CONFIDENCE INTERVALS FOR THE MEAN OF A FINITE POPULATION

By

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1. Introduction.

The admissibility of estimates of the population total with the Squared Error as the loss function were considered by the author (1965, II) and a certain estimate was shown to be always admissible whatever be the sampling design, in the entire class of all estimates. This estimate is equivalent to using the sample mean as estimate of the population mean. Here we consider the allied question of admissibility of the confidence intervals for the population mean based on the sample mean and the sample standard deviation, which are also commonly used in practice. These confidence intervals are here shown to be also always admissible whatever be the sampling design. The result is proved for a wider class of sets of symmetrical confidence intervals of which the above mentioned confidence intervals are a particular case.

2. Notation and Definitions.

The population u consists of N units $u_1, u_2, \ldots, u_N$; with the unit u_i is associated the variate value x_i, $i = 1, 2, \ldots, N$; $x = (x_1, x_2, \ldots, x_N)$ denotes a point in the Euclidean N-space R_N; a sample s means any subset of u; S denotes the set of all possible samples s; on s is defined a function p such that

$$p(s) \geq 0 \text{ for all } s \text{ and } \sum_{s \in S} p(s) = 1$$

The samples s are drawn at random with probability p(s) and the sampling design $d = (s, p)$. Then we define

DEFINITION 2.1 An estimate $e(s,x)$ is a real function e defined on $S \times R_N$ which depends on x through only those x_i for which $u_i \in s$.

The above definitions of sampling design and estimate are wide enough to cover all sampling procedures and classes of estimates; for a brief account we refer to Godambe and Joshi (1965, I, Section 5).

We next define admissibility of a set of confidence intervals for the population mean,

(1) $$\bar{X}_N = \frac{1}{N} \sum_{i=1}^{N} x_i$$

For a given sampling design d, we denote by $\bar{S}$, the subset of S, $\bar{S} \subset S$, consisting of all those samples s for which $p(s) > 0$. Now let $e_1(s, x)$, $e_2(s, x)$ be two estimates (Definition 2.1) such that $e_1(s,x) \leq e_2(s,x)$ for all $x \in R_N$ and all $s \in \bar{S}$; then $\{e_1(s,x), e_2(s,x)\}$ denotes the set of confidence intervals

(2) $$e_1(s,x) \leq \bar{X}_N \leq e_2(s,x)$$

For every $x \in R_N$, let $\bar{S}_{e_1, e_2, x}$ denote the subset of $\bar{S}$, $\bar{S}_{e_1, e_2, x} \subset \bar{S}$, consisting of all those samples s for which (2) holds, and for any alternative set of confidence intervals $\{e_1'(s,x), e_2'(s,x)\}$ let $\bar{S}_{e_1', e_2', x}$, denote the subset of $\bar{S}$, $\bar{S}_{e_1', e_2', x} \subset \bar{S}$, consisting of all those samples s for which

(3) $$e_1'(s,x) \leq \bar{X}_N \leq e_2'(s,x)$$

We now define

DEFINITION 2.2 The set of confidence intervals $\{e_1(s,x), e_2(s,x)\}$ for the population mean is admissible if there exists no other set of confidence intervals $\{e_1'(s,x), e_2'(s,x)\}$ such that

(i) $e_2'(s,x) - e_1'(s,x) \leq e_2(s,x) - e_1(s,x)$ for all $x \in R_N$ and for all $s \in \bar{S}$

and

(ii) $$\sum_{s \in \bar{S}_{e_1', e_2', x}} p(s) \geq \sum_{s \in \bar{S}_{e_1, e_2, x}} \qquad \text{for all } x \in R_N$$

the strict inequality in (ii) holding for at least one $x \in R_N$. The sums in (ii) are obviously the inclusion probabilities for the confidence intervals.

We also define a weaker version of admissibility by,

DEFINITION 2.3 The set of confidence intervals $\{e_1(s,x), e_2(s,x)\}$ for the population mean is weakly admissible if there exists no other set of confidence intervals $\{e_1'(s,x), e_2'(s,x)\}$ such that

(i) $e_2'(s,x) - e_1'(s,x) \leq e_2(s,x) - e_1(s,x)$ for all $x \in R_N$ and all $s \in \bar{S}$,

and (ii) $$\sum_{s \in \bar{S}_{e_1', e_2', x}} p(s) \geq \sum_{s \in \bar{S}_{e_1, e_2, x}} p(s) ,$$

for almost all (Lebesgue measure) $x \in R_N$, the strict inequality in (ii) holding on a non null subset of R_N.

To distinguish the admissibility as defined in Definition 2.2 from weak admissibility we shall refer to the former as strict admissibility.

