ON CERTAIN TYPES OF ASYMPTOTIC EQUIVALENCE OF REAL PROBABILITY DISTRIBUTIONS

II

FURTHER RESULTS ON THE PROPERTIES OF TYPE($\mathcal{S}$) ASYMPTOTIC EQUIVALENCE IN THE CASE OF EQUAL BASIC SPACES

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Summary

The author introduced in [1] some types of asymptotic equivalence of real probability distributions and discussed some of their properties. In the present article some additional results are stated on the properties of one of those types of asymptotic equivalence, type($\mathcal{S}$), in the case of equal basic spaces. In this case, type($\mathcal{S}$) and type ($\mathcal{M}$) asymptotic equivalence are the mutually equivalent notions, and hence the results obtained here are applicable equally to type ($\mathcal{M}$) asymptotic equivalence.

In the first section, some fundamental properties of type ($\mathcal{S}$) asymptotic equivalence are discussed in connection mainly with the usual in probability convergence. Section 2 is devoted to show that a finite linear sum of the marginal random variables preserves the type ($\mathcal{S}$) asymptotic equivalence of the original sequence of random variables.

In section 3, some results are shown on type ($\mathcal{S}$) asymptotic equivalence of marginal random variables when the other marginals are replaced in a sense by another random variables which are asymptotically equivalent ($\mathcal{S}$) to these marginals, or when some of them converge in probability in a certain manner.

1. Some of the fundamental properties of type(S) asymptotic equivalence.

Throughout this paper, the dimensions of the basic spaces are fixed independently of the limiting process under consideration.

As for the terminologies and definitions of type ($\mathcal{S}$) or type ($\mathcal{M}$) asymptotic equivalence , reference should be made to [1].

In the first place, we shall define two kinds of property of a sequence of real random variables, one is a sort of uniform absolute continuity of a sequence, and the other is a sort of boundedness in probability.

Let $\{X^s_{(n)} = (X^s_1, \ldots, X^s_n)\}(s = 1,2, \ldots)$ be a sequence of n-dimensional random variables defined over the euclidean space $(\mathfrak{R}_{(n)}, \mathfrak{B}_{(n)}, \mu_{(n)})$, where $\mathfrak{B}_{(n)}$ is the Borel field and $\mu_{(n)}$ stand for the ordinary Lebesgue measure over $(\mathfrak{R}_{(n)}, \mathfrak{B}_{(n)})$.

For this sequence we shall give the following definitions.

DEFINITION 1.1 Let N be any given positive integer. Then, a sequence of random variables, $\{X^s_{(n)}\}(s = 1,2, \ldots)$, is said to have the property $C_N(s)$, if, for any given $\epsilon > 0$, there exist a constant $\delta > 0$ and a positive integer s_o such that

$$\sup_{E_{(n)} \in \mathcal{S}_{(n)}(N,\delta)} P^{X^s_{(n)}}(E_{(n)}) < \epsilon \tag{1.1}$$

for all s not less than s_o, where $\mathcal{S}_{(n)}(N,\delta)$ is a class of subsets of $R_{(n)}$ defined by

$$\mathcal{S}_{(n)}(N,\delta) = \{E_{(n)} = \sum_{i=1}^{N} E_{(n)i} \mid \mu_{(n)}(E_{(n)}) < \delta, E_{(n)i} \cap E_{(n)j} = 0 (i \neq j), E_{(n)i} \in \mathcal{S}_{(n)}, \quad i=1, \ldots, N\}. \tag{1.2}$$

DEFINITION 1.2 (a) A sequence of random variables, $\{X^s_{(n)}\}$ ($s = 1,2, \ldots$), is said to have the property $B(\mathcal{S})$, if, for any given $\epsilon > 0$, there exist a positive integer s_o and a member, $E_{(n)}$, of $\mathcal{S}_{(n)}$, whose closure being compact, such that

$$(1.3) \qquad P^{X^s_{(n)}}(E_{(n)}) > 1-\epsilon$$

for all s not less than s_o, where $E_{(n)}$ does not depend on s.

(b) If, in the above definition (a), the subset $E_{(n)}$ is allowable to depend on s under the limiting, i.e., if, for any given $\epsilon > 0$, there exist a psoitive integer s_o and a member, $E^s_{(n)}$, of $\mathcal{S}_{(n)}$, whose closure being compact, such that

$$(1.4) \qquad P^{X^s_{(n)}}(E^s_{(n)}) > 1-\epsilon$$

for all s not less than s_o, then we say that the sequence $\{X^s_{(n)}\}$($s=1,2,\ldots,$) has the property $B^*(\mathcal{S})$.

For these definitions, it is easy to see the following results.

LEMMA 1.1 (a) If $N' < N$, and if $\{X^s_{(n)}\}$($s=1,2,\ldots$) has the property $C_N(\mathcal{S})$, then it has the property $C_{N'}(\mathcal{S})$.

(b) If $\{X^s_{(n)}\}$($s=1,2,\ldots$) and $\{Y^s_{(n)}\}$($s=1,2, \ldots$) are asymptotically equivalent ($\mathcal{S}$) as $s \to \infty$, and if one of them has the property $C_N(\mathcal{S})$, then the other has the same property, for any given N.

(c) If $\{X^s_{(n)}\}$($s=1,2, \ldots$) converges in law to some probability distribution of the continuous type (this is equivalent to the type ($\mathcal{S}$) convergence), then $\{X^s_{(n)}\}$($s=1,2, \ldots$) has the property $C_N(\mathcal{S})$ for any fixed N.

(d) Let $\{X^s_{(n)}\}(s=1,2, \ldots)$ and $\{Y^s_{(n)}\}(s=1,2, \ldots)$ be two sequences of random variables such that

$$Y_{(n)} = (Y^s_1, \ldots, Y^s_n), \ Y^s_i = c^s_i \, X^s_i + d^s_i, \ i=1, \ldots, n,$$

where $c^s_i \geq c^o_i$, for some positive constant c^o_i, $i = 1, \ldots, n$. Then, the property $C_N(\mathcal{S})$ of $\{X^s_{(n)}\}(s=1,2, \ldots)$ implies that of $\{Y^s_{(n)}\}(s=1,2, \ldots,)$ for every fixed N.

(e) If the sequence $\{X^s_{(n)}\}(s=1,2, \ldots)$ is such that the variables

$$(c_1 X^s_1 + d_1, \ c_2 \, X^s_2 + d_2, \ \ldots, \ c_n \, X^s_n + d_n),$$

$$s=1,2, \ldots$$

converges in law to some probability distribution of the continuous type, then the sequence

$$\{(c_1 \, X^s_1, \ \ldots, \ c_n \, X^s_n)\}(s=1,2,\ldots)$$

has the property $C_N(S)$ for any given N.

<u>LEMMA 1.2. (a)</u> If $\{X^s_{(n)}\}(s=1,2, \ldots)$ and $\{Y^s_{(n)}\}(s=1,2, \ldots)$ are asymptotically equivalent $(\mathcal{S})$ as $s \to \infty$, and if one of the sequences has the property $B(\mathcal{S})$, then the other has the same property. The same holds for the property $B^*(\mathcal{S})$ too.

(b) If $\{X^s_{(n)}\}(s=1,2, \ldots)$ converges in law to some probability distribution as $s \to \infty$, then the sequence has the property $B(\mathcal{S})$.

(c) Suppsoe that the sequence $\{X^s_{(n)}\}(s=1,2, \dots)$ has the property $B(\mathcal{S})$, and let $\{(c^s_1, \dots, c^s_n)\}$ and $\{(d^s_1, \dots, d^s_n)\}(s=1,2, \dots)$ be two sequences of real vectors. Let us consider the sequence $\{Y^s_{(n)}\}(s=1,2, \dots)$ defined by

$$(1.5) \qquad Y^s_{(n)} = (Y^s_1, \dots, Y^s_n),\ Y^s_i = c^s_i X^s_i + d^s_i,\ i=1, \dots, n$$

Then, if $\sum_{i=1}^{n} (c^s_i)^2$ and $\sum_{i=1}^{n} (d^s_i)^2$ are both bounded uniformly for all s, the sequence $\{Y^s_{(n)}\}(s=1,2, \dots)$ has the property $B(\mathcal{S})$. In general $\{Y^s_{(n)}\}(s=1,2,\dots)$ has the property $B^*(\mathcal{S})$.

