ON THE NON-NULL DISTRIBUTION OF THE F-STATISTIC FOR TESTING A PARTIAL NULL-HYPOTHESIS IN A RANDOMIZED PBIB DESIGN WITH m ASSOCIATE CLASSES UNDER THE NEYMAN MODEL

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Institute of Statistics Mimeo Series No. 466

March 1966

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This research was supported by the Mathematics Division of the Air Force Office of Scientific Research Contract No. AF-AFOSR-760-65

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Chapel Hill, N. C.

1. Introduction

In this section we shall give a brief sketch of the problem which is treated in the present article. The notations and terminologies are the same as those of the preceding papers [1], [2], unless otherwise stated. As for the notions of asymptotic equivalence, references should be made to [3], [4].

As in [1], we are concerned with a PBIB design with m associate classes, which has v treatments with the association, b blocks of size k each, r replications of each treatment, and the number of incidence of any pairs of treatments, λ_u, if they are u-th associates.

Let us take a special numbering of the whole $n(= vr = bk)$ plots such that the i-th plot of the p-th block receives the number $f = (p-1)k + i$, and this numbering is fixed throughout the present paper.

Let Φ and ψ be the incidence matrices of treatments and blocks respectively, and put $B = \psi\psi'$ and $N = \Phi'\psi$, N being the incidence matrix of the design.

Further let $\tau' = (\tau_1, \ldots, \tau_v)$ and $\beta' = (\beta_1, \ldots, \beta_b)$ be the treatment-effect vector and the block-effect vector satisfying the restrictions

$$\sum_{\alpha=1}^{v} \tau_\alpha = 0 \qquad \text{and} \qquad \sum_{p=1}^{\beta} \beta_p = 0$$

respectively, and let $\pi' = (\pi_1, \ldots, \pi_n)$ be the plot-effect vector subject to the restraints

$$\sum_{i=1}^{k} \pi_i^{(p)} = 0, \qquad p = 1, \ldots, b$$

where $\pi_i^{(p)} = \pi_f$ for $f = (p-1)k + i$.

Summary

This article treats the problem of finding a probability distribution which is asymptotically equivalent in the sense of type (S) under some limiting conditions to the non-null distribution of the F-statistic for testing a 'partial' null-hypothesis (and hence, the 'total' null-hypothesis in a special case) in a randomized partially balanced incomplete block design with $m(\geq 1)$ associate classes under the Neyman model with both unit and technical errors.

The result is satisfactory to some extent.

The Neyman model assuming the existence of plot-effects which have no interaction with treatments is given by

$$(1.1) \qquad x = \gamma 1 + \Phi\tau + \psi\beta + \pi + e,$$

where $x' = (x_1, \ldots, x_n)$ is the observation vector, γ is the general mean, $1' = (1, \ldots, 1)$, and $e' = (e_1, \ldots, e_n)$ stands for the technical errors, which is assumed to be distributed according to $N(0, \sigma^2 I_n)$ with unknown σ^2.

We consider the null-hypothesis

$$(1.2) \qquad H_{o(h)}: \; A_u^{\#}\tau = 0, \qquad u = 1, \ldots, h,$$

where h is a positive integer not greater than m. This is called the 'partial' null-hypothesis [1], and when $h = m$ this reduces to the 'total' null-hypothesis

$$(1.3) \qquad H_o: \; \tau = 0.$$

To test the null-hypothesis $H_{o(h)}$, we take the F-statistic

$$(1.4) \qquad \bar{F} = \frac{n-b-v+1}{\bar{\alpha}} \; \frac{S^2_{t(h)}}{S^2_e}$$

with $\bar{\alpha} = \Sigma_{u=1}^{h} \alpha_u$ and

$$(1.5) \qquad \begin{cases} S^2_{t(h)} = x' \left(\sum_{u=1}^{h} V_u^{\#} \right) x \\[2ex] S^2_e = x'(I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#})x \end{cases}$$

where

$$V_u^{\#} = (I - \frac{1}{k} B)\Phi(c_u \, A_u^{\#})\Phi'(I - \frac{1}{k} B), \qquad u = 1, \ldots, m$$

$$c_u = \frac{k}{rk - \rho_u}, \qquad u = 1, \ldots, m.$$

Here, $A_o^{\#} = \frac{1}{v} G_v, A_1^{\#}, \ldots, A_m^{\#}$ are mutually orthogonal idempotents, with respective ranks $\alpha_o(=1), \alpha_1, \ldots, \alpha_m$, and $\rho_o(=rk), \rho_1, \ldots, \rho_m$ are the characteristic roots of $N N'$ with respective multiplicities $\alpha_o, \alpha_1, \ldots, \alpha_m$, for which it is known that

$$\sum_{u=o}^{m} \alpha_u = v, \quad \sum_{u=o}^{m} A_u^{\#} = I_v \quad \text{and} \quad N N' = \sum_{u=o}^{m} \rho_u A_u^{\#}.$$

In the special case when $h = m$, $\overline{F}$ in (1.4) reduces to the usual F-statistic

$$(1.6) \qquad F = \frac{n-b-v+1}{v-1} \frac{s_t^2}{s_e^2}$$

with

$$(1.7) \qquad \begin{cases} s_t^2 = x'(\sum_{u=1}^{m} V_u^{\#})x \\ s_e^2 = x'(I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#})x, \end{cases}$$

which is used to test the total null-hypothesis (1.3).