The above definitions of admissibility of confidence intervals are based on the definition for infinite frequency functions with an unknown parameter formulated by Godambe (1961) and subsequently slightly modified by the author (1966)

3. Main Result.

For a sample s the sample mean $\bar{x}_s$ is given by

$$(4) \qquad \bar{x}_s = \frac{1}{n(s)} \sum_{i \epsilon s} x_i$$

where $i \in s$ is written shortly for $u_i \epsilon s$ and is so written hereafter and n(s) denotes the sample size i.e. the number of units in the sample s.

The sample standard deviation is

$$(5) \qquad v'(s,x) = \left[\frac{1}{n(s)} \sum_{i \epsilon s} (x_i - \bar{x}_s)^2 \right]^{\frac{1}{2}}$$

The usual confidence intervals based on the sample standard deviation are then $\{e_1(s,x),\ e_2(s,x)\}$ where

$$(6)\qquad \begin{aligned} e_1(s,x) &= \bar{x}_s - k\ v'(s,x) \\ e_2(s,x) &= \bar{x}_s + k\ v'(s,x) \end{aligned}$$

where k is a positive constant, usually = 2 or 3. However we shall prove the result for the more general class of confidence intervals given by

$$(7)\qquad \begin{aligned} e_1(s,x) &= \bar{x}_s - v(s,x) \\ e_2(s,x) &= \bar{x}_s + v(s,x) \end{aligned}$$

where $v(s,x)$ is any arbitrary non-negative estimate (definition 2.1). As a first step towards proving strict admissibility we prove

THEOREM 3.1 The set of confidence intervals $\{e_1(s,x),\ e_2(s,x)\}$ in (7) is weakly admissible.

PROOF Suppose the set is not weakly admissible. Then by definition 2.3, there exists an alternative set of confidence intervals $\{e_1'(s,x),\ e_2'(s,x)\}$, satisfying

$$(8)\qquad e_2'(s,x) - e_1'(s,x) \leq e_2(s,x) - e_1(s,x) \quad \text{for all} \quad x \in R_N \text{ and all } s \in \bar{S},$$

and

$$(9)\qquad \sum_{s \in \bar{S}_{e_1, e_2, x}} p(s) \geq \sum_{s \in \bar{S}_{e_1', e_2', x}} p(s)$$

for almost all $x \in R_N$, the strict inequality in (9) holding on a non-null set.

Note: Throughout the rest of this paper the measure considered will be the Lebesgue measure on R_N. So also for any k demensional subspace of R_N, the measure considered will be the Lebesgue measure for the k dimensional subspace. When the measure considered is for a k-dimensional subspace, it will be indicated by (μ_k). These points will not be repeated each time.

Now put

$$e_1''(s,x) = \frac{e_1'(s,x) + e_2'(s,x)}{2} - v(s,x)$$

(10) and

$$e_2''(s,x) = \frac{e_1'(s,x) + e_2'(s,x)}{2} + v(s,x)$$

Then

(11) $$e_1''(s,x) = e_1'(s,x) + \tfrac{1}{2}[e_2'(s,x) - e_1'(s,x) - 2v(s,x)] \le e_1'(s,x) \quad \text{by (8)}$$

and

(12) $$e_2''(s,x) = e_2'(s,x) - \tfrac{1}{2}[e_2'(s,x) - e_1'(s,x) - 2v(s,x)] \ge e_2'(s,x) \quad \text{by (8)},$$

so that denoting by $\bar{S}_{e_1'',e_2'',x}$ for every $x \in R_N$, the subset of $\bar{S}$, on which

(13) $$e_1''(s,x) \le \bar{X}_N \le e_2''(s,x)$$

we have from (11), (12) and (3)

(14) $$\bar{S}_{e_1'',e_2'',x} \supset \bar{S}_{e_1',e_2',x}$$

(14) combined with (9) now gives

(15) $$\sum_{s\epsilon\bar{S}_{e_1'',e_2'',x}} p(s) \ge \sum_{s\epsilon\bar{S}_{e_1,e_2,x}} p(s) \quad ,$$

for almost all $x \in R_N$, the strict inequality in (15) holding on a non-null subset of R_N.

We now take the expectations of both sides of (15) w.r.t. a prior distribution ξ on R_N, such that all the x_i, i=1,2,...,N are distributed identically. We denote as usual by P[D], the probability assigned under the distribution ξ to any subset D of R_N; further the set D will be denoted by specifying the relation satisfied by the co-ordinates of the points $x \in D$. Then on taking expectations of both sides of (15) we have

$$(16) \qquad \sum_{s \in \bar{S}} p(s)\, P[e_1''(s,x) \leq \bar{X}_N \leq e_2''(s,x)] \geq \sum_{s \in \bar{S}} p(s)\, P[e_1(s,x) \leq \bar{X}_N \leq e_2(s,x)]$$

Now putting in (10)

$$(17) \qquad \bar{e}(s,x) = \frac{e_1'(s,x) + e_2'(s,x)}{2}$$

and using (7), (16) is reduced to

$$(18) \qquad \sum_{s \in \bar{S}} p(s)\, P[\bar{e}(s,x) - v(s,x) \leq \bar{X}_N \leq \bar{e}(s,x) + v(s,x)]$$

$$\geq \sum_{s \in \bar{S}} p(s)\, P[\bar{x}_s - v(s,x) \leq \bar{X}_N \leq \bar{x}_s + v(s,x)]$$