(d) If $\{X^s_{(n)}\}(s=1,2, \dots)$ has the property $B^*(\mathcal{S})$, then, there can be found two sequences $\{(c^s_1, \dots, c^s_n)\}(s=1,2, \dots)$ and $\{d^s_1, \dots, d^s_n)\}$ $(s=1,2, \dots)$ of real vectors such that the sequence $\{Y^s_{(n)}\}(s=1,2, \dots)$ defined by (1.5) has the property $B(\mathcal{S})$.

Now, we shall show the following

<u>THEOREM 1.1</u> Let $\{(X^s_1, X^s_2)\}(s=1,2, \dots)$ be a sequence of two-dimensional random variables such that

(i) the first marginals, $\{X^s_1\}(s=1,2, \dots)$, has the property $C_1(\mathcal{S})$, and

(ii) the second marginals, $\{X^s_2\}(s=1,2, \dots)$, converges in probability to some constant λ.

Then, it holds that

$$(1.6) \qquad X^s_1 + X^s_2 \sim X^s_1 + \lambda\ (\mathcal{S}_{(1)}),\ (s \to \infty)$$

PROOF: Since

$$P(X_1^s + X_2^s < a) = P(X_1^s < a-X_2^s, \left| X_2^s-\lambda \right| > \delta) + P(X_1^s < a-X_2^s, \left| X_2^s-\lambda \right| \leq \delta)$$

and

$$\begin{cases} P(X_1^s < a-\lambda-\delta) \leq P(X_1^s < a-X_2^s, \left| X_2^s-\lambda \right| \leq \delta) \leq P(X_1^s < a-\lambda+\delta) \\ \qquad\qquad P(X_1^s < a-X_2^s, \left|X_2^s-\lambda\right| \leq \delta) \leq P(\left| X_2^s-\lambda\right| \leq \delta), \end{cases}$$

for any given $\delta > 0$, we have

$$(1.7) \left| P(X_1^s + X_2^s < a) - P(X_1^s + \lambda < a) \right| \leq P(a-\lambda-\delta \leq X_1^s < a-\lambda+\delta) + P(\left|X_2^s-\lambda\right| \geq \delta).$$

Since the sequence $\{X_1^s\}(s = 1,2, \dots)$ has the property $C_1(\mathcal{S})$, there exist, for any given $\epsilon > 0$, a constant $\delta_o > 0$ and an integer $s_o > 0$, such that

$$\sup_{E \,\epsilon\, \mathcal{S}_{(1)}(1,\delta_o)} \mathbf{p}^{X_1^s}(E) < \epsilon$$

for all $s \geq s_o$. Hence, if we choose δ in (1.7) such that $2\delta \leq \delta_o$, then

$$(1.8) \qquad \sup_{-\infty \leq a \leq \infty} \mathbf{p}(a-\lambda-\delta \leq X_1^s < a-\lambda+\delta) < \epsilon.$$

On the other hand, since X_2^s converges in probability to λ as $s \to \infty$, there exists, for such a choice of δ as above, a positive integer s_o' such that

(1.9) $$P(|X_2^s-\lambda| > \delta) < \epsilon$$

for all $s > s_o'$.

From (1.7), (1.8) and (1.9) it follows that

(1.10) $$\sup_{-\infty \leq a \leq \infty} \left| P(X_1^s + X_2^s < a) - P(X_1^s + \lambda < a)\right| < 2\epsilon,$$

for all $s \geq \max(s_o, s_o')$, which means that both of the sequences in (1.6) are asymptotically equivalent (**M**) as $s \to \infty$, and hence, in the sense of type (**S**).

This completes the proof of the theorem.

As a direct consequence of this theorem, we have the following

COROLLARY 1.1 (a) In the above theorem, if $\{X_1^s\}(s=1,2, \ldots)$ is asymptotically equivalent (**S**) to another sequence, $\{Y^s\}(s=1,2, \ldots)$, then it holds that

(1.11) $$X_1^s + X_2^s \sim Y^s + \lambda \ (S_{(1)}), \ (s \to \infty)$$

(b) In the above theorem, if we replace the condition (i) by the condition (i)' $\{X_1^s\}(s=1,2, \ldots)$ converges in law to some fixed distribution, Z say, of the continuous type,

Then it holds that

(1.12) $$X_1^s + X_2^s \sim Z + \lambda \ (S_{(1)}), \ (s \to \infty).$$

In the next place, we state and prove the following

THEOREM 1.2 Let $\{(x_1^s, X_2^s)\}(s=1,2, \ldots,)$ be a sequence of two-dimensional random variables satisfying the following conditions:

(i) $\{X_1^s\}(s=1,2, \ldots)$ has the property $C_1(\mathfrak{S})$ and $B(\mathfrak{S})$ simultaneously.

(ii) $\{X_2^s\}(s=1,2, \ldots)$ converges in probability to a non-zero constant λ.

Then, it holds that

$$\frac{X_1^s}{X_2^s} \sim \frac{X_1^s}{\lambda} \ (S_{(1)}), \ (s \to \infty) \tag{1.13}$$

and

$$X_1^s X_2^s \sim \lambda X_1^s \ (S_{(1)}), \ (s \to \infty) \tag{1.14}$$

PROOF: Since the in probability convergence of X_2^s to λ implies that of $1/X_2^s$ to $1/\lambda$, one may prove (1.13) only, and in doing this, there is no harm in assuming that $\lambda = 1$.

From the inequalities

$$\left| P(\frac{X_1^s}{X_2^s} < a) - P(X_1^s < a X_2^s, X_2^s > 0) \right| \leq P(X_2^s \leq 0)$$

and

$$\left| P(X_1^s < a X_2^s) - P(X_1^s < a X_2^s, \left| X_2^s - 1 \right| < \delta) \right| \leq P(\left| X_2^s - 1 \right| > \delta),$$

δ being any positive constant less than unity, it follows that

$$\left| P(\frac{X_1^s}{X_2^s} < a) - P(X_1^s < a) \right| \leq \sup_{|x-x'| < 2|a|\delta} (x \leq X_1^s < x') + P(X_2^s \leq 0) + P(\left|X_2^s - 1\right| > \delta) \tag{1.15}$$

Since $\{X_1^s\}(s=1,2, \ldots)$ has property $B(\mathcal{S})$, it can be assumed that, for any given $\delta>0$, there exists a constant $M(>0)$, such that

$$P(|X_1^s| \geq M) < \epsilon/4$$

for all $s \geq s_o$, for some s_o. Moreover, the property $C_1(\mathcal{S})$ of $\{X_1^s\}(s=1,2, \ldots)$ assures us that there exists a constant $\delta_o > 0$ such that

$$\sup_{|(x-x'| < \delta_o} P(x \leq X_1^s < x') < \epsilon/4$$

for all $s \geq s_o'$ for some s_o'.

Because of the in probability convergence of $\{X_2^s\}(s=1,2, \ldots)$ to unity, there exists a positive integer s_o'' such that

$$P(X_2^s \leq 0) < \epsilon/4 \text{ and } P(|X_2^s-1| > \frac{\delta_o}{2M}) < \epsilon/4,$$

for all $s \geq s_o''$.

Hence if we take the value of δ in (1.15) as $\delta = \delta_o/2M$, then it holds that

$$\sup_{-\infty \leq a \leq \infty} \left| P(\frac{X_1^s}{X_2^s} < a) - P(X_1^s < a) \right| < \epsilon \tag{1.16}$$

for all $s \geq \max(s_o, s_o', s_o'')$, which proves the theorem.

The following is straightforward from this theorem.