Now, if the existence of the plot-effects is not assumed, it is known that the non-null distribution of the F-statistic given by (1.4) under the Neyman model (1.1) with $\pi = 0$ is the non-central F-distribution of degrees of freedom $(\overline{\alpha}, n-b-v+1)$ with non-centrality parameter $\overline{T}/\sigma^2$, where

$$(1.8) \qquad \bar{T} = \Big(\sum_{u=1}^{h} A_u^{\#} \tau \Big)'\Big(r\, I - \frac{1}{k} N\, N'\Big) \Big(\sum_{u=1}^{h} A_u^{\#} \tau \Big).$$

Hence the probability element of the $\bar{F}$ is given by

$$(1.9) \qquad \exp\Big(-\frac{\bar{T}}{2\sigma^2}\Big) \sum_{\ell=0}^{\infty} \frac{\left(\frac{\bar{T}}{2\sigma^2}\right)^{\ell}}{\ell!} \cdot \frac{\Gamma\left(\frac{n-b-\bar{\bar{\alpha}}}{2} + \ell\right)}{\Gamma\left(\frac{\bar{\alpha}}{2} + \ell\right)\Gamma\left(\frac{n-b-v+1}{2}\right)}$$

$$\left(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right)^{\frac{\bar{\alpha}}{2} + \ell - 1} \left(1 + \frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right)^{-\left(\frac{n-b-\bar{\bar{\alpha}}}{2} + \ell\right)} d\left(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right),$$

where we have put $\bar{\bar{\alpha}} = v-1-\bar{\alpha}$.

In the case when $h = m$, this reduces to

$$(1.10) \qquad \exp\Big(-\frac{T}{2\sigma^2}\Big) \sum_{\ell=0}^{\infty} \frac{\left(\frac{T}{2\sigma^2}\right)^{\ell}}{\ell!} \cdot \frac{\Gamma\left(\frac{n-b}{2} + \ell\right)}{\Gamma\left(\frac{v-1}{2} + \ell\right)\Gamma\left(\frac{n-b-v+1}{2}\right)}$$

$$\left(\frac{v-1}{n-b-v+1}F\right)^{\frac{v-1}{2} + \ell - 1} \left(1 + \frac{v-1}{n-b-v+1}F\right)^{-\left(\frac{n-b}{2} + \ell\right)} d\left(\frac{v-1}{n-b-v+1}F\right),$$

where

$$(1.11) \qquad T = \tau'\Big(r\, I - \frac{1}{k} N\, N'\Big)\tau.$$

This gives the probability element of the non-null distribution of the F-statistic given in (1.6) under the Neyman model (1.1) with $\pi = 0$.

However, once we assume the existence of the plot-effects, these are no longer true, and, as is shown in the following section, the non-null distribution of the $\bar{F}$ or F is a non-central F-distribution, whose non-centrality parameter depends on both τ and π. It is interesting to investigate whether the influence of π on the non-null distribution of the $\bar{F}$ or F can be eliminated under certain limiting conditions by yielding a randomization procedure in the allocation of treatments to the plots within each block, as it was so in the null case [2].

2. The non-null distribution of the F-statistic before the randomization.

In this section, we derive the non-null distribution of the $\overline{F}$ given by (1.4) and (1.5), and of the F given by (1.6) with (1.7), before the randomization.

Since

$$S^2_{t(h)} = x'(\sum_{u=1}^{h} V^{\#}_u)\, \underline{x}$$

$$= (\gamma 1 + \Phi\tau + \psi\beta + \pi + e)'(\sum_{u=1}^{h} V^{\#}_u)(\gamma 1 + \Phi\tau + \psi\beta + \pi + e)$$

$$= (\Phi\tau + \pi + e)'(\sum_{u=1}^{h} V^{\#}_u)(\Phi\tau + \pi + e)$$

$$= (\Phi\tau + \pi)'(\sum_{u=1}^{h} V^{\#}_u)(\Phi\tau + \pi) + 2(\Phi\tau + \pi)'(\sum_{u=1}^{h} V^{\#}_u)\, e + e'(\sum_{u=1}^{h} V^{\#}_u)e,$$

and the rank of $\sum_{u=1}^{h} V^{\#}_u$ is $\overline{\alpha}$, the non-null distribution of the variate

$$\overline{\chi}^2_1 = S^2_{t(h)}/\sigma^2 \tag{2.1}$$

before the randomization is the non-central chi-square distribution of degrees of freedom $\overline{\alpha}$ with the non-centrality parameter $\overline{\delta}_1/\sigma^2$, whose probability element being given by

$$\exp(-\frac{\overline{\delta}_1}{2\sigma^2})\sum_{\mu=0}^{\infty}\frac{(\frac{\overline{\delta}_1}{2\sigma^2})^{\mu}}{\mu!}\cdot\frac{(\frac{\overline{\chi}^2_1}{2})^{\frac{\overline{\alpha}}{2}+\mu-1}}{\Gamma(\frac{\overline{\alpha}}{2}+\mu)}\exp(-\frac{\overline{\chi}^2_1}{2})\, d\,(\frac{\overline{\chi}^2_1}{2}), \tag{2.2}$$

where

(2.3) $$\bar{\delta}_1 = (\Phi\tau + \pi)' \left(\sum_{u=1}^{h} V_u^{\#} \right)(\Phi\tau + \pi).$$

Hence, in the case when $h = m$, the non-null distribution of the variate

(2.4) $$\chi_1^2 = S_t^2/\sigma^2$$

is the non-central chi-square distribution of degrees of freedom v-1 with non-centrality parameter δ_1/σ^2, whose probability element being

(2.5) $$\exp\left(-\frac{\delta_1}{2\sigma^2}\right) \sum_{\mu=0}^{\infty} \frac{\left(\frac{\delta_1}{2\sigma^2}\right)^{\mu}}{\mu!} \frac{\left(\frac{\chi_1^2}{2}\right)^{\frac{v-1}{2}+\mu-1}}{\Gamma\left(\frac{v-1}{2}+\mu\right)} \exp\left(-\frac{\chi_1^2}{2}\right),$$

where

(2.6) $$\delta_1 = (\Phi\tau + \pi)'\left(\sum_{u=1}^{m} V_u^{\#} \right)(\Phi\tau + \pi).$$