As the x_i, i = 1, 2, ..., N, are by assumption distributed identically, the value of each probability occurring in (18), remains unaltered if in each of the functions $\bar{e}(s,x)$, $v(s,x)$ and $\bar{x}_s$, we replace the x_i for $i \in s$ taken in some particular order by $x_1, x_2, \ldots, x_{n(s)}$. Let by this substitution, the functions $\bar{e}(s,x)$, and $v(s,x)$ be respectively transformed into functions $f_s(x)$ and $w_s(x)$, which depend on x only through the co-ordinates $x_1, x_2, \ldots, x_{n(s)}$. $\bar{x}_s$ is obviously transformed into

$$(19) \qquad \bar{x}_{n(s)} = \frac{1}{n(s)} \sum_{i=1}^{n(s)} x_i$$

Then (18) is transformed into

(20) $$\sum_{s\in\bar{S}} p(s)\, P[f_s(x) - w_s(x) \leq \bar{X}_N \leq f_s(x) + w_s(x)]$$

$$\geq \sum_{s\in\bar{S}} p(s)\, P[\bar{x}_{n(s)} - w_s(x) \leq \bar{X}_N \leq \bar{x}_{n(s)} + w_{s(x)}]$$

Now putting

$$\bar{X}_{N-n(s)} = \frac{1}{N-n(s)} \sum_{i=n(s)+1}^{N} x_i$$

(21) $$g_s(x) = \frac{f_s(x) - \frac{n(s)}{N}\bar{x}_{n(s)}}{1 - \frac{n(s)}{N}}$$

$$u_{s(x)} = \frac{w_{s(x)}}{1 - \frac{n(s)}{N}}$$

and noting that

$$\bar{X}_N = \frac{n(s)\ \bar{x}_{n(s)}}{N} + \frac{(N-n(s))\ \bar{X}_{N-n(s)}}{N}$$

(20) is seen to reduce to

(22) $$\sum_{s\in\bar{S}} p(s)\, P[g_s(x) - u_s(x) \leq \bar{X}_{N-n(s)} \leq g_s(x) + u_s(x)]$$

$$\geq \sum_{s\in\bar{S}} p(s)\, P[\bar{x}_{n(s)} - u_s(x) \leq \bar{X}_{N-n(s)} \leq \bar{x}_{n(s)} + u_s(x)]$$

So far the only assumption regarding the prior distribution ξ on R_N is that all the x_i, i = 1, 2, ..., N are distributed identically. We now make the further assumption that all the x_i, i = 1, 2, ..., N are distributed idependently with a Normal distribution with mean θ and variance σ^2 so that

$$x_i \frown N(\theta,\sigma). \quad i, = 1, 2, \ldots, N$$

$$\bar{x}_{n(s)} \frown N(\theta, \frac{\sigma}{\sqrt{n(s)}})$$

and

$$\bar{X}_{N-n(s)} \frown N(\theta, \frac{\sigma}{\sqrt{N-n(s)}})$$

For given values of the 1^{st} $n(s)$ co-ordinates $x_1, x_2, \ldots, x_{n(s)}$ let

$$g_s(x) - \theta = t_s$$

and

$$\bar{x}_{n(s)} - \theta = \tau_s$$

Then for the given values of $x_1, x_2, \ldots, x_{n(s)}$ the conditional probability

(23) $$P[g_s(x)-u_s(x) \leq \bar{X}_{N-n(s)} \leq g_s(x)+u_s(x) | x_1, x_2, \ldots, x_{n(s)}]$$

$$= \frac{\sqrt{N-n(s)}}{\sqrt{2\pi}\ \sigma} \int_{t_s-u_s(x)}^{t_s+u_s(x)} e^{-\frac{(N-n(s))^2}{2\sigma^2} z^2} dz$$

$= \phi(|t_s|)$ say where $\phi(|x|)$ is strictly decreasing as $|x|$ increases. Here we have written $\phi(|t_s|)$ briefly for $\phi(|t_s|, u_s(x))$, where $u_s(x)$ may depend on $\bar{x}_{n(s)}$. By $\phi(|t_s|)$ being strictly decreasing is meant that $\phi(|t_s|, u_s(x))$ strictly decreases when $|t_s|$ alone is increased, the values of $u_s(x)$ being held fixed.