COROLLARY 1.2 In the above theorem, if we replace the condition (i) by the following

(i)' $\{X_1^s\}(s=1,2, \dots)$ converges in law to some probability distribution, Z say, of the continuous type,

Then it holds that

$$\frac{X_1^s}{X_2^s} \sim \frac{Z}{\lambda} \quad (\mathcal{S}_{(1)}), \ (s \to \infty) \tag{1.17}$$

and

$$X_1^s \, X_2^s \sim \lambda \, Z \ (\mathcal{S}_{(1)}), \ (s \to \infty). \tag{1.18}$$

Now, in the next place, we shall discuss another type of problem. We often meet with the following type of problem : Let $\{X_{(n)}^s\}(s=1,2, \dots)$ and $\{Y_{(n)}^s\}(s=1,2, \dots)$ be two sequences of random variables which are asymptotically equivalent $(\mathcal{S})$ as $s \to \infty$, and let $\{(\rho_1^s, \rho_2^s, \dots, \rho_n^s)\}(s=1,2, \dots)$ be a sequence of real n-vectors such that $\rho_i^s \to 1$, $i=1, \dots, n$. We are asked whether it is true or not that

$$X_{(n)}^s \sim Z_{(n)}^s \ (\mathcal{S}_{(n)}), \ (s \to \infty), \tag{1.19}$$

where

$$Z_{(n)}^s = (Z_1^s, \dots, Z_n^s), \ Z_i^s = \rho_i^s \, Y_i^s, \ i = 1, \dots, n; \ s=1,2, \dots . \tag{1.20}$$

The answer to this question would be 'no' in general, but, in some special cases, (1.19) is true. In fact, if we take the variable X_2^s in

Theorem 1.2 as the unit distribution whose mass point being ρ^s, then the theorem states that (1.19) holds true for the case when $n = 1$. This result can be shown to be true for the n-dimensional case.

<u>THEOREM 1.3</u> Let $\{X^s_{(n)}\}(s=1,2, \dots)$ and $\{Y^s_{(n)}\}(s=1,2, \dots)$be two sequences of random variables which are asymptotically equivalent ($\mathfrak{S}$) as $s\to\infty$, and let $\{Z^s_{(n)}\}(s=1,2, \dots)$ be the same as defined in (1.20) with the same vectors $(\rho^s_1, \dots, \rho^s_n)$ given there.

Suppose that one of the original sequences, $\{X^s_{(n)}\}(s=1,2, \dots)$ say, has the properties $C_n(\mathfrak{S})$ and $B(\mathfrak{S})$ simultaneously. Then (1.19) holds true.

<u>PROOF</u>: It is noted that $\{Y^s_{(n)}\}(s=1,2,\dots)$ has also the same properties as $\{X^s_{(n)}\}(s=1,2, \dots)$, and moreover, in order to prove (1.19) it sufficies to show the following

$$Y^s_{(n)}\sim Z^s_{(n)} \; (\mathfrak{S}_{(n)}), \; (s\to\infty). \tag{1.21}$$

Furthermore, there is no loss of generality in assuming that

$$\rho^s_i > 0, \; i = 1, \dots, n; \; s = 1,2, \dots .$$

Let $\epsilon > 0$ be any given constant. Then, the property $B(\mathfrak{S})$ of $\{Y^s_{(n)}\}(s=1,2, \dots,)$ assures the existence of a positive integer s_o and a member of $\mathfrak{S}_{(n)}$, $M_{(n)}$ say, whose closure being compact, such that

$$P^{Y^s_{(n)}}(M_{(n)}) > 1-\epsilon \tag{1.22}$$

for all $s \geq s_o$.

For any given $(a_1, \ldots, a_n)$, let us define as

$$E_{(n)}(a_1, \ldots, a_n) = \{x_{(n)} = (x_1, \ldots, x_n) \mid x_i < a_i,\ i = 1, \ldots, n\}.$$

Then, since

$$P^{Z^s_{(n)}}(E_{(n)}(a_1, \ldots, a_n)) = P^{Y^s_{(n)}}(E_{(n)}(a_1 \rho^s_1, \ldots, a_n \rho^s_n)),$$

we have

$$(1.23) \quad \left| P^{Z^s_{(n)}}(E_{(n)}(a_1, \ldots, a_n)) - P^{Y^s_{(n)}}(E_{(n)}(a_1, \ldots, a_n)) \right| \leq P^{Y^s_{(n)}}(A^s_{(n)}),$$

where we have put

$$A^s_{(n)} = E_{(n)}(a_1 + |a_1(\rho^s_1 - 1)|, \ldots, a_n + |a_n(\rho^s_n - 1)|) - E_{(n)}(a_1 - |a_1(\rho^s_1 - 1)|, \ldots, a_n - |a_n(\rho^s_n - 1)|).$$

Let us put

$$F^s_{(n)} = M_{(n)} \cap A^s_{(n)},$$

then, this must be a disjoint sum of at most n-members of $\mathcal{S}_{(n)}$, and be the empty set when $a_1, \ldots, a_n$ are sufficiently large. Thus, from (1.22) and (1.23) it follows that

$$(1.24) \quad \left| P^{Z^s_{(n)}}(E_{(n)}(a_1, \ldots, a_n)) - P^{Y^s_{(n)}}(E_{(n)}(a_1, \ldots, a_n)) \right| \leq P^{Y^s_{(n)}}(F^s_{(n)}) + \epsilon$$

for all $s \geq s_o$.

Since $\{Y^s_{(n)}\}(s=1,2, \ldots)$ has the property $C_n(\mathcal{S})$, there exists a positive integer s'_o such that

$$P^{Y^s_{(n)}}(F^s_{(n)}) < \epsilon$$

for all $s \geq s'_o$.

Hence, from (1.24) we obtain

$$\delta_{\mathcal{M}(n)} (Z^s_{(n)}, Y^s_{(n)}) < 2\,\epsilon \tag{1.25}$$

for all $s \geq \max(s_o, s'_o)$, which proves (1.21) and hence the theorem.

In the final place, we note that

<u>LEMMA 1.3</u> If $\{X^s_{(n)}\}(s=1,2, \ldots)$ and $\{Y^s_{(n)}\}(s=1,2, \ldots)$ are asymptotically equivalent $(\mathcal{S})$ as $s \to \infty$, and $\{X^s_{(n)}\}(s=1,2, \ldots)$ converges in probability to a point $\lambda_{(n)} = (\lambda_1, \ldots, \lambda_n)$, then $\{Y^s_{(n)}\}(s=1,2,\ldots)$ converges in probability to $\lambda_{(n)}$ as $s \to \infty$.

2. On a class of measurable transformations preserving type (S) asymptotic equivalence.

In this section we shall treat the following problem: Suppose $\{X^s_{(n)} = (X^s_1, \ldots, X^s_n)\}(s=1,2, \ldots)$ and $\{Y^s_{(n)} = (Y^s_1, \ldots, Y^s_n)\}(s=1,2, \ldots)$ are asymptotically equivalent ($\mathcal{S}$) as $s\to\infty$, and let

(2.1) $$f^s_{(n,1)}(z_{(n)}) = \sum_{i=1}^{n} \lambda^s_i z_i$$

be a transformation from $\Re_{(n)}$ to $\Re_{(1)}$, where $z_{(n)} = (z_1, \ldots, z_n)$, and $\{\lambda^s_1, \ldots, \lambda^s_n\}$ $(s=1,2,\ldots,)$ is a sequence of real vectors. Put

(2.2) $$U^s_{(1)} = f^s_{(n,1)}(X^s_{(n)}) \text{ and } V^s_{(1)} = f^s_{(n,1)}(Y^s_{(n)}),$$

for each s. Then, what conditions should be imposed in order that these two random variables are asymptotically equivalent in the sense of type ($\mathcal{S}$) as $s\to\infty$.

An answer is given by the following theorem.

THEOREM 2.1 Under the situation stated above, suppose the following condition is satisfied: One of the sequences $\{X^s_{(n)}\}(s=1,2,\ldots)$ and $\{Y^s_{(n)}\}(s=1,2,\ldots)$ has the properties $C_N(\mathcal{S})$ for any given N and $B(\mathcal{S})$ simultaneously. Then it holds that

(2.3) $$U^s_{(1)} \sim V^s_{(1)} \; (\mathcal{S}_{(1)}), \; (s\to\infty)$$

PROOF: Put

$$\rho^2_s = \sum_{i=1}^{n} (\lambda^s_i)^2, \; s=1,2,\ldots.$$

Then, it is clear that there exist a subsequence $\{s'\}$ of $\{s\}$ and a vector $(\mu_1, \ldots, \mu_n)$ such that

$$(2.4)\quad \mu_i^{s'} = \frac{\lambda_i^{s'}}{\rho_{s'}} \to \mu_i,\ (s' \to \infty),\ i=1,2,\ldots,n, \text{ and } \sum_{i=1}^{n} \mu_i^2 = 1$$

For our proof below, there is no harm in assuming that (2.4) holds for the original sequence $\{s\}$.