Similarly we obtain

$$S_e^2 = x'(I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#})x$$

$$= (\Phi\tau + \pi)'(I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#})(\Phi\tau + \pi)$$

$$+ 2(\Phi\tau + \pi)'(I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#})e + e'(I - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#})e,$$

where the matrix $I - \frac{1}{k} - \Sigma_{u=1}^{m} V_u^{\#}$ is an idempotent of rank n-b-v+1. Hence the non-null distribution of the variate

(2.7) $$\chi_1^2 = S_e^2/\sigma^2$$

before the randomization is the non-central chi-square distribution of degrees of freedom n-b-v+1 with non-centrality parameter δ_2/σ^2 whose probability element being

(2.8) $$\exp(-\frac{\delta_2}{2\sigma^2}) \sum_{\nu=0}^{\infty} \frac{(\frac{\delta_2}{2\sigma^2})^{\nu}}{\nu!} \frac{(\frac{\chi_2^2}{2})^{\frac{n-b-v+1}{2}+\nu-1}}{\Gamma(\frac{n-b-v+1}{2}+\nu)} \exp(-\frac{\chi_2^2}{2})\, d(\frac{\chi_2^2}{2}),$$

where

(2.9) $$\delta_2 = (\Phi\tau + \pi)'(1 - \frac{1}{k} B - \sum_{u=1}^{m} V_u^{\#}) (\Phi\tau + \pi).$$

This is seen to be independent of τ, as is shown later.

Since the variates $\bar{\chi}_1^2$ and χ_2^2 are mutually independent in the stochastic sense before the randomization, the non-null distribution of the $\bar{F}$ statistic given by (1.4) with (1.5) before the randomization is a non-central F-distribution of degrees of freedom ($\bar{\alpha}$, n-b-v+1) whose probability element being given by

(2.10) $$\exp(-\frac{\bar{\delta}_1 + \delta_2}{2\sigma^2}) \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(\frac{\bar{\delta}_1}{2\sigma^2})^{\mu}}{\mu!} \frac{(\frac{\delta_2}{2\sigma^2})^{\nu}}{\nu!} \frac{\Gamma(\frac{n-b-\bar{\bar{\alpha}}}{2}+\mu+\nu)}{\Gamma(\frac{\bar{\alpha}}{2}+\mu)\Gamma(\frac{n-b-v+1}{2}+\nu)} .$$

$$(\frac{\bar{\alpha}}{n-b-v+1}\bar{F})^{\frac{\bar{\alpha}}{2}+\mu-1} (1 + \frac{\bar{\alpha}}{n-b-v+1}\bar{F})^{-(\frac{n-b-\bar{\bar{\alpha}}}{2}+\mu+\nu)} \alpha(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}).$$

Putting

(2.11) $$\bar{\bar{\delta}}_1 = \delta_1 - \bar{\delta}_1 = (\Phi\tau + \pi)'(\sum_{u=h+1}^{m} V_u^{\#})(\Phi\tau + \pi),$$

the above can be rewritten as

$$\exp(-\frac{\xi}{2\sigma^2}) \sum_{\ell=0}^{\infty} \frac{(\frac{\xi}{2\sigma^2})^{\ell}}{\ell!} \sum_{\mu+\nu+\gamma=\ell} \frac{\ell!}{\mu!\ \nu!\ \gamma!} \ \bar{\eta}^{\mu} \bar{\bar{\eta}}^{\nu} (1-\bar{\eta}-\bar{\bar{\eta}})^{\gamma} \cdot$$

(2.12)

$$\cdot \frac{\Gamma(\frac{n-b-\bar{\bar{\alpha}}}{2}+\mu+\nu)}{\Gamma(\frac{\bar{\alpha}}{2}+\mu)\Gamma(\frac{n-b-v+1}{2}+\nu)} \left(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right)^{\frac{\bar{\alpha}}{2}+\mu-1} \left(1+\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right)^{-(\frac{n-b-\bar{\bar{\alpha}}}{2}+\mu+\nu)} d\left(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right),$$

where

(2.13) $\xi = \delta_1 + \delta_2 = \bar{\delta}_1 + \bar{\bar{\delta}}_1 + \delta_2$, $\bar{\eta} = \frac{\bar{\delta}_1}{\delta_1 + \delta_2}$ and $\bar{\bar{\eta}} = \frac{\bar{\bar{\delta}}_1}{\delta_1 + \delta_2}$

In the case $h = m$, putting

(2.14) $$\eta = \frac{\delta_1}{\delta_1 + \delta_2},$$

the non-null distribution of the F-statistic given by (1.6) with (1.7) before the randomization is a non-central F-distribution of degrees of freedom $(v - 1, n - b - v + 1)$, whose probability element being given by

(2.15) $$\exp(-\frac{\xi}{2\sigma^2}) \sum_{\ell=0}^{\infty} \frac{(\frac{\xi}{2\sigma^2})^{\ell}}{\ell!} \sum_{\mu+\nu=\ell} \frac{\ell!}{\mu!\ \nu!} \eta^{\mu}(1-\eta)^{\nu} \frac{\Gamma(\frac{n-b}{2}+\ell)}{\Gamma(\frac{v-1}{2}+\mu)\Gamma(\frac{n-b-v+1}{2}+\nu)} \cdot$$

$$\cdot \left(\frac{v-1}{n-b-v+1}F\right)^{\frac{v-1}{2}+\mu-1} \left(1+\frac{v-1}{n-b-v+1}F\right)^{-(\frac{n-b}{2}+\ell)} d\left(\frac{v-1}{n-b-v+1}F\right).$$

Since the quantities $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ or (ξ, η) contain the incidence matrix of treatments, Φ, if we adopt a randomization procedure in allocating k treatments to k plots within each block, then Φ, and hence the quantities $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ or (ξ, η) become the random variables under the permutation distribution due to the randomization adopted.