Similarly the conditional probability

(24) $$P[\bar{x}_{n(s)}-u_s(x) \leq \bar{X}_{N-n(s)} \leq \bar{x}_{n(s)}+u_s(x) | x_1, x_2, \ldots, x_{n(s)}]$$

$$= \phi(|\tau_s|)$$

Also the frequency function of the variables $x_1, x_2, \ldots, x_{n(s)}$ may be written in the form

$$(25)\qquad L(x_1, \ldots, x_{n(s)} | \bar{x}_{n(s)})\, e^{-\frac{n(s)}{2\sigma^2}(\bar{x}_{n(s)}-\theta)^2}$$

Hence transposing the r.h.s. of (22) to its l.h.s. and using (23), (24) and (25), we have, writing x for short for $x_1, x_2, \ldots, x_{n(s)}$ and dx for short for $dx_1, dx_2, \ldots, dx_{n(s)}$

$$(26)\qquad \sum_{s\in\bar{S}} p(s) \int L(x|\bar{x}_{n(s)})\, e^{-\frac{n(s)}{2\sigma^2}(\bar{x}_{n(s)}-\theta)^2} [\phi(|g_s(x)-\theta|-\phi(|\bar{x}_{n(s)}-\theta|)]dx \geq 0$$

We now integrate the l.h.s. of (26) w.r.t. θ from - a to + a. As the integrand is non-negative we may by Fubini's theorem interchange the order of integration w.r.t. $x=(x_1, x_2, \ldots, x_{n(s)})$ and θ. We therefore have

$$(27)\qquad \sum_{s\in\bar{S}} p(s) \int L(x|\bar{x}_{n(s)})dx \int_{-a}^{a} d\theta\, e^{-\frac{n(s)}{2\sigma^2}(\bar{x}_{n(s)}-\theta)^2} [\phi(|g_s(x)-\theta|)-\phi(|\bar{x}_s-\theta|)] \geq 0$$

The l.h.s. in (27) is non-decreasing as a increases. So as $a \to \infty$, it either converges to a limit or $\to \infty$. In either case we have

$$(28)\qquad \sum_{s\in\bar{S}} p(s) \int L(x|\bar{x}_{n(s)})dx \int_{-\infty}^{\infty} d\theta\, e^{-\frac{n(s)}{2\sigma^2}[\bar{x}_{n(s)}-\theta]^2} [\phi(|g_s(x)-\theta|)-\phi(|\bar{x}_s-\theta|)] \geq 0$$

In the l.h.s. of (28), the integral w.r.t. θ vanishes at any point $x=(x_1, x_2, \ldots, x_{n(s)}$ at which $g_s(x) = \bar{x}_{n(s)}$. We shall now show that at any point at which $g_s(x) \neq \bar{x}_{n(s)}$, this integral is < 0. For denoting the integrand by I, the integral w.r.t. θ in (28) may be expressed as

$$(29)\qquad \int_{-\infty}^{\frac{\bar{x}_{n(s)}+g_s(x)}{2}} I\, d\theta + \int_{\frac{x_{n(s)}+g_s(x)}{2}}^{\infty} I\, d\theta$$

Now suppose that $g_s(x) > \bar{x}_n(s)$. Then in the second integral in (29), transform the variable of integration by putting

$$\theta - g_s(x) = \bar{x}_{n(s)} - \theta_1$$

so that θ_1 goes from $-\infty$ to $\frac{\bar{x}_{n(s)}+g_s(x)}{2}$.

Also $\theta - \bar{x}_{n(s)} = g_s(x) - \theta_1$, so that the second integral becomes

$$(30) \qquad \int_{-\infty}^{\frac{g_s(x)+\bar{x}_{n(s)}}{2}} d\theta_1 \, e^{-\frac{n(s)}{2\sigma^2}(g_s(x)-\theta_1)^2} [\phi(|\bar{x}_{n(s)}-\theta_1|)-\phi(|g_s(x)-\theta_1|)]$$

Writing in (30), θ for θ_1, and combining with the first integral in (29), (29) reduces to

$$(31) \qquad \int_{-\infty}^{\frac{g_s(x)+\bar{x}_{n(s)}}{2}} d\theta\left[\phi(|g_s(x)-\theta|)-\phi(|\bar{x}_{n(s)}-\theta|)\right]\left[e^{-\frac{n(s)}{2\sigma^2}[\bar{x}_{n(s)}-\theta)^2} - e^{-\frac{n(s)}{2\sigma^2}(g_s(x)-\theta)^2}\right]$$

Since by assumption $g_s(x) > \bar{x}_{n(s)}$, we have in the range of integration in (31),

$$|g_s(x) - \theta| > |\bar{x}_{n(s)} - \theta|$$

so that since $\phi(|x|)$ strictly decreases as $|x|$ increases the first factor in the integrand of (31) is always < 0, while the second factor is > 0 and hence (31) and so (29) < 0. By a similar argument (29) is shown to be < 0 if at the point x, if $g_s(x) < \bar{x}_{n(s)}$. As (29) is the same as the integral w.r.t. θ in the l.h.s. of (27), it follows from (27) that the set of points $x = (x_1, x_2, \ldots x_{n(s)})$ at which

$$(32) \qquad g_s(x) \neq \bar{x}_{n(s)}$$

is for every $s \in \bar{S}$ a null $(\mu_{n(s)})$ set. Consequently since $g_s(x)$ and $\bar{x}_{n(s)}$ depend on $x = (x_1, x_2, \ldots, x_N)$ only through $x_1, x_2, \ldots, x_{n(s)}$, the set of points $x \in R_N$, at which (32) holds is a null (μ_N) set. It follows from the definition of $g_s(x)$ in (21) that