Now, put

$$(2.5)\qquad g^s_{(n,1)}(z_{(n)}) = \frac{1}{\rho_s} f^s_{(n,1)} = \sum_{i=1}^{n} \mu_i^s z_i,\ (s = 1,2, \ldots),$$

and

$$(2.6)\qquad \tilde{U}^s_{(1)} = g^s_{(n,1)}(X^s_{(n)}) \text{ and } \tilde{V}^s_{(1)} = g^s_{(n,1)}(Y^s_{(n)}),\ (s = 1,2, \ldots).$$

Then, by Theorem 4.1 of [1], it is seen that (2.3) is equivalent to

$$(2.7)\qquad \tilde{U}^s_{(1)} \sim \tilde{V}^s_{(1)}\ (S_{(1)}),\ (s \to \infty).$$

Suppose (2.7) is false. Then, without any loss of generality, we can assume that there exist a sequence of the members of $\mathcal{M}_{(1)}$, $\{E^s_{(1)}\}(s=1,2,\ldots)$, and a positive number η such that

$$(2.8)\qquad \left| P^{\tilde{U}^s_{(1)}}(E^s_{(1)}) - P^{\tilde{V}^s_{(1)}}(E^s_{(1)}) \right| > \eta,\ (s = 1, 2, \ldots).$$

For this sequence $\{E^s_{(1)}\}(s=1,2, \ldots)$, let us put

$$G^s_{(n)} = g^{s^{-1}}_{(n,1)} (E^s_{(1)}), \ (s = 1, 2, \dots).$$

Since, by the assumption of the theorem, Lemmas 1.1 and 1.2, for any given $\epsilon > 0$, there exists a member $B_{(n)}$ of $\mathcal{S}_{(n)}$, whose closure being compact, such that both of the inequalities

$$(2.9) \qquad P^{X^s_{(n)}}(B_{(n)}) > 1 - \epsilon \text{ and } P^{Y^s_{(n)}}(B_{(n)}) > 1 - \epsilon$$

hold simultaneously for sufficiently large values of s. Let us put

$$A^s_{(n)} = G^s_{(n)} \cap B_{(n)}.$$

Then, by (2.4) and the compactness of the closure of $B_{(n)}$, for any given $\delta > 0$, there exist positive integers N and N' ($N<N'$) depending only on δ, and a set of N' mutually disjoint members of $\mathcal{S}_{(n)}$, $\{E_{(n)1}, \dots, E_{(n)N'}\}$ say, such that

$$(2.10) \qquad \sum_{k=1}^{N} E_{(n)k} \subseteq A^s_{(n)} \subseteq \sum_{k=1}^{N'} E_{(n)k}$$

for sufficiently large s, and

$$(2.11) \qquad \mu_{(n)}\left(\sum_{k=N+1}^{N'} E_{(n)k}\right) < \delta.$$

But, since $\{X^s_{(n)}\}(s=1,2, \dots)$ and hence $\{Y^s_{(n)}\}(s=1,2, \dots)$ were assumed to have the property $C_N(S)$ for any fixed N, for ϵ given in (2.9) there can be found a positive number $\delta_o > 0$ such that, for large s,

$$P^{X^s_{(n)}}\left(\sum_{k=N+1}^{N'} E_{(n)k}\right) < \epsilon \text{ and } P^{Y^s_{(n)}}\left(\sum_{k=N+1}^{N'} E_{(n)k}\right) < \epsilon$$

if we choose δ in (2.11) such that $\delta \leq \delta_o$.

Hence from (2.10) we have

$$(2.12) \qquad \left| P^{Y^s_{(n)}}\left(\sum_{k=1}^{N} E_{(n)k}\right) - P^{X^s_{(n)}}(A^s_{(n)}) \right| < \epsilon$$

and

$$(2.13) \qquad \left| P^{Y^s_{(n)}}\left(\sum_{k=1}^{N} E_{(n)k}\right) - P^{Y^s_{(n)}}(A^s_{(n)}) \right| < \epsilon$$

for sufficiently large s. Furthermore, from (2.9) it follows that

$$(2.14) \qquad \left| P^{X^s_{(n)}}(A^s_{(n)}) - P^{X^s_{(n)}}(G^s_{(n)}) \right| < \epsilon$$

and

$$(2.15) \qquad \left| P^{Y^s_{(n)}}(A^s_{(n)}) - P^{Y^s_{(n)}}(G^s_{(n)}) \right| < \epsilon$$

for sufficiently large s.

Thus, by (2.12), (2.13), (2.14) and (2.15), we have

$$(2.16) \left| P^{X^s_{(n)}}(E^s_{(n)}) - P^{Y^s_{(n)}}(E^s_{(n)}) \right| < 4\,\epsilon + \left| P^{X^s_{(n)}}\left(\sum_{k=1}^{N} E_{(n)k}\right) - P^{Y^s_{(n)}}\left(\sum_{k=1}^{N} E_{(n)k}\right) \right|$$

$$\leq 4\,\epsilon + N.\ \delta_{\mathcal{S}(n)}\ (X^s_{(n)}, Y^s_{(n)}).$$

Since $\delta_{\mathfrak{S}(n)}\ (X^s_{(n)}, Y^s_{(n)}) \to o\ (s \to \infty)$, it follows from (2.16) that

$$(2.17) \qquad \left| P^{X^s_{(n)}}(E^s_{(n)}) - P^{Y^s_{(n)}}(E^s_{(n)}) \right| < 5\ \epsilon$$

for sufficiently large values of s, or

$$(2.18) \qquad \left| P^{\tilde{U}^s_{(1)}}(E^s_{(1)}) - P^{\tilde{V}^s_{(1)}}(E^s_{(1)}) \right| < 5\ \epsilon\ .$$

If we choose ϵ small, (2.18) contradicts (2.8), which proves the theorem.

<u>COROLLARY 2.1</u> If $\{X^s_{(n)}\}(s=1,2,\ \dots\ )$ converges in law to some $Y_{(n)} = (Y_1,\ \dots,\ Y_n)$ of the continuous type, then it holds that

$$(2.19) \qquad \sum_{i=1}^{n} \lambda^s_i\ X^s_i \sim \sum_{i=1}^{n} \lambda^s_i\ Y_i\ \ (S_{(1)}),\ (s \to \infty)$$

for any sequence of real vectors $\{(\lambda^s_i,\ \dots,\ \lambda^s_n)\}(s=1,2,\ \dots\ )$.

This follows from Lemma 1.1(c), Lemma 1.2(b) and the above theorem.

The problem we discussed in this section originates in the following application.

<u>EXAMPLE</u>: Let $Y_{(n)} = (Y_1,\ \dots,\ Y_n)$ be a random variable which is distributed according to the n-dimensional independent normal distribution $N(0,\ I_n)$, and suppose that a sequence, $\{X^s_{(n)} = (X^s_1,\ \dots,\ X^s_n)\}(s=1,2,\ \dots,)$, converges in law to $Y_{(n)}$ as $s \to \infty$. Furthermore, let $\{(\lambda^s_1,\ \dots,\ \lambda^s_n)\}(s=1,2,\ \dots)$ be a sequence of real vectors such that $\sum_{i=1}^{n} (\lambda^s_i)^2 \to \infty$ as $s \to \infty$.

Then, the variables

$$(2.20) \qquad \tilde{X}^s = \sum_{i=1}^{n} (X^s_i + \lambda^s_i)^2,\ s = 1,2,\ \dots$$

are shown to be asymptotically equivalent ($\mathfrak{P}$) as $s\to\infty$ to the non-central chi-square distribution of degrees of freedom n with the non-centrality parameter $\rho_s^2 = \sum_{i=1}^{n} (\lambda_i^s)^2$.

