{1c}): \; S_N(\theta) \leq S_{N,\alpha}\}$

will also be a closed convex set in $(\Delta_{12}, \dots, \Delta_{1c})$, having the property that

(3.19) $\quad P\{(\Delta_{12}, \dots, \Delta_{1c}) \in C(\underset{\sim}{\Delta}_{1.}) \mid \underset{\sim}{\theta}\} = 1 - \alpha.$

For small samples, S_N in (3.4) will have essentially a finite number of discrete mass points and hence on the interior boundary of the ellipsoid $S_N < S_{N,\alpha}$, there will be only a finite number of points $\{\underset{\sim}{V}\}$. So, if for each of these points, we find out (by the process in (3.15) and (3.16),) the coreesponding (finite number of) points $\{\underset{\sim}{\Delta}^*_{1.}\}$, in (3.17), then the convex hull of these set of points will be our desired simultaneous confidence region for $(\Delta_{12}, \dots, \Delta_{1c})$. For large samples, we get by a straight forward generalization of theorem 4 of Hodges and Lehmann [11] along with an extension of theorem 1 of Sen [22] (to more than one parameter) that the set $c(\underset{\sim}{\Delta}_{1.})$ in (3.18) reduces to

(3.20) $$c(\underset{\sim}{\Delta}_{1.}) = \left\{\underset{\sim}{\Delta}_{1.}: \frac{B^2(J,G)}{A^2} \sum_{i=2}^{c} \sum_{j=2}^{c} \rho_i(\delta_{ij} - \rho_j) N(\Delta_{1i} - \Delta_{1i})(\hat{\Delta}_{1j} - \Delta_{1j}) \leq \chi^2_{c-1,\alpha}\right\},$$

where $\underset{\sim}{\Delta}_{1.} = (\Delta_{12}, \dots, \Delta_{1c})$ and $\hat{\Delta}_{1.} = (\hat{\Delta}_{12}, \dots, \hat{\Delta}_{1c})$ is the Hodges-Lehmann [11] estimate of $\underset{\sim}{\Delta}_{1.}$. Thus, if as in (2.25) we write (replacing n by N)

(3.21) $\quad N^{\frac{1}{2}}(\Delta_{ij} - \Delta^0_{ij}) \to \lambda_{ij}$ as $N \to \infty$, for i, j = 1, ..., c,

(3.22) $\quad \hat{\lambda}_{ij} = N^{\frac{1}{2}}(\hat{\Delta}_{ij} - \Delta^o_{ij})$ for $i \neq j = 1, \dots, c$, $\underset{\sim}{\lambda}_1 = (\lambda_{12}, \dots, \lambda_{1c})$.

where λ_{ij}'s are all real and finite, then (3.20) reduces to

$$(3.23) \qquad c(\underset{\sim}{\Delta}_{1.}) = \left\{ \underset{\sim}{\lambda}_1 : \sum_{i=2}^{c} \sum_{j=2}^{c} \rho_i(\delta_{ij} - \rho_j)(\hat{\lambda}_{1i} - \lambda_{1i})(\hat{\lambda}_{1j} - \lambda_{1j}) \leq \frac{A^2 \chi^2_{c-1,\alpha}}{B^2(J,G)} \right\} .$$

Consequently, if we attempt to estimate B(J,G) as in [22], separately for each combination (1, i)th (i = 2, ..., c) samples, and pool these together into a single estimate $\hat{B}(J,G)$, then an asymptotic simultaneous confidence region to $\underset{\sim}{\lambda}_{1.}$ will be

$$(3.24) \qquad c(\underset{\sim}{\lambda}_1) : \left\{ \underset{\sim}{\lambda}_1 : \sum_{i=2}^{c} \sum_{j=2}^{c} \rho_i(\delta_{ij} - \rho_j)(\hat{\lambda}_{1i} - \lambda_{1i})(\hat{\lambda}_{1j} - \lambda_{1j}) \leq \frac{A^2 \chi^2_{c-1,\alpha}}{\hat{B}^2(J,G)} \right\} .$$