(33) $$f_s(x) = \bar{x}_{n(s)} \text{ a.e. in } R_N, \text{ for all } s \in \bar{S}$$

and since $f_s(x)$ is the transformed form of $\bar{e}(s,x)$ in (18), we have from (33)

(34) $$\bar{e}(s,x) = \bar{x}_s \text{ a.e. in } R_N, \text{ for all } s \in \bar{S}$$

Hence the strict inequality in (15) and consequently in (9), cannot hold on a non-null set. The assertion in Theorem 3.1 is thus proved.

4. Extention of Theorem 3.1.

We have now to prove that the weak admissibility of the confidence intervals proved in Theorem 3.1 implies their strict admissibility. A similar result in connection with the admissibility of a certain estimate was proved by the author (1965, III, sections 4 and 5) and the argument here runs closely parallel. We consider the hyperplanes in R_N obtained by assigning fixed values to some k of the variates. Let Q^{α}_{N-k} be the hyperplane in which say the last k variates x_{N-k+t}, t = 1, 2, ..., k have fixed values α_{N-k+t} respectively. Let $\bar{S}_k$ be the subset of $\bar{S}$, consisting of all those samples which contains each of the last k units u_{N-k+t}, t=1,2,...,k, i.e. $s \in \bar{S}_k$, if, and only if, $u_{n-k+t} \in s$ for all t = 1, 2, ..., k and $s \in \bar{S}$. We denote by $\bar{s}_k \cdot \bar{s}_{e_1,e_2,x}$ the intersection of the set $\bar{s}_k$ with the set $\bar{s}_{e_1,e_2,x}$ on which (2) holds; the intersections of $\bar{S}_k$ with the sets $\bar{S}_{e_1',e_2',x}$ on which (3) holds, and $S_{e_1'',e_2'',x}$ on which (13) holds, are denoted similarly. Now suppose that for $x \in Q^{\alpha}_{N-k}$, and $s \in \bar{S}_k$, estimates $e_1'(s,x)$, $e_2'(s,x)$ exist such that,

(35) $$\sum_{s \in \bar{S}_k \cdot \bar{S}_{e_1',e_2',x}} p(s) \geq \sum_{s \in \bar{S}_k \cdot \bar{S}_{e_1,e_2,x}} p(s)$$

holds for almost all (μ_{N-k}) $x \in Q^{\alpha}_{N-k}$ and that further $\bar{e}(s,x)$ being as in (17)

$$h(s,x) = \bar{e}(s,x) - \bar{x}_s \neq 0 \text{ for at least one } s \in \bar{S}_k$$

for a non-null (μ_{N-k}) subset of points $x \in Q^{\alpha}_{N-k}$. We shall show that this supposition leads to a contradiction. As before (35) implies

$$(36) \qquad \sum_{s \in \bar{S}_k \cdot \bar{S}_{e_1'', e_2'', x}} p(s) \geq \sum_{s \in \bar{S}_k \cdot \bar{S}_{e_1, e_2, x}} p(s)$$

for almost all (μ_{N-k}) $x \in Q^{\alpha}_{N-k}$. Now in (35) and (36), the estimates $e_1'(s,x)$, $e_2'(s,x)$, and hence $e_1''(s,x)$, $e_2''(s,x)$ and $e(s,x)$ are defined only for samples $s \in \bar{S}_k$ and points $x \in Q^{\alpha}_{N-k}$. We now extend their definitions to other points $x \in R_N$ and to samples $s \notin \bar{S}_k$ as follows.

Let $Q^{\alpha'}_{N-k}$ be the hyperplane $\subset R_N$ given by $x_{N-k+t} = \alpha'_{N-k+t}$, $t = 1, 2, \ldots, k$. We establish a 1-1 correspondence between the points $x \in Q^{\alpha}_{N-k}$ and $x' \in Q^{\alpha'}_{N-k}$ by putting

$$(37) \qquad x_r' = x_r + a, \quad r=1, 2, \ldots, N-k$$

the constant a being so fixed, that denoting by

$$(38) \qquad \bar{x}_s' = \frac{1}{n(s)} \sum_{i \in s} x_i' \qquad \bar{X}_N' = \frac{1}{N} \sum_{i=1}^{N} x_i'$$

we have, for all $x \in Q^{\alpha}_{N-k}$ and all $s \in \bar{S}_k$,

$$(39) \qquad \bar{X}_N' - \bar{x}_s' = \bar{X}_N - \bar{x}_s$$

This can always be done, for putting

$$\bar{\alpha}' = \frac{1}{k} \sum_{r=N-k+1}^{N} \alpha'_r$$

(40)