In fact, this is shown as follows: Put

$$\tilde{Y}^s = \sum_{i=1}^{n}(Y_i + \lambda_i^s)^2 , \tag{2.21}$$

then, this is distributed according to the non-central chi-square distribution of degrees of freedom n with non-centrality parameter ρ_s^2.

Now, since

$$\tilde{X}^s = \sum_{i=1}^{n}(X_i^s)^2 + 2 \sum_{i=1}^{n} \lambda_i^s X_i^s + \rho_s^2$$

and

$$\tilde{Y}^s = \sum_{i=1}^{n} Y_i^2 + 2 \sum_{i=1}^{n} \lambda_i^s Y_i + \rho_s^2,$$

the asymptotic equivalence

$$\tilde{X}^s \sim \tilde{Y}^s \ (\mathfrak{P}_{(1)}), \ (s\to\infty), \tag{2.22}$$

is equivalent to

$$\sum_{i=1}^{n}(X_i^s)^2 + 2 \sum_{i=1}^{n} \lambda_i^s X_i^s \sim \sum_{i=1}^{n} Y_i^2 + 2 \sum_{i=1}^{n} \lambda_i^s Y_i, \ (\mathfrak{P}_{(1)}), \ (s\to\infty). \tag{2.23}$$

Moreover, the two dimensional random variables, $\{(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} \lambda_i^s Y_i)\}(s=1,2,\dots)$ has the property $C_N(\mathfrak{P})$ for any given N. Hence, if it holds that

$$(2.24)\quad \left(\sum_{i=1}^{n}(X_i^s)^2, \sum_{i=1}^{n} \lambda_i^s X_i^s\right) \sim \left(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} \lambda_i^s Y_i\right) \quad (S_{(2)}), \quad (s \to \infty)$$

or equivalently

$$(2.25)\quad \left(\sum_{i=1}^{n} (X_i^s)^2, \sum_{i=1}^{n} \mu_i^s X_i^s\right) \sim \left(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} \mu_i^s Y_i\right) (S_{(2)}), \ (s \to \infty)$$

with $\mu_i^s = \lambda_i^s / \rho_s$, $i=1,\ldots, n$, assuming that the sequence $\{(\mu_1^s, \ldots, \mu_n^s)\}(s=1,2,\ldots)$ is convergent, then Theorem 4.4 of [1] is applicable to show that (2.23) holds true.

It is not so difficult, by using an analogous method to the proof of Theorem 2.1 above, to prove (2.25).

3. Asymptotic equivalence ($\mathcal{S}$) of marginal random variables.

We shall be concerned, in this section, with the following type of problem: Let $\{X^s_{(k)}, Y^s_{(\ell)}, Z^s_{(m)}\}(s=1,2, \ldots)$ be a sequence of $n(=k+\ell+m)$ dimensional random variables, where k, ℓ and m are fixed independently of s, and let

$$(3.1) \qquad H_s(z_{(m)}) = \int_{\Re_{(k)} \times \Re_{(\ell)}} F_s(z_{(m)} \mid x_{(k)}, y_{(\ell)}) \, d\, G_s(x_{(k)}, y_{(\ell)})$$

be the cumulative distribution function of the marginal $Z^s_{(m)}$, where $F_s(z_{(m)} \mid x_{(k)}, y_{(\ell)})$ with $z_{(m)} = (z_1, \ldots, z_m)$, $x_{(k)} = (x_1, \ldots, x_k)$ and $y_{(\ell)} = (y_1, \ldots, y_\ell)$, and $G_s(x_{(k)}, y_{(\ell)})$ stands for the conditional cumulative distribution function of $Z^s_{(m)}$ given $(X^s_{(k)}, Y^s_{(\ell)}) = (x_{(k)}, y_{(\ell)})$ and the cumulative distribution function of $(X^s_{(k)}, Y^s_{(\ell)})$, respectively.

How, suppose that the following conditions are satisfied:

(i) The marginals $\{X^s_{(k)}\}(s=1,2, \ldots)$ are asymptotically equivalent ($\mathcal{S}$) to some other sequence of random variables, $\{X^s_{(k)}\}(s=1,2, \ldots)$ say.

(ii) For the marginals $\{Y^s_{(\ell)}\}(s=1,2, \ldots)$ there can be found a sequence of real vectors, $\{\lambda^s_{(\ell)} = (\lambda^s_1, \ldots, \lambda^s_\ell)\}(s=1,2, \ldots,)$, such that the sequence

$$\{(\frac{Y^s_1}{\lambda^s_1}, \ldots, \frac{Y^s_\ell}{\lambda^s_\ell})\}(s= 1,2, \ldots)$$

converges in probability to the point $(1, \ldots, 1)$ as $s \to \infty$, where $Y^s_{(\ell)} = (Y^s_1, \ldots, Y^s_\ell)$. Let us designate the cumulative distribution function of $\widetilde{X}^s_{(k)}$ by $\widetilde{G}_s(x_{(k)})$, and put

(3.2) $$\tilde{H}_s(z_{(m)}) = \int_{\mathfrak{R}_{(k)}} F_s(z_{(m)}, \lambda^s_{(\ell)})\, d\, \tilde{G}_s(x_{(k)}).$$

Then, clearly this is a cumulative distribution function of some m-dimensional random variable, $\tilde{Z}^s_{(m)}$ say.

Under the situation stated above, what additional conditions should be imposed in order that the asymptotic equivalence

(3.3) $$Z^s_{(m)} \sim \tilde{Z}^s_{(m)}(\beta_{(m)}),\ (s \to \infty)$$

holds true?

Answers in the special cases when m = 1; k = 1,2; $\ell = 0$ has been given in [2].

In the first place, we shall state and prove the following theorem, which is an extension of Theorems 3 and 4 of [2].

THEOREM 3.1 Let $\{(X^s_{(k)}, Z^s_{(m)})\}(s=1,2, \dots)$ be a sequence of n-dimensional random variables (n = k + m) for which k and m are fixed independently of s, and let $H_s(z_{(m)})$, $F_s(z_{(m)} \mid x_{(k)})$ and $G_s(x_{(k)})$ be the cumulative distribution functions of $Z^s_{(m)}$, of the conditional variable $\bar{Z}^s_{(m)}$ given $X^s_{(k)}=x_{(k)}$, and of $X^s_{(k)}$, respectively.

Suppose the following conditions are satisfied:

(i) For the marginals $\{X^s_{(k)} = (X^s_1, \dots, X^s_k)\}(s=1,2,\dots)$, there exists another sequence of random variables $\{\tilde{X}^s_{(k)} = (\tilde{X}^s_1, \dots, \tilde{X}^s_k)\}(s=1,2,\dots)$, which is asymptotically equivalent (β) to $\{X^s_{(k)}\}(s=1,2, \dots)$. For this sequence $\{\tilde{X}^s_{(k)}\}(s=1,2, \dots)$ there can be found two sequences of real vectors, $\{c^s_{(k)} = (c^s_1, \dots, c^s_k)\}(s=1,2, \dots)$ with $c^s_i > 0$ (i = 1, ..., k), and $\{d^s_{(k)}=(d^s_1,\dots,d^s_k)\}$ (s=1,2, ...) such that the sequence of random variables $\{\bar{U}^s_{(k)}=\bar{U}^s_1,\dots,\bar{U}^s_k)\}(s=1,2,\dots)$

given by

$$(3.4) \qquad U_i^s = \frac{1}{c_i^s}(\widetilde{X}_i^s - d_i^s),\ i = 1, \ldots, k;\ s = 1,2, \ldots$$

converges in the sense of type ($\mathcal{S}$) to some fixed k-dimensional random variables $U_{(k)}$ as $s \to \infty$.

(ii) The conditional cumulative distribution $F_s(z_{(m)} | x_{(k)})$ is continuous in $(z_{(m)}, x_{(k)})$ jointly over $\mathfrak{R}_{(m)} \times \mathfrak{R}_{(k)}$.