The null-distribution of the F-statistic, $\overline{F}$ or F, after the randomization can be obtained by taking expectation of the probability element (2.12) or (2.15) with respect to the permutation distribution of $(\xi, \overline{\eta}, \overline{\overline{\eta}})$ or (ξ, η) due to the randomization. For this, exact calculations are difficult to do and so we shall seek for another probability distribution which is asymptotically equivalent to the non-null distribution of the F after the randomization in the sense of type (S) under certain limiting conditions, which is done in the following two sections.

3. Asymptotic behavior of the permutation distributions of $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ and (ξ, η) due to the randomization

The purpose of this section is to derive the asymptotic distributions of the marginals of $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ or of (ξ, η) under a certain limiting process, and give some results which are useful in the next section.

Since

$$\bar{\delta}_1 = (\Phi\tau)'(\sum_{u=1}^{h} V_u^{\#})(\Phi\tau) + 2(\Phi\tau)'(\sum_{u=1}^{h} V_u^{\#})\pi + \pi'(\sum_{u=1}^{h} V_u^{\#})\pi,$$

and

$$(\Phi\tau)'(\sum_{u=1}^{h} V_u^{\#})(\Phi\tau) = \tau'\Phi'(I - \frac{1}{k} B)\Phi(\sum_{u=1}^{h} c_u A_u^{\#})\Phi'(I - \frac{1}{k} B)\Phi\tau$$

$$= \tau'(r I - \frac{1}{k} N N')(\sum_{u=1}^{h} c_u A_u^{\#})(r I - \frac{1}{k} N N')\tau$$

$$= \tau'(\sum_{u=1}^{m} (r - \frac{1}{k} \rho_u)A_u^{\#})(\sum_{u=1}^{h} c_u A_u^{\#})(r I - \frac{1}{k} N N')\tau$$

$$= \tau'(\sum_{u=1}^{h} A_u^{\#})(r I - \frac{1}{k} N N')\tau$$

$$= \tau'(\sum_{u=1}^{h} A_u^{\#})(r I - \frac{1}{k} N N')(\sum_{u=1}^{h} A_u^{\#})\tau$$

$$= (\sum_{u=1}^{h} A_u^{\#} \tau)'(r I - \frac{1}{k} N N')(\sum_{u=1}^{h} A_u^{\#} \tau),$$

$$(\Phi\tau)'(\sum_{u=1}^{h} V_u^{\#})\pi = \tau' \Phi'(I - \frac{1}{k} B)\Phi(\sum_{u=1}^{h} c_u A_u^{\#})\Phi' (I - \frac{1}{k} B)\pi$$

$$= \tau'(r I - \frac{1}{k} N N')(\sum_{u=1}^{h} c_u A_u^{\#})\Phi'\pi$$

$$= \tau'(\sum_{u=1}^{h} A_u^{\#})\Phi'\pi$$

$$= (\sum_{u=1}^{h} A_u^{\#} \tau)'\Phi'\pi,$$

we get

$$\bar{\delta}_1 = \bar{T} + 2(\sum_{u=1}^{h} A_u^{\#} \tau)'\Phi'\pi + \pi'(\sum_{u=1}^{h} V_u^{\#})\pi, \tag{3.1}$$

and similarly

$$\bar{\bar{\delta}}_1 = \bar{\bar{T}} + 2(\sum_{u=h+1}^{m} A_u^{\#} \tau)'\Phi'\pi + \pi'(\sum_{u=h+1}^{m} V_u^{\#})\pi, \tag{3.2}$$

where we have put

$$\bar{T} = (\sum_{u=1}^{h} A_u^{\#} \tau)'(r I - \frac{1}{k} N N')(\sum_{u=1}^{h} A_u^{\#} \tau) \tag{3.3}$$

and

$$\bar{\bar{T}} = (\sum_{u=h+1}^{m} A_u^{\#} \tau)'(r I - \frac{1}{k} N N')(\sum_{u=h+1}^{m} A_u^{\#} \tau), \tag{3.4}$$

for which it holds that

(3.5) $\bar{T} + \bar{\bar{T}} = T,$

$\bar{T}$ and T being the same as those given by (1.8) and (1.11).

From (2.1) and (2.2), it follows that

$$(3.6) \qquad \delta_1 = \bar{\delta}_1 + \bar{\bar{\delta}}_1 = T + 2\,\tau'\,\Phi'\,\pi + \pi'(\sum_{u=1}^{m} V_u^{\#})\pi.$$

The quantity δ_2 is easily calculated as

$$\delta_2 = (\Phi\tau + \pi)'(I - \frac{1}{k}B)(\Phi\tau + \pi) - \delta_1$$

$$= \tau'\Phi'(I - \frac{1}{k}B)\Phi\,\tau + 2\tau'\Phi'(I - \frac{1}{k}B)\pi + \pi'(I - \frac{1}{k}B)\pi - \delta_1$$

$$= \tau'(r\,I - \frac{1}{k}B)\tau + 2\tau'\Phi'\pi + \pi'\pi - \delta_1,$$

and therefore

$$(3.7) \qquad \delta_2 = \Delta - \pi'(\sum_{u=1}^{m} V_u^{\#})\pi,$$

where $\Delta = \pi'\pi$. This is independent of the values of τ, which means that the distribution of the variate $\chi_2^2 = S_e^2/\sigma^2$ given by (2.7) before the randomization is independent of the hypotheses.