We shall now use (3.18) or (3.23)(or (3.24),) to derive a simultaneous confidence region to any number of contrasts in $\underset{\sim}{\theta}$. Any contrast $\phi = \underset{\sim}{\ell} \cdot \underset{\sim}{\theta}$ may also be written as $-\sum_{1}^{c} \ell_i \Delta_{1i} = -\sum_{2}^{c} \ell_i \Delta_{1i}$. Since $c(\underset{\sim}{\Delta}_{1.})$ in (3.18) is a closed convex set, its convex hull is contained within the intersection of all the supporting hyperplanes of $c(\underset{\sim}{\Delta}_{1.})$. Thus, given any $\sum_{2}^{c} \ell_i \Delta_{1i}$, (which represents the equation of a (c-2)-dimensional hyperplane,) we can always find two parallel (c-2)-dimensional hyperplanes haiing the equations $\sum_{2}^{c} \ell_i \Delta_{1i} = c_j$, j = 1, 2; such that the convex hull of $c(\underset{\sim}{\Delta}_{1.})$ is contained within the (c-1)-dimensional steip defined by these two hyperplanes. Then, if we let $c_1 < c_2$ (without any loss of generality), we get the condidence interval

$$(3.25) \qquad -c_2 \leq \phi = -\sum_{2}^{c} \ell_i \Delta_{1i} \leq -c_1,$$

whose confidence coefficient will be at least as large as $1 - \alpha$, whatever be the number of contrasts, we work with. In general, c_1, c_2 will depend not only on $(\ell_1, \ldots, \ell_c)$ but also the convex hull of the set $c(\underset{\sim}{\Delta}_{1.})$. For large samples, (3.25) simplifies to a great extent. For this, let us define

(3.26) $\delta^2 = A^2\chi^2_{c-1,\alpha}/B^2(J,G)$ and $\hat{\delta}^2 = A^2\chi^2_{c-1,\alpha}/\hat{B}^2(J,G)$.

Then, it follows from (3.23) and a few simple adjustments that the equations of the two parallel (c-2)-dimensional hyperplanes $\sum_2^c \ell_i\Delta_{1i} = c_j$, j=1,2, reduce to

$$(3.27) \qquad \sum_{j=2}^{c} \ell_j N^{\frac{1}{2}}(\hat{\Delta}_{1j}-\Delta_{1j}) = \sum_{j=2}^{c} \ell_j(\hat{\lambda}_{1j}-\lambda_{1j}) = \pm\delta\sqrt{\sum_2^c\sum_2^c \ell_i\ell_j(1/\varrho_1+\delta_{ij}/\varrho_i)} =$$

$$= \pm\delta\sqrt{\sum_1^c \ell_i^2/\varrho_i} \quad .$$

Consequently, asymptotically the probability is 1 - α that the inequalities

$$(3.28) \qquad \sum_2^c \ell_i\hat{\lambda}_{1i} - \delta(\sum_1^c \ell_i^2/\varrho_i)^{\frac{1}{2}} \leq \sum_2^c \ell_i\lambda_{1i} \leq \sum_2^c \ell_i\hat{\lambda}_{1i} + \delta(\sum_1^c \ell_i^2/\varrho_i)^{\frac{1}{2}}$$

hold simultaneously for all $\phi = \sum_1^c \ell_i\theta_i$, $\phi \varepsilon \Phi$, where δ is defined in (3.26). Since $\hat{\delta}$, in (3.26), converges stochastically to δ independently of $\phi \varepsilon \Phi$, we get that <u>asymptotically the probability is</u> 1 - α <u>that</u> <u>the</u> <u>inequalities</u>

$$(3.29) \qquad \sum_2^c \ell_i\hat{\lambda}_{1i} - \hat{\delta}(\sum_1^c \ell_i^2/\varrho_i)^{\frac{1}{2}} \leq \phi = \sum_1^c \ell_i\theta_i \leq \sum_2^c \ell_i\hat{\lambda}_{1i} + \hat{\delta}(\sum_1^c \ell_i^2/\varrho_i)^{\frac{1}{2}}$$

hold simultaneously for all $0 \varepsilon o$.

(3.29) may also be regarded as an extension of a similar result by Lehmann [14, p. 1500] to a much wider class of rank order statistics.

Again, comparing (3.29) with the simultaneous confidence region provided by the S-method of multiple comparison (cf. Scheffé [20, pp. 68-71]), we conclude that the asymptotic relative efficiency (A.R.E.) of the nonparametric generalization with respect to the parametric S-method is equal to

$$e(J,G) = \sigma^2 B^2(J,G)/A_1^2 \tag{3.30}$$

which is the same as in (2.32). Hence, the theorem.

<u>THEOREM 3.1.</u> The conclusions of theorem 2.2 also hold for the nonparametric generalizations of S-method of multiple comparison.

Thus, the discussion made at the end of section 2 also apply to this case.

Finally, regarding the comparison of this method with the one proposed by Dunn [5], we would like to point out the following.