$$\bar{\alpha} = \frac{1}{k} \sum_{r=N-k+1}^{N} \alpha_r$$

we have

$$\bar{x}'_s = \frac{1}{n(s)} [k\, \bar{\alpha}' + n(s)\, \bar{x}_s - k\, \bar{\alpha} + (n(s)-k)\, a]$$

and

$$\bar{X}'_N = \frac{1}{N}[k\, \bar{\alpha}' + N\, \bar{X}_N - k\, \bar{\alpha} + (N-k)a\ ]$$

so that

$$\bar{X}'_N - \bar{x}'_s = \bar{X}_N - \bar{x}_s - k(\frac{1}{n(s)} - \frac{1}{N})(\bar{\alpha}' - \bar{\alpha} - a)$$

so that (39) is satisfied if

(41) $\quad a = \bar{\alpha}' - \bar{\alpha}$

We now introduce a new estimate $\overset{*}{v}(s,x)$ defined as follows; for $x \in Q^{\alpha}_{N-k}$ and $s \in \bar{S}_k$,

(42) $\quad \overset{*}{v}(s,x) = v(s,x)$

and for $x' \in Q^{\alpha'}_{N-k}$, $s \in \bar{S}_k$

(43) $\quad \overset{*}{v}(s,x') = v(s,x)$

Since by (42) and (43) $\overset{*}{v}(s,x)$ is defined for every hyperplane $Q^{\alpha'}_{N-k}$, it is defined for all $x \in R_N$, for all $s \in \bar{S}_k$. In place of the original set of condidence intervals $\{e_1(s,x),\ e_2(s,x)\}$ in (7) we consider a new set $\{e^*_1(s,x),\ e^*_2(s,x)\}$ where

(44) $$e_1^*(s,x) = \bar{x}_s - \overset{*}{v}(s,x)$$
$$e_2^*(s,x) = \bar{x}_s + \overset{*}{v}(s,x)$$

We denote by $\bar{S}_{e_1^*, e_2^*, x,}$ the subset of $\bar{S}$, for each $x \in R_N$, for which

(45) $$e_1^*(s,x) \leq \bar{X}_N \leq e_2^*(s,x) \text{ holds.}$$

We now extend the definitions of $\bar{e}(s,x)$, $e_1''(s,x)$ and $e_2''(s,x)$ to other hyperplanes $Q_{N-k}^{\alpha'}$ by putting

(46) $$\bar{e}(s,x') = \bar{e}(s,x) + \bar{x}_s' - \bar{x}_s$$
$$e_1''(s,x') = \bar{e}(s,x') - \overset{*}{v}(s,x')$$
$$e_2''(s,x') = \bar{e}(s,x') + \overset{*}{v}(s,x')$$

Since for $x \in Q_{N-k}^{\alpha}$, $e_1^*(s,x)$ and $e_2^*(s,x)$ coincide with $e_1(s,x)$ and $e_2(s,x)$, for every $x \in Q_{N-k}^{\alpha}$

$$\bar{S}_{e_1^*, e_2^*, x} = \bar{S}_{e_1, e_2, x}$$

and hence from (36)

(47) $$\sum_{s \in \bar{S}_k \cdot \bar{S}_{e_1'', e_2'', x}} p(s) \geq \sum_{s \in \bar{S}_k \cdot \bar{S}_{e_1^*, e_2^*, x}} p(s)$$

for almost all (μ_{N-k}) $x \in Q_{N-k}^{\alpha}$ and further from the observation below (35)

(48) $$h(s,x) = \bar{e}(s,x) - \bar{x}_s \neq 0 \quad \text{for at least one } s \in \bar{S}_k ,$$

for a non-null (μ_{N-k}) subset of $Q_{N-k}^{\alpha'}$.

Now (47) and (48) are easily seen to hold for every other hyperplane $Q_{N-k}^{\alpha'}$. For, for any point $x' \in Q_{N-k}^{\alpha'}$, by (44), $e_1^*(s,x') = \bar{x}_s' - \overset{*}{v}(s,x') = \bar{x}_s - \overset{*}{v}(s,x) + (\bar{x}_s' - \bar{x}_s)$

$$e_2^*(s,x') = \bar{x}_s' + \overset{*}{v}(s,x') = \bar{x}_s + \overset{*}{v}(s,x) + (\bar{x}_s' - \bar{x}_s) ,$$

by (39), $\bar{x}_N' = \bar{x}_N + (\bar{x}_s' - \bar{x}_s)$

and by (46), and (43), $e_1''(s,x') = e_1''(s,x) + (\bar{x}_s' - \bar{x}_s)$

$$e_2''(s,x') = e_2''(s,x) + (\bar{x}_s' - \bar{x}_s)$$

so that for every sample $s \in \bar{S}_k$, at every point $x' \in Q_{N-k}^{\alpha'}$,