From (2.13), (2.14), (3.1), (3.2), and (3.7), we have

$$(3.8) \qquad \begin{cases} \xi = \Delta + T + 2\tau'\Phi^{-1}\pi, \\ \eta = (T + 2\tau'\Phi'\pi + \pi'(\sum_{u=1}^{m} V_u^{\#})\pi)/(\Delta + T + 2\tau'\Phi'\pi) \\ \bar{\eta} = (\bar{T} + 2(\sum_{u=1}^{h} A_u^{\#}\tau)'\Phi'\pi + \pi'(\sum_{u=1}^{h} V_u^{\#})\pi)/(\Delta + T + 2\tau'\Phi'\pi) \\ \bar{\bar{\eta}} = (\bar{\bar{T}} + 2(\sum_{u=h+1}^{m} A_u^{\#}\tau)'\Phi'\pi + \pi'(\sum_{u=h+1}^{m} V_u^{\#})\pi)/(\Delta + T + 2\tau'\Phi'\pi). \end{cases}$$

Now, the randomization adopted is represented by the following $n \times n$ permutation matrix

$$U_\sigma = \begin{Vmatrix} S_{\sigma_1} & & & 0 \\ & S_{\sigma_2} & & \\ & & \ddots & \\ 0 & & & S_{\sigma_b} \end{Vmatrix},$$

where S_{σ_p} is a $k \times k$ permutation matrix corresponding to the randomization within the p-th block. For this matrix it is clear that

$$(3.10) \qquad U_\sigma B = B\, U_\sigma = B.$$

Let Φ be an arbitrary but fixed incidence matrix of treatments. Then any other incidence matrix of the treatments is given by $\Phi_\sigma = U'_\sigma \Phi$. Since $N N'$ is invariant under this transformation, i.e.,

$$(N N')_\sigma = \Phi'_\sigma \psi \psi' \Phi_\sigma = \Phi' \psi \psi' \Phi = N N',$$

the quantities $\overline{T}$, $\overline{\overline{T}}$ and T are regarded as constants under the permutation distribution of Φ_σ.

Now, let us put

$$(3.11) \qquad X = \tau' \Phi'_\sigma \pi, \quad X_1 = (\sum_{u=1}^{h} A_u^{\#} \tau)' \Phi_\sigma' \pi, \quad X_2 = (\sum_{u=h+1}^{m} A_u^{\#} \tau)' \Phi_\sigma' \pi$$

and

$$(3.12) \qquad \Upsilon = \pi' (\sum_{u=1}^{m} V_u^{\#})_\sigma \pi = \pi' (I - \frac{1}{k} B) \Phi_\sigma (\sum_{u=1}^{m} c_u A_u^{\#}) \Phi'_\sigma (I - \frac{1}{k} B) \pi,$$

and consider the following limiting process:

(3.13) $\quad b \to \infty$, keeping v, k, n_1, ..., n_m, p^i_{jk} fixed.

Furthermore, the following conditions are assumed to be satisfied:

$$(3.14) \qquad \bar{\Delta} = \frac{1}{b} \sum_{p=1}^{b} \Delta_p \to \Delta_0 (>0) \quad \text{and} \quad \frac{1}{b} \sum_{p=1}^{b} (\Delta_p - \bar{\Delta})^2 \to 0$$

under the limiting (3.13), where Δ_p's are the same as in [1], [2].

It has been known [2] that the permutation distribution of the variate

$$(3.15) \qquad \frac{k-1}{\Delta_0} Y = \frac{k-1}{\Delta_0} \pi' \left(\sum_{u=1}^{m} V_u^{\#} \right) \sigma^{\pi}$$

converges in the sense of type (S) to the central chi-square distribution of degrees of freedom v-1, as $b \to \infty$, provides that the conditions (3.14) are satisfied.

Let us consider the asymptotic distributions of X_1, X_2 and X given by (3.11), for which $X = X_1 + X_2$.

Since

$$E(\pi^{(p)}_{\sigma(i)}) = 0, \; E(\pi^{(p)2}_{\sigma(i)}) = \frac{1}{k} \Delta_p \quad \text{and} \quad E(\pi^{(p)}_{\sigma(i)} \; \pi^{(q)}_{\sigma(j)}) = - \frac{1}{k(k-1)} \Delta_p,$$

where E stands for the expectation with respect to the permutation distribution under consideration, it is evident that

$$(3.16) \qquad E(X_1) = 0, \quad E(X_2) = 0 \text{ and } E(X) = 0$$

It is also clear that the covariance of X_1 and X_2 vanishes from the orthogonality of the matrices $\Sigma_{u=1}^{h} A_u^{\#}$ and $\Sigma_{u=h+1}^{m} A_u^{\#}$. Hence

$$(3.17) \qquad \text{Var}(X) = \text{Var}(X_1) + \text{Var}(X_2).$$

The variance Var (X_1) with respect to the permutation distribution is calculated as follows:

$$\text{Var}(X_1) = E(\tau' \sum_{u=1}^{h} A_u^{\#} \Phi_\sigma' \pi \cdot \pi' \Phi_\sigma \sum_{u=1}^{h} A_u^{\#} \tau)$$

$$= (\sum_{u=1}^{h} A_u^{\#} \tau)' \Phi' E[U_\sigma \pi \cdot \pi' U_\sigma'] \Phi (\sum_{u=1}^{h} A_u^{\#} \tau)$$