Then, it holds that

$$(3.5) \qquad Z_{(m)}^s \sim \widetilde{Z}_{(m)}^s(\mathcal{S}_{(m)}),\ (s \to \infty)$$

where $Z_{(m)}^s$ is the marginal of $(X_{(k)}^s, Z_{(m)}^s)$ and $\widetilde{Z}_{(m)}^s$ stands for a random variable whose cumulative distribution function is given by

$$(3.6) \qquad \widetilde{H}_s(z_{(m)}) = \int_{\mathfrak{R}_{(k)}} F_s(z_{(m)} | x_{(k)})\, d\,\widetilde{G}_s(x_{(k)}),$$

$\widetilde{G}_s(x_{(k)})$ being the cumulative distribution function of $\widetilde{X}_{(k)}^s$.

<u>PROOF</u>: Firstly, it is noted that the sequence of random variables $\{U_{(k)}^s\}(s=1,2, \ldots)$ given by (3.4) has the property $B(\mathcal{S})$.

The proof of the theorem goes quite similarly to that of Theorem 3 of [2].

In order to show (3.5), it is sufficient to prove that

$$(3.7) \qquad \sup_{z_{(m)} \in \mathfrak{R}_{(m)}} \left| H_s(z_{(m)}) - \widetilde{H}_s(z_{(m)}) \right| \to 0,\ (s \to \infty),$$

because, in the case of equal basic spaces, type ($\mathfrak{B}$) and type ($\mathfrak{M}$) asymptotic equivalence are mutually equivalent.

Since, for each s,

$$0 \leq \left| H_s(z_{(m)}) - \widetilde{H}_s(z_{(m)}) \right| \leq 1 \quad \text{on } \Re_{(m)}$$

and

$$\left| H_s(z_{(m)}) - \widetilde{H}_s(z_{(m)}) \right| \to 0$$

as $z_i \to -\infty$ for some i or as $z_i \to +\infty$ for all i, i = 1, ..., m, there exists a point $z^s_{(m)} = (z^s_1, \ldots, z^s_m)$ such that

$$(3.8) \quad \left| H_s(z^s_{(m)}) - \widetilde{H}_s(z^s_{(m)}) \right| = \sup_{z_{(m)} \in \Re_{(m)}} \left| H_s(z_{(m)}) - \widetilde{H}_s(z_{(m)}) \right|, \quad s=1,2, \ldots ,$$

where $z^s_{(m)}$ is allowable to take the value $+\infty$ as some but not all of its components.

Since the cumulative distribution function of $U^s_{(k)}$ is given by

$$\widetilde{G}_s(x^s_{(k)}) \text{ with } x^s_{(k)} = (c^s_1 x_1 + d^s_1, \ldots, c^s_k x_k + d^s_k)$$

for each s, it follows that

$$(3.9) \quad \sup_{x_{(k)} \in \Re_{(k)}} \left| \widetilde{G}_s(x^s_{(k)}) - \widetilde{G}(x_{(k)}) \right| \to 0, \quad (s \to \infty)$$

where $x_{(k)} = (x_1, \ldots, x_k)$ and $\check{G}(x_{(k)})$ stands for the cumulative distribution function of $U_{(k)}$. It also follows from the condition (i) of the theorem that

$$(3.10) \qquad \sup_{x_{(k)} \in \mathfrak{R}_{(k)}} \left| G_s(x^s_{(k)}) - \check{G}(x_{(k)}) \right| \to 0, \ (s \to \infty).$$

Now, we can see that

$$H_s(z^s_{(m)}) - \widetilde{H}_s(z^s_{(m)})$$

$$(3.11) \quad = \int_{\mathfrak{R}_{(k)}} F_s(z^s_{(m)} | x_{(k)}) \, d \ G_s(x_{(k)}) - \int_{\mathfrak{R}_{(k)}} F_s(z^s_{(m)} | x_{(k)}) \, d \ \widetilde{G}_s(x_{(k)})$$

$$= \int_{\mathfrak{R}_{(k)}} F_s(z^s_{(m)} | x^s_{(k)}) \, d \ G_s(x^s_{(k)}) - \int_{\mathfrak{R}_{(k)}} F_s(z^s_{(m)} | x^s_{(k)}) \, d \ \widetilde{G}_s(x^s_{(k)}).$$

Since the function of the variable $x_{(k)}$, $F_s(z^s_{(m)} | x^s_{(k)})$, is bounded and continuous over $\mathfrak{R}_{(k)}$, there exist a convergent subsequence of $\{F_s(z^s_{(m)} | x^s_{(k)})\}$ $(s=1,2,\ldots,)$, $\{F_{s'}(z^{s'}_{(m)} | x^{s'}_{(m)})\}(s' \to \infty)$ say, and a limit function $\varphi_o(x_{(k)})$ such that $0 \leq \varphi_o(x_{(k)}) \leq 1$ for all $x_{(k)}$ in $\mathfrak{R}_{(k)}$, and $\varphi_o(x_{(k)})$ is continuous over $\mathfrak{R}_{(k)}$. Moreover, the convergence

$$F_{s'}(z^{s'}_{(m)} | x^{s'}_{(k)}) \to \varphi_o(x_{(k)}), \ (s' \to \infty)$$

can be regarded to be uniform on any given compact subset of $\mathfrak{R}_{(k)}$.

Since the sequence $\{U^s_{(k)}\}(s=1,2,\ldots)$, and hence $U_{(k)}$, has the property $B(\mathfrak{B})$, there exists, for any given $\epsilon > 0$, a member of $\mathfrak{B}_{(k)}$, $B_{(k)}$ say, whose closure being compact, such that the inequalities

$$P^{U^s_{(k)}}(B_{(k)}) > 1-\epsilon,\ P^{U_{(k)}}(B_{(k)}) > 1-\epsilon \text{ and } P^{\overline{X}^s_{(k)}}(B_{(k)}) > 1-\epsilon$$

hold simultaneously for sufficiently large values of s, where

$$(3.12) \quad \overline{X}^s_{(k)} = (\overline{X}^s_1, \ldots, \overline{X}^s_k) \text{ with } \overline{X}^s_i = \frac{1}{c^s_i}(X^s_i - d^s_i),\ i = 1, \ldots, k,$$

whose cumulative distribution function being $G_s(x^s_{(k)})$. Hence, it holds that, for large s,

$$(3.13) \left| \int_{\mathfrak{R}_{(k)}} F_s(z^s_{(m)} | x^s_{(k)})\, d\, G_s(x^s_{(k)}) - \int_{B_{(k)}} F_s(z^s_{(m)} | x^s_{(k)})\, d\, G_s(x^s_{(k)}) \right| < \epsilon$$

and

$$(3.14) \left| \int_{\mathfrak{R}_{(k)}} F_s(z^s_{(m)} | x^s_{(k)})\, d\, \widetilde{G}_s(x^s_{(k)}) - \int_{B_{(k)}} F_s(z^s_{(m)} | x^s_{(k)})\, d\, \widetilde{G}_s(x^s_{(k)}) \right| < \epsilon.$$

On the other hand, we have

$$(3.15) \left| \int_{B_{(k)}} F_{s'}(z^{s'}_{(m)} | x^{s'}_{(k)})\, d\, G_{s'}(x^{s'}_{(k)}) - \int_{B_{(k)}} \varphi_o(x_{(k)})\, d\, G_{s'}(x^{s'}_{(k)}) \right| < \epsilon$$

and

$$(3.16) \left| \int_{B_{(k)}} F_{s'}(z^{s'}_{(m)} | x^{s'}_{(k)})\, d\, \widetilde{G}_{s'}(x^{s'}_{(k)}) - \int_{B_{(k)}} \varphi_o(x_{(k)})\, d\, \widetilde{G}_{s'}(x^{s'}_{(k)}) \right| < \epsilon$$

for sufficiently large s'.

But, it is shown by using the multi-dimensional extension of the Helly-Bray Theorem [3], (3.9) and (3.10) we get

$$(3.17)\quad \left| \int_{B_{(k)}} \varphi_o(x_{(k)})\, d\, G_s(x^s_{(k)}) - \int_{B_{(k)}} \varphi_o(x_{(k)})\, d\, \widetilde{G}(x_{(k)}) \right| < \epsilon$$

and

$$(3.18)\quad \left| \int_{B_{(k)}} \varphi_o(x_{(k)})\, d\, \widetilde{G}_s(x^s_{(k)}) - \int_{B_{(k)}} \varphi_o(x_{(k)})\, d\, \widetilde{G}(x_{(k)}) \right| < \epsilon$$

for large values of s.