$$e_1^*(s,x') \leq \bar{X}_N \leq e_2^*(s,x')$$

holds, if and only if, at the corresponding point $x \in Q_{N-k}^{\alpha}$ defined by (37)

$$e_1(s,x) \leq \bar{X}_N \leq e_2(s,x)$$

and similarly

$$e_1''(s,x') \leq \bar{X}_N \leq e_2(s,x')$$

holds, if and only if

$$e_1''(s,x) \leq \bar{X}_N \leq e_2(s,x) \text{ holds.}$$

Hence

$$\bar{S}_k \cdot \bar{S}_{e_1^*, e_2^*, x'} = \bar{S}_k \cdot \bar{S}_{e_1^*, e_2^*, x} \qquad \text{(Note remark below (46))}$$

(49)

$$\bar{S}_k \cdot \bar{S}_{e_1'', e_2'', x'} = \bar{S}_k \cdot \bar{S}_{e_1'', e_2'', x}$$

From (49) it follows that (47) holds for almost all (μ_N) $x \in R_N$ and since every hyperplane $Q_{N-k}^{\alpha'}$ contains a non-null (μ_{N-k}) subset on which (48) holds, the subset of R_N for which (48) holds is non-null (μ_N).

Now consider samples $s \notin \bar{S}_k$, i.e. $s \in (\bar{S} - \bar{S}_k)$. We have so far not defined the values of $\bar{e}(s,x)$, $e_1'(s,x)$, $e_2''(s,x)$, and $\overset{*}{v}(s,x)$ for $s \in (\bar{S} - \bar{S}_k)$. We now put,

for $s \in (\bar{S}-\bar{S}_k)$ and all $x \in R_N$

$$\bar{e}(s,x) = \bar{x}_s$$

$$\overset{*}{v}(s,x) = v(s,x)$$

$$e_1''(s,x) = \bar{e}(s,x) - \overset{*}{v}(s,x)$$

$$e_2''(s,x) = \bar{e}(s,x) + \overset{*}{v}(s,x)$$

$e_1^*(s,x)$ and $e_2^*(s,x)$ being given by (44). Then for $s \in (\bar{S} - \bar{S}_k)$, and all $x \in R_N$

$$e_1''(s,x) = e_1^*(s,x)$$
$$e_2''(s,x) = e_2^*(s,x)$$

Hence denoting by $(\bar{S} - \bar{S}_k)\cdot\bar{S}_{e_1^*, e_2^*, x}$ and $(\bar{S} - \bar{S}_k)\cdot\bar{S}_{e_1'', e_2'', x}$ the intersection sets of $(\bar{S} - \bar{S}_k)$ with $\bar{S}_{e_1^*, e_2^*, x}$ and $\bar{S}_{e_1'', e_2'', x}$ respectively, we have

$$(\bar{S} - \bar{S}_k)\cdot\bar{S}_{e_1'', e_2'', x} = (\bar{S} - \bar{S}_k)\cdot\bar{S}_{e_1^*, e_2^*, x},$$

so that

$$(50) \qquad \sum_{s\epsilon(\bar{S}-\bar{S}_k)\cdot\bar{S}_{e_1'', e_2'', x}} p(s) = \sum_{s\epsilon(\bar{S}-\bar{S}_k)\cdot\bar{S}_{e_1^*, e_2^*, x}} p(s)$$

combining (47), (48) and (50)

$$(51) \qquad \sum_{s\epsilon\bar{S}_{e_1'', e_2'', x}} p(s) \geq \sum_{s\epsilon\bar{S}_{e_1^*, e_2^*, x}} p(s)$$

for almost all $x \epsilon R_N$, and

$$(52) \qquad h(s,x) = \bar{e}(s,x) - \bar{x}_s \neq 0$$

for at least one $s \epsilon \bar{S}$, for a non-null (μ_N) subset of R_N.

But Theorem 3.1 applies to the set of confidence intervals $\{e_1^*(s,x), e_2^*(s,x)\}$. Hence by (34), the set of points on which (52) holds must be a null (μ_N) set of R_N. Thus our original assumption below (35) that

$h(s,x) = \bar{e}(s,x) - \bar{x}_s \neq 0$ for at least one $s \epsilon \bar{S}_k$ for a non-null (μ_{N-k}) subset of Q^{α}_{N-k} must itself be false. We thus have

THEOREM 4.1 If estimates $e_1'(s,x)$, $e_2'(s,x)$ exist, such that (35) holds for almost all (μ_{N-k}) $x \epsilon Q^{\alpha}_{N-k}$, then with $\bar{e}(s,x)$ as in (17), $h(s,x) = \bar{e}(s,x) - \bar{x}_s$, vanishes for all $s \epsilon \bar{S}_k$, for almost all (μ_{N-k}) $x \epsilon Q^{\alpha}_{N-k}$.