$$(3.18) \qquad = (\sum_{u=1}^{h} A_u^{\#} \tau)' \Phi' \left\| \begin{matrix} \frac{\Delta_1}{k-1}(I_k - \frac{1}{k} G_k) & & 0 \\ & \ddots & \\ 0 & & \frac{\Delta_b}{k-1}(I_k - \frac{1}{k} G_k) \end{matrix} \right\| \Phi (\sum_{u=1}^{h} A_u^{\#} \tau)$$

$$= \frac{1}{k-1} \bar{\Delta} \cdot \bar{T} + \bar{\rho}(\tau, \Delta_1, \ldots, \Delta_b),$$

where

$$(3.19) \qquad \bar{\rho}(\tau, \Delta_1, \ldots, \Delta_b) = \frac{1}{k-1} (\sum_{u=1}^{h} A_u^{\#} \tau)' \Phi' \left\| \begin{matrix} \frac{\Delta_1 - \bar{\Delta}}{k-1}(I_k - \frac{1}{k} G_k) & & 0 \\ & \ddots & \\ 0 & & \frac{\Delta_b - \bar{\Delta}}{k-1}(I_k - \frac{1}{k} G_k) \end{matrix} \right\| \Phi (\sum_{u=1}^{h} A_u^{\#} \tau).$$

Likewise, we have

$$(3.20) \qquad \begin{cases} \text{Var}(X) = \frac{1}{k-1} \bar{\Delta} T + \rho(\tau, \Delta_1, \ldots, \Delta_b) \\ \text{Var}(X_2) = \frac{1}{k-1} \bar{\Delta} \bar{\bar{T}} + \bar{\bar{\rho}}(\tau, \Delta_1, \ldots, \Delta_b) \end{cases}$$

where

$$(3.21)\quad \rho(\tau, \Delta_1, \ldots, \Delta_b) = \tau'\Phi' \left\| \begin{matrix} \frac{\Delta_1-\bar{\Delta}}{k-1}(I_k - \frac{1}{k}G_k) & & 0 \\ & \ddots & \\ 0 & & \frac{\Delta_b-\bar{\Delta}}{k-1}(I_k - \frac{1}{k}G_k) \end{matrix} \right\| \Phi\tau$$

and

$$(3.22)\quad \bar{\bar{\rho}}(\tau, \Delta_1, \ldots, \Delta_b) = \frac{1}{k-1}\Big(\sum_{u=n+1}^{m} A_u^{\#}\tau\Big)'\Phi' \left\| \begin{matrix} \frac{\Delta_1-\bar{\Delta}}{k-1}(I_k - \frac{1}{k}G_k) & & 0 \\ & \ddots & \\ 0 & & \frac{\Delta_b-\bar{\Delta}}{k-1}(I_k - \frac{1}{k}G_k) \end{matrix} \right\| \Phi\Big(\sum_{u=n+1}^{m} A_u^{\#}\tau\Big).$$

From the practical point of view, there is no harm in assuming that there exist constants $\bar{T}_0$, $\bar{\bar{T}}_0$ and T_0 such that

$$(3.23)\quad \bar{T}_0 = \lim_{b\to\infty} \frac{1}{b}\bar{T}, \quad \bar{\bar{T}}_0 = \lim_{b\to\infty} \frac{1}{b}\bar{\bar{T}} \text{ and } T_0 = \lim_{b\to\infty} \frac{1}{b} T,$$

for which, of course, it holds that $T_0 = \bar{T}_0 + \bar{\bar{T}}_0$ and these values are all positive.

Since

$$\Phi' \left\| \begin{matrix} \frac{\Delta_1-\bar{\Delta}}{k-1}(I_k - \frac{1}{k}G_k) & & 0 \\ & \ddots & \\ 0 & & \frac{\Delta_b-\bar{\Delta}}{k-1}(I_k - \frac{1}{k}G_k) \end{matrix} \right\| \Phi$$

$$= \frac{1}{k-1}\sum_{p=1}^{b}(\Delta_p-\bar{\Delta}) \left\| \begin{matrix} n_{1p} & -n_{1p}n_{2p} & \cdots & -n_{1p}n_{vp} \\ -n_{1p}n_{2p} & n_{2p} & \cdots & -n_{2p}n_{vp} \\ \vdots & & & \\ -n_{1p}n_{vp} & \cdots\cdots & & n_{vp} \end{matrix} \right\| ,$$

it is easy to see that, under the conditions (3.14),

$$\lim_{b\to\infty} \frac{1}{b} \bar{\rho}(\tau, \Delta_1, \ldots, \Delta_b) = 0, \quad \lim_{b\to\infty} \frac{1}{b} \bar{\bar{\rho}}(\tau, \Delta_1,\ldots,\Delta_b) = 0 \quad \text{and}$$

$$\lim_{b\to\infty} \frac{1}{b} \rho(\tau, \Delta_1, \ldots, \Delta_b) = 0.$$

Hence we obtain

$$(3.24) \quad \lim_{b\to\infty} \frac{\mathrm{Var}\,(X_1)}{b} = \frac{\Delta_o \bar{T}_o}{k-1}, \quad \lim_{b\to\infty} \frac{\mathrm{Var}\,(X_2)}{b} = \frac{\Delta_o \bar{\bar{T}}_o}{k-1} \text{ and } \lim_{b\to\infty} \frac{\mathrm{Var}\,(X)}{b} = \frac{\Delta_0 T_0}{k-1}.$$

Now, we shall show the asymptotic normality of these random variables. For this, put

$$\sum_{u=1}^{h} A_u^{\#} \tau = \bar{\tau} = \begin{bmatrix} \bar{\tau}_1 \\ \vdots \\ \bar{\tau}_v \end{bmatrix} .$$