From (3.10), (3.13), (3.14), (3.15), (3.16), (3.17) and (3.18), we obtain

$$(3.19)\qquad \left| H_{s'}(z^{s'}_{(m)}) - \widetilde{H}_{s'}(z^{s'}_{(m)}) \right| < 6\,\epsilon$$

for sufficiently large values of s', which means that

$$(3.20)\qquad \sup_{z_{(m)} \in \mathfrak{R}_{(m)}} \left| H_{s'}(z_{(m)}) - \widetilde{H}_{s'}(z_{(m)}) \right| \to 0, \ (s' \to \infty).$$

Hence (3.7) should be true, which proves the theorem.

The following is a direct consequence of this theorem.

COROLLARY 3.1 Under the same situation as in the theorem, suppose that the marginal $Z^s_{(m)}$ is of the continuous type, i.e., is absolutely continuous with respect to the Lebesque measure $\mu_{(m)}$ over $(\mathfrak{R}_{(m)}, B_{(m)})$ for each s, whose probability element being given by

$$(3.21) \qquad p_s(z_{(m)}| x_{(k)}) \, d \, G_s \, (x_{(k)}) \, d \, z_{(m)},$$

where $p_s(z_{(m)}| x_{(k)})$ designates the conditional probability density function given $X^s_{(k)} = x_{(k)}$.

Suppose the following conditions are satisfied:

(i) $\{X^s_{(k)}\}(s=1,2, \dots)$ satisfies the same condition as (i) of the theorem.

(ii) $p_s(z_{(m)}| x_{(k)})$ is continuous in $(z_{(m)}, x_{(k)})$ jointly over $\mathfrak{R}_m \times \mathfrak{R}_{(k)}$.

Then, it holds that

$$(3.22) \qquad Z^s_{(m)} \sim \tilde{Z}^s_{(m)} \; (\mathcal{S}_{(m)}), \; (s \to \infty),$$

where $\tilde{Z}^s_{(m)}$ stands for a variable whose probability element being

$$(3.23) \qquad p_s(z_{(m)}| x_{(k)}) \, d \, \tilde{G}_s \, (x_{(k)}) \, d \, z_{(m)}.$$

Now, in the next place, we shall prove the following theorem.

THEOREM 3.2 Let $\{(Y^s_{(\ell)}, Z^s_{(m)})\}(s=1,2, \dots)$ be a sequence of n-dimensional random variables $(n = \ell + m)$, for which it is assumed that

(i) the conditional cumulative distribution function of $Z^s_{(m)}$

given $Y^s_{(\ell)} = y_{(\ell)}$, $F_s(z_{(m)}|y_{(\ell)})$ is continuous in $(z_{(m)}, y_{(\ell)})$ jointly over $\Re_{(n)}$ for each s, and

(ii) for the marginals $\{Y^s_{(\ell)}\}(s=1,2, \dots)$, there exists a sequence of real vectors $\{\lambda^s_1, \dots, \lambda^s_\ell)\}(s=1,2, \dots)$ such that $\lambda^s_i > 0$, $i=1,\dots,\ell$, and the sequence $\{\overline{Y}^s_{(\ell)} = (Y^s_1/\lambda^s_1, \dots, Y^s_\ell/\lambda^s_\ell)\}(s=1,2, \dots)$ converges in probability to $(1, \dots, 1)$ as $s \to \infty$.

Then, two random variables, $Z^s_{(m)}$ and $\widetilde{Z}^s_{(m)}$, whose cumulative distribution functions being

$$(3.24) \qquad H_s(z_{(m)}) = \int_{\Re_{(\ell)}} F_s(z_{(m)}|y_{(\ell)}) \, d\, G_s\,(y_{(\ell)})$$

where $G_s(y_{(\ell)})$ stands for the cumulative distribution function of $Y^s_{(\ell)}$, and

$$(3.25) \qquad \widetilde{H}_s(z_{(m)}) = F_s(z_{(m)}|\lambda^s_{(\ell)}), \text{ with } \lambda^s_{(\ell)} = (\lambda^s_1, \dots, \lambda^s_\ell),$$

are asymptotically equivalent ($\mathcal{S}$) as $s \to \infty$.

<u>PROOF</u>: For $y_{(\ell)} = (y_1, \dots, y_\ell)$, put $y^s_{(\ell)} = (\lambda^s_1\, y_1, \dots, \lambda^s_\ell\, y_\ell)$. Then the cumulative distribution function of $\overline{Y}^s_{(\ell)}$ is given by $G_s(y^s_{(\ell)})$, and it becomes that

$$(3.26) \qquad H_s(z_{(m)}) = \int_{\Re_{(\ell)}} F_s(z_{(m)}|\, y^s_{(\ell)}) \, d\, G_s\,(y^s_{(\ell)}).$$

For each s, there exists a point $z^s_{(m)}$ such that

(3.27) $$\left| H_s(z^s_{(m)}) - \widetilde{H}_s(z^s_{(m)}) \right| = \sup_{z_{(m)} \in \Re_{(m)}} \left| H_s(z_{(m)}) - \widetilde{H}_s(z_{(m)}) \right|$$

The left-hand member of this equality is, for any given $\delta > 0$,

$$\left| H_s(z^s_{(m)}) - \widetilde{H}_s(z^s_{(m)}) \right|$$

$$= \left| \int_{\Re_{(\ell)}} \{F_s(z^s_{(m)} | y^s_{(\ell)}) - F_s(z^s_{(m)} | \lambda^s_{(\ell)})\} \, d\, G_s(y^s_{(\ell)}) \right|$$

(3.28) $$\leqq \int_{|y_{(\ell)} - 1_{(\ell)}| \leq \delta} \left| F_s(z^s_{(m)} | y^s_{(\ell)}) - F_s(z^s_{(m)} | \lambda^s_{(\ell)}) \right| d\, \overline{G}_s(y_{(\ell)})$$

$$+ \int_{|y_{(\ell)} - 1_{(\ell)}| > \delta} \left| F_s(z^s_{(m)} | y^s_{(\ell)}) - F_s(z^s_{(m)} | \lambda^s_{(\ell)}) \right| d\, \overline{G}_s(y_{(\ell)}) ,$$

where we have put $1_{(\ell)} = (1, \ldots, 1)$ and $\overline{G}_s(y_{(\ell)}) = G_s(y^s_{(\ell)})$.

For each s, the function

$$\varphi_s(y_{(\ell)}) = F_s(z^s_{(m)} | y^s_{(\ell)}) - F_s(z^s_{(m)} | \lambda^s_{(\ell)}).$$

is continuous in $y_{(\ell)}$ and bounded over $\Re_{(\ell)}$, and consequently, the sequence $\{\varphi_s(y_{(\ell)})\}(s=1,2, \ldots)$ has a uniformly convergent subsequence on the compact domain $\{y_{(\ell)} \mid |y_{(\ell)} - 1_{(\ell)}| \leq \delta\}$, i.e., there exist a subsequence $\{s'\}$ (which may depend on δ) of $\{s\}$ and a limit function $\varphi_o(y_{(\ell)})$, defined over $\Re_{(\ell)}$, such that

$$(3.29) \quad \sup_{|y_{(\ell)}-1_{(\ell)}|<\delta} \left| \varphi_{s'}(y_{(\ell)}) - \varphi_o(y_{(\ell)})\right| \to 0, \ (s' \to \infty).$$

It is clear that $\varphi_o(y_{(\ell)})$ is continuous and bounded over $\Re_{(\ell)}$ and $\varphi_o(1_{(\ell)}) = 0$, and, for any given $\epsilon > 0$, we can choose $\delta_o > 0$ such that

$$(3.30) \qquad \sup_{|y_{(\ell)}-1_{(\ell)}|<\delta_o} \left| \varphi_o(y_{(\ell)})\right| < \epsilon$$

Hence, for the first member of the last expression of (3.28), we have

$$(3.31) \quad \begin{aligned} &\int_{|y_{(\ell)}-1_{(\ell)}|\le\delta_o} \left| F_s(z^{s'}_{(m)}| y^{s'}_{(\ell)}) - F_{s'}(z^{s'}_{(m)}| \lambda^{s'}_{(\ell)})\right| d\,\overline{G}_{s'}(y_{(\ell)}) \\ &= \int_{|y_{(\ell)}-1_{(\ell)}|\le\delta_o} \left| \varphi_{s'}(y_{(\ell)})\right| d\,\overline{G}_{s'}(y_{(\ell)}) < 2\,\epsilon \end{aligned}$$

for sufficiently large values of s'.