(i) Our method is valid even when Δ_{ij}^0's, defined in (2.25), are not necessarily zero, while as the procedure by Dunn assumes that θ is a null vector (under H_0) against the set of alternatives that at least one of the θ_i in each group is different from at least one from the other. Our method is naturally applicable in a more wider class of situations.

(ii) This is essentially a simultaneous confidence region based on the principle of S-method of multiple comparisons, while Dunn's technique requires a selection of a fixed number of contrasts on which the level of significance to each contrast depends. This is not really justifiable in many cases, where we may desire to test for any arbitrary number of contrasts and in that case, her method will have some difficulty to apply.

REFERENCES

[1] BHUCHONGKUL, S., and PURI, M. L. (1965). On the estimation of contrasts in linear models. Ann. Math. Statist 36, 198-202.

[2] BOSE, R. C., and Roy, S. N. (1953). Simultaneous confidence interval estimation. Ann. Math. Statist 24, 513-536.

[3] CHERNOFF, H., and SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. Ann. Math. Statist. 29, 972-994.

[4] DUNCAN, D. B. (1952). On the properties of the multiple comparison test. Virginia Jour. of Sci. 3, 49-67.

[5] DUNN, O. J. (1964). Multiple comparisons using rank sums. Technometrics 6, 241-252.

[6] DWASS, M. (1959). Multiple confidence procedures. Ann. Inst. Statist Math. 10, 277-282.

[7] DWASS, M. (1960). Some k sample rank order tests. Harold Hotelling volume. Stanford Univ. Press. 198-202.

[8] GHOSH, M. N. (1955). Simultaneous tests of linear hypotheses. Biometrike 42, 441-449.

[9] HODGES, J. L. Jr., and LEHMANN, E. L. (1956). The efficiency of some nonparametric competitors of the t-test. Ann. Math. Statist 27, 324-335.

[10] HODGES, J. L. Jr., and LEHMANN, E. L. (1960). Comparison of the normal score and Wilcoxon tests. Proc. 4th Berkeley Symp. Math. Statist. Prob. 1, 307-317.

[11] HODGES, J. L. Jr., and LEHMANN, E. L. (1963). Estimates of location based on rank tests. Ann. Math. Statist. 34, 598-611.

[12] LEHMANN, E. L. (1963). Robust estimation in analysis of variance. Ann. Math. Statist. 34, 957-966.

[13] LEHMANN, E. L. (1963). Nonparametric confidence interval for a shift parameter. Ann. Math. Statist. 34, 1507-1512.

[14] LEHMANN, E. L. (1963). Asymptotically nonparametric inference: an alternative approach to linear models. Ann. Math. Statist. 34, 1494-1505.

[15] NANDI, H. K. (1951). On the analysis of variance tests. Calcutta Statist. Asso. Bull. 3, 103-114.

[16] PEARSON, E. S., and HARTLEY, H. O. (1954). Biometrika tables for statisticians. (Edited). Univ. Press London. vol. 1:

[17] PURI, M. L. (1964). Asymptotic efficiency of a class of c sample tests. Ann. Math. Statist. 35, 102-121.

[18] ROY, S. N. (1954). Some further results in simultaneous confidence interval estimation. Ann. Math. Statist. 25, 752-761.

[19] SCHEFFE, H. (1953). A method for judging all contrasts in analysis of variance. Biometrika 40, 87-104.

[20] SCHEFFÉ, H. (1959). The Analysis of Variance. John Wiley and Sons Inc. New York

[21] SEN, P. K. (1963). On the estimation of relative potency in dilution (-direct) assays by distribution free methods. Biometrics 19, 532-552.

[22] SEN, P. K. (1966). On pooled estimation and testing of heterogeneity of shift parameters by distribution-free methods. Part one: Univariate two factor case. Calcutta Statist. Asso. Bull. (Submitted).

[23] SEN, P. K. (1966). On a distribution free method of estimating asymptotic efficiency of a class of nonparametric tests. Ann. Math. Statist. (Submitted)

[24] STEEL, R. G. D. (1960). A rank sum tests for comparing all pairs of treatments. Technometrics 2, 197-208.

[25] STEEL, R. G. D. (1961). Some rank sum multiple comparison tests. Biometrics 17, 539-552.

[26] TUKEY, J. W. (1953). The problem of multiple comparisons. Unpublished manuscript. Princeton Univ.

[27] WILCOXON, F. (1945). Individual comparioon by ranking methods. Biometrics 1, 80-83.

[28] WILKS, S. S. (1962). Mathematical Statistics. John Wiley and Sons, Inc. New York.