Then X_1 is the sum of b independent random variables:

$$X_1 = \sum_{p=1}^{b} \Big(\sum_{\alpha=1}^{v} \bar{\tau}_\alpha \sum_{i=1}^{k} \zeta_{\alpha i}^{(p)} \pi_{\sigma(i)}^{(p)} \Big)$$

$$= \sum_{p=1}^{b} \Big(\sum_{i=1}^{k} \sum_{\alpha=1}^{v} \bar{\tau}_\alpha \zeta_{\alpha i}^{(p)} \pi_{\sigma(i)}^{(p)} \Big),$$

where $\zeta_{\alpha i}^{(p)}$'s are the elements of the incidence matrix of treatments, Φ. Since

$$\Big(\sum_{i=1}^{k} \sum_{\alpha=1}^{v} \bar{\tau}_\alpha \zeta_{\alpha i}^{(p)} \cdot \pi_{\sigma(i)}^{(p)} \Big)^2 \leq \sum_{i=1}^{k} \Big(\sum_{\alpha=1}^{v} \bar{\tau}_\alpha \zeta_{\alpha i}^{(p)} \Big)^2 \sum_{i=1}^{k} \pi_{\sigma(i)}^{(p)\,2}$$

$$\leq k \sum_{\alpha=1}^{v} \bar{\tau}_\alpha^2 \quad \Delta_p,$$

and, under the conditions (3.14),

$$\frac{1}{b}\sum_{p=1}^{b} \Delta_p^{\frac{3}{2}} \to \Delta_o^{\frac{3}{2}}, \quad (b \to \infty),$$

from (3.24) it follows that

$$(3.25) \qquad \frac{\left[\sum_{p=1}^{b} E\left(\left|\sum_{i=1}^{k}\sum_{\alpha=1}^{v} \bar{\tau}_\alpha \zeta_{\alpha i}^{(p)} \pi_{\sigma(i)}^{(p)}\right|^3\right)\right]^{\frac{1}{3}}}{\left[\sum_{p=1}^{b} \mathrm{Var}\left(\sum_{i=1}^{k}\sum_{\alpha=1}^{v} \bar{\tau}_\alpha \zeta_{\alpha i}^{(p)} \pi_{\sigma(i)}^{(p)}\right)\right]^{\frac{1}{2}}} \to 0, \ (b \to \infty)$$

under the conditions (3.14). This shows that Liapounoff's condition is satisfied.

Hence from (3.16) and (3.24) it follows that, under the conditions (3.14), the permutation distribution of $X_1/\sqrt{\Delta_o \bar{T}_o b/(k-1)}$ converges in law, and hence, in the sense of type (S), to the standard normal N(0, 1), as $b \to \infty$. Similarly, under the conditions (3.14), the permutation distribution of $X_2/\sqrt{\Delta_o \bar{\bar{T}} b/(k-1)}$ and of $X/\sqrt{\Delta_o T_o b/(k-1)}$ converges in the sense of type (S) to the standard normal N(0, 1).

Thus, from (3.8) we can easily obtain the following

Lemma 1 Under the conditions (3.14),

(a) the permutation distribution of $\xi/(\Delta+T)$ converges in probability to unity as $b \to \infty$,

(b) the permutation distribution of $\bar{\eta}/(\bar{T}/(\Delta+T))$ converges in probability to unity as $b \to \infty$,

(c) the permutation distribution of $\bar{\bar{\eta}}/(\bar{\bar{T}}/(\Delta+T))$ converges in probability to unity as $b \to \infty$,

(d) the permutation distribution of $\eta/(T/(\Delta+T))$ converges in probability to unity as $b \to \infty$, and

(e) the joint distribution of

$$\left(\frac{\xi}{\Delta + T}, \frac{\bar{\eta}}{\bar{T}/(\Delta+T)}, \frac{\bar{\bar{\eta}}}{\bar{\bar{T}}/(\Delta+T)} \right), \text{ or of } \left(\frac{\xi}{\Delta + T}, \frac{\eta}{T/(\Delta+T)} \right)$$

converges in probability to (1,1,1) or to (1,1) respectively as $b \to \infty$.

4. A probability distribution which is asymptotically equivalent (S) to the non-null distribution of the F-statistic after the randomization

In this section, we derive a probability distribution which is asymptotically equivalent (S) to the non-null distribution of the F-statistic, $\bar{F}$ or F, under the conditions (3.14).

As was mentioned in the last of Section 2, the non-null distribution of $\bar{F}$ after the randomization is obtained by integrating out the quantities ξ, $\bar{\eta}$ and $\bar{\bar{\eta}}$ in (2.12) with respect to their permutation distribution due to randomization, for which theorem 3.2 or Corollary 3.2 of [4] and Lemma 1 of the preceding section can be applied to obtain a probability distribution which is asymptotically equivalent (S) to the non-null distribution of $\bar{F}$, i.e., a probability distribution whose probability element being

$$\exp\left(-\frac{\Delta + T}{2\sigma^2}\right) \sum_{\ell=0}^{\infty} \frac{\left(\frac{\Delta + T}{2\sigma^2}\right)^{\ell}}{\ell!} \sum_{\mu+\nu+\gamma=\ell} \frac{\ell!}{\mu!\ \nu!\ \gamma!} \left(\frac{\bar{T}}{\Delta + T}\right)^{\mu} \left(\frac{\bar{\bar{T}}}{\Delta + T}\right)^{\nu} \left(1 - \frac{T}{\Delta + T}\right)^{\gamma}$$

(4.1)

$$\frac{\Gamma\left(\frac{n-b-\bar{\bar{\alpha}}}{2} + \mu + \nu\right)}{\Gamma\left(\frac{\bar{\alpha}}{2} + \mu\right)\Gamma\left(\frac{n-b-v+1}{2} + \nu\right)} \cdot \left(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right)^{\frac{\bar{\alpha}}{2} + \mu - 1} \left(1 + \frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right)^{-\left(\frac{n-b-\bar{\bar{\alpha}}}{2} + \mu + \nu\right)}$$

$$d\left(\frac{\bar{\alpha}}{n-b-v+1}\bar{F}\right),$$

is asymptotically equivalent in the sense of type (S) to the non-null distribution of $\bar{F}$ after the randomization, as $b \to \infty$, provided that the conditions (3.14) are satisfied.