For the second member of the last expression of (3.28), it is clear that

$$(3.32) \quad \begin{aligned} &\int_{|y_{(\ell)}-1_{(\ell)}|>\delta_o} \left| F_{s'}(z^{s'}_{(m)}| y^{s'}_{(\ell)}) - F_{s'}(z^{s'}_{(m)}| \lambda^{s'}_{(\ell)})\right| d\,\overline{G}_{s'}(y_{(\ell)}) \\ &\le \int_{|y_{(\ell)}-1_{(\ell)}|>\delta_o} d\,\overline{G}_{s'}(y_{(\ell)}) < \epsilon \end{aligned}$$

for sufficiently large values of s'.

It follows from (3.31) and (3.32) that

$$(3.33) \qquad \sup_{z_{(m)} \in \Re_{(m)}} \left| H_{s'}(z_{(m)}) - \tilde{H}_{s'}(z_{(m)}) \right| \to 0, \quad (s' \to \infty).$$

Hence, if we assume that

$$(3.34) \qquad \sup_{z_{(m)} \in \Re_{(m)}} \left| H_s(z_{(m)}) - H_s(z_{(m)}) \right| \not\to 0, \quad (s \to \infty),$$

then this leads us to a contradiction, which proves the theorem.

The following is a direct consequence of this theorem.

COROLLARY 3.2 In the above theorem, if, instead of condition (i), we impose the following

(i)' the conditional probability density function of $Z^s_{(m)}$ given $Y^s_{(\ell)} = y_{(\ell)}$, $p_s(z_{(m)} | y_{(\ell)})$ is continuous in $(z_{(m)}, y_{(\ell)})$ jointly over $\Re_{(n)}$, for each s.

Then, two random variables, $Z^s_{(m)}$ and $\tilde{Z}^s_{(m)}$, whose probability density function being respectively given by

$$(3.35) \qquad h_s(z_{(m)}) = \int_{\Re_{(\ell)}} p_s(z_{(m)} | y_{(\ell)}) \, d\, G_s(y_{(\ell)})$$

and

$$(3.36) \qquad \tilde{h}_s(z_{(m)}) = p_s(z_{(m)} | \lambda^s_{(\ell)}) ,$$

are asymptotically equivalent ($\mathcal{S}$) as $s \to \infty$.

In the final place, we state the following theorem without proof, which is a easy consequence of the above two theorems.

THEOREM 3.3 As was stated in the beginning of this section, let $\{(X^s_{(k)}, Y^s_{(\ell)}, Z^s_{(m)})\}(s=1,2, \dots)$ be a sequence of n-dimensional random variables ($n = k + \ell + m$), for which the following conditions are satisfied:

(i) For the second marginals, $\{Y^s_{(\ell)} = (Y^s_1, \dots, Y^s_{(\ell)})\}$ $(s=1,2, \dots)$, there exists a sequence of real vectors $\{\lambda^s_{(\ell)}=(\lambda^s_1,\dots, \lambda^s_\ell)\}$ $(s=1,2, \dots)$ with $\lambda^s_i > 0$, $i = 1, \dots, \ell$, such that the sequence $\{\overline{Y}^s_{(\ell)} = (Y^s_1/\lambda^s_1, \dots, Y^s_\ell/\lambda^s_\ell)\}(s=1,2, \dots)$ converges in probability to $(1, \dots, 1)$ as $s \to \infty$.

(ii) For the conditional distributions of $X^s_{(k)}$ given $Y^s_{(\ell)} = \lambda^s_{(\ell)}$ respectively for each s, $\{\overline{X}^s_{(k)}\}(s=1,2, \dots)$, there exist two sequences of real vectors, $\{(c^s_1, \dots, c^s_k)\}(s=1,2, \dots)$ with $c^s_i > 0$, $i = 1, \dots, k$, and $\{(d^s_1, \dots, d^s_k)\}(s=1,2, \dots)$, and a sequence of k-dimensional random variables $\{U^s_{(k)} = (U^s_1, \dots, U^s_k)\}(s=1,2, \dots)$, having the property $B(\mathcal{S})$, such that $\{\overline{X}^s_{(k)} = (\overline{X}^s_1, \dots, \overline{X}^s_k)\}(s=1,2, \dots)$ and $\{\widetilde{X}^s_{(k)} = (\widetilde{X}^s_1, \dots, \widetilde{X}^s_k)\}(s=1,2, \dots)$ are asymptotically equivalent ($\mathcal{S}$) as $s \to \infty$, where

$$\widetilde{X}^s_i = c^s_i U^s_i + d^s_i, \quad i = 1, \dots, k; \; s = 1,2, \dots \; . \tag{3.37}$$

(iii) The conditional cumulative distribution function of $Z^s_{(m)}$ given $X^s_{(k)} = x_{(k)}$ and $Y^s_{(\ell)} = y_{(\ell)}$, $F_s(z_{(m)} \mid x_{(k)}, y_{(\ell)})$ is continuous in $(z_{(m)}, x_{(k)}, y_{(\ell)})$ jointly over $\Re_{(n)}$, for each s.

Then, two random variables, $Z^s_{(m)}$ and $\widetilde{Z}^s_{(m)}$, whose cumulative distribution functions being given respectively by

(3.38) $$H_s(z_{(m)}) = \int_{\mathfrak{R}_{(k+\ell)}} F_s(z_{(m)} | x_{(k)}, y_{(\ell)}) \, d\, G_s(x_{(k)}, y_{(\ell)})$$

and

(3.39) $$\widetilde{H}_s(z_{(m)}) = \int_{\mathfrak{R}_{(k)}} F_s(z_{(m)} | x_{(k)}, \lambda^s_{(\ell)}) \, d\, \widetilde{K}_s(x_{(k)}),$$

are asymptotically equivalent ($\mathfrak{S}$) as $s \to \infty$, where $G_s(x_{(k)}, y_{(\ell)})$ and $\widetilde{K}_s(x_{(k)})$ stand for the cumulative distribution functions of $(X^s_{(k)}, Y^s_{(\ell)})$ and $\{\widetilde{X}^s_{(k)}\}$ $(s=1,2, \ldots)$ given by (3.37) respectively. Hence, if $Z^s_{(m)}$'s are absolutely continuous, and the conditional probability density functions $p_s(z_{(m)} | x_{(k)}, y_{(\ell)})$'s given $X^s_{(k)} = x_{(k)}$ and $Y^s_{(\ell)} = y_{(\ell)}$ are continuous in $(z_{(m)}, x_{(k)}, y_{(\ell)})$ jointly over $\mathfrak{R}_{(n)}$, then, under the conditions (i) and (ii) given above, $Z^s_{(m)}$ and $\widetilde{Z}^s_{(m)}$ are asymptotically equivalent ($\mathfrak{S}$) as $s \to \infty$, with

(3.40) $$F_s(z_{(m)} | x_{(k)}, y_{(\ell)}) = \int_{\substack{z_i < z_i \\ i=1,\ldots,m}} p_s(z'_{(m)} | x_{(k)}, y_{(\ell)}) \, d\, z'_{(m)}.$$

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R E F E R E N C E S

[1] S. Ikeda "On certain types of asymptotic equivalence of real probability distributions, I. Definitions and some of their properties," Univ. of North Carolina, Inst. of Statistics Mimeo Series, 455 (1965).

[2] S. Ikeda, J. Ogawa and M. Ogasawara "On the asymptotic distribution of the F-statistic under the null-hypothesis in a randomized PBIB design with m associate classes under the Neyman model," Unif. of North Carolina, Inst. of Statistics Mimeo Series, 454 (1965).

[3] M. Loève Probability Theory, Van Nost., 1955.