It is easily checked that (4.1) gives a probability element of some probability distribution.

Quite analogously, a probability distribution, which is asymptotically equivalent in the sense of type (S) to the non-null distribution of F after the randomization, is given by

$$\exp\left(-\frac{\Delta + T}{2\sigma^2}\right) \sum_{\ell=0}^{\infty} \frac{\left(\frac{\Delta + T}{2\sigma^2}\right)^{\ell}}{\ell!} \sum_{\mu+\nu=\ell} \frac{\ell!}{\mu!\ \nu!} \left(\frac{T}{\Delta + T}\right)^{\mu} \left(1 - \frac{T}{\Delta + T}\right)^{\nu} .$$

(4.2)

$$\frac{\Gamma(\frac{n-b}{2} + \ell)}{\Gamma(\frac{v-1}{2} + \mu)\Gamma(\frac{n-b-v+1}{2} + \nu)} \left(\frac{v-1}{n-b-v+1} F\right)^{\frac{v-1}{2} + \mu - 1} \left(1 + \frac{v-1}{n-b-v+1} F\right)^{-(\frac{n-b}{2} + \ell)}$$

$$d\left(\frac{v-1}{n-b-v+1} F\right),$$

under the conditions (3.14).

These are the results we wanted to derive, for which some discussions will be given below.

(i) The results show that, as far as we use these probability distributions as the asymptotic distributions of $\bar{\bar{F}}$ and of F, the influence of the existence of plot-effects on the non-null distributions of these F-statistics can not be eleminated by the randomization procedure adopted, even if the strong limiting conditions (3.14) are assumed to be satisfied.

(ii) In the case when $\Delta = 0$, (4.2) gives the same probability element as that given by (1.10), i.e., as that of the non-null distribution of the F-statistic under the usual Neyman model (without plot-effects). Likewise, if we put $\Delta = 0$ in (4.1), the resulting probability element

$$\exp\left(- \frac{T}{2\sigma^2}\right) \sum_{\ell=0}^{\infty} \frac{\left(\frac{T}{2\sigma^2}\right)^{\ell}}{\ell!} \sum_{\mu+\nu=\ell}^{\infty} \frac{\ell!}{\mu!\ \nu!} \left(\frac{\bar{T}}{T}\right)^{\mu} \left(1 - \frac{\bar{T}}{T}\right)^{\nu} .$$

(4.3)

$$\frac{\Gamma(\frac{n-b-\bar{\bar{\alpha}}}{2}+\ell)}{\Gamma(\frac{\bar{\alpha}}{2} + \mu)\Gamma(\frac{n-b-v+1}{2} + \nu)} \left(\frac{\bar{\alpha}}{n-b-v+1} \bar{\bar{F}}\right)^{\frac{\bar{\alpha}}{2} + \mu - 1} \left(1 + \frac{\bar{\alpha}}{n-b-v+1} \bar{\bar{F}}\right)^{-(\frac{n-b-\bar{\bar{\alpha}}}{2}\ \ell)} d\left(\frac{\bar{\alpha}}{n-b-v+1} \bar{\bar{F}}\right),$$

gives the probability element of the exact non-null distribution of $\overline{F}$ given by (1.4) with (1.5) under the usual Neyman model.

(iii) There may be other probability distributions which are asymptotically equivalent in the sense of type (S) to the non-null distribution of $\overline{F}$ or of F. One of them will be derived by investigating the conditional distribution of $(\overline{\eta}, \overline{\overline{\eta}})$ or of η given $\xi = \Delta + T$ and applying theorem 3.1 of [4]. In this case, the conditional variable of η, for example, given $\xi = \Delta + \Gamma$, becomes

$$(\eta | \xi = \Delta + \Gamma) = (T + \pi'(\sum_{u=1}^{m} V_u^{\#})\pi)/(\Delta + T).$$

But, since $\tau'\Phi'\pi$ and $\pi'(\sum_{u=1}^{m} V_u^{\#})\pi$ are not mutually independent in the stochastic sense under the permutation distribution due to the randomization in general, the conditional distribution of $\frac{k-1}{\Delta_0}\pi'(\sum_{u=1}^{m} V_u^{\#})\pi$ given $\tau'\Phi'\pi = 0$ is no longer asymptotically approximated (S) by the central chi-square distribution of degrees of freedom v-1, but by certain weighted chi-square distribution whose weighing coefficients depends on the matrix C given in the section 2 of the previous paper [2]. This causes a difficulty in practical application, because it is difficult to express the elements of the matrix C by the parameters of association and the design under consideration.

In connection with this, it should be noted that, in order to derive the results of this section by using Theorem 3.2 of [4], it is not necessary to state the asymptotic normality of the variables X_1, X_2 or X given by (3.11), but merely to obtain the variances of these variables. However, if we intend to work out another asymptotic distribution of the non-null distribution of $\overline{F}$ or F by the method stated above, the asymptotic normality of X_1, X_2 or X will be needed.

Acknowledgements

The authors want to express their gratitude to Professor N. L. Johnson for his encouragements. Thanks are also due to Mrs. Kay Herring for her nice typewriting of this manuscript.

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