5. Strict Admissibility.

We now come to the final part of the argument. The argument is closely similar to that of Theorem 5.1 proved by the author (1965, III), but as the proof was given there for a fixed sample size design, while here we have a varying size design, we shall again indicate the main steps in the proof.

Let $E \subset R_N$ be the set of all points $x \in R_N$, for which $h(s,x) = \bar{e}(s,x)-\bar{x}_s \neq 0$ for at least one $s \in \bar{S}$. By (34), E is a null set, and we have to show that it is empty. Suppose it is not empty, then there exists at least one point $x = a = (a_1, a_2, \ldots, a_N)$ and one sample $s_o \in \bar{S}$ such that $h(s_o,a) = h_o \neq o$. Without loss of generality we may suppose the sample s_o to consist of the first m units $u_1, u_2, \ldots, u_m$. Consider the (N-m) dimensional hyperplane P^a_{N-m} defined by

(53) $$x \in P^a_{N-m} \text{ if and only if, } x_i = a_i, \; i = 1, 2, \ldots, m.$$

For every $x \in P^a_{N-m}$

$$\bar{x}_{s_o} = \frac{1}{m} \sum_{i=1}^{m} a_i = \bar{a}_o \quad \text{say}$$

$$\bar{e}(s_o, x) = \bar{a}_o + h_o$$

$$v(s_o, x) = v(s_o, a)$$

and

$$\bar{X}_N = \frac{m\,\bar{a}_o}{N} + \frac{1}{N} \sum_{i=m+1}^{N} x_i$$

Hence for $x \in P^a_{N-m}$

$$\bar{x}_{s_o} - v(s_o,x) \leq \bar{X}_N \leq \bar{x}_{s_o} + v(s_o,x) \text{ holds if, and only if,}$$

(54) $$\bar{a}_o(1 - \frac{m}{N}) - v(s_o,a) \leq \frac{1}{N} \sum_{i=m+1}^{N} x_i \leq \bar{a}_o(1 - \frac{m}{N}) + v(s_o,a)$$

and similarly

$$e_1''(s,x) = \bar{e}(s,x) - v(s,x) \leq \bar{X}_N \leq \bar{e}(s,x) + v(s,x) = e_2''(s,x)$$

holds, if and only if

$$(55) \qquad \bar{a}_o(1 - \frac{m}{N}) - v(s_o,a) + h_o \leq \frac{1}{N} \sum_{i=m+1}^{N} x_i \leq \bar{a}_o(1 - \frac{m}{N}) + v(s_o,a) + h_o$$

We now determine a subset Q^a_{N-m} of P^a_{N-m} such that for $x \in Q^a_{N-m}$, (54) holds but (55) does not, as follows:

(56) If $h_o > o$, $x \in Q^a_{N-m}$, if and only if,

$$\bar{a}_o(1 - \frac{m}{N}) + v(s_o,a) < \frac{1}{N} \sum_{i=m+1}^{N} x_i \leq \bar{a}_o(1 - \frac{m}{N}) + v(s_o,a) + h_o$$

and if $h_o < o$

$$\bar{a}_o(1 - \frac{m}{N}) - v(s_o,a) + h_o \leq \frac{1}{N} \sum_{i=m+1}^{N} x_i < \bar{a}_o(1 - \frac{m}{N}) - v(s_o,a)$$

Since $h_o \neq o$, Q^a_{N-m} is always determined and has infinite measure (μ_{N-m}).

Since we are considering strict admissibility now, we assume that the alternative confidence intervals $(e_1'(s,x)\ e_2'(s,x))$ are such that as required by condition (ii) of Definition 2.2,

$$(57) \qquad \sum_{s \in \bar{S}_{e_1',e_2',x}} p(s) \geq \sum_{s \in \bar{S}_{e_1,e_2,x}} p(s)$$

holds for all $x \in R_N$ and not merely for almost all $x \in R_N$. Then by (14)

$$(58) \qquad \sum_{s \in \bar{S}_{e_1'',e_2'',x}} p(s) \geq \sum_{s \in \bar{S}_{e_1,e_2,x}} p(s)$$

holds for all $x \in R_N$, the strict inequality holding for at least one $x \in R_N$. Now by (56) for every point $x \in Q^a_{N-m}$,

$$s_o \in \bar{S}_{e_1, e_2, x}$$

and $s_o \notin \bar{S}_{e_1'', e_2'', x}$

and hence at every such point, there must be at least one other sample $s \in \bar{S}$, for which $h(s,x) = \bar{e}(s,x) - \bar{x}_s \neq 0$ as otherwise at this point the r.h.s. of (58) will exceed its l.h.s.

Let E^a_{N-m} be the subset of all the points $x \in P^a_{N-m}$, for which $h(s,x) \neq 0$ for at least one $s \in \bar{S}$. Obviously

(59) $$Q^a_{N-m} \subset E^a_{N-m}$$

and since Q^a_{N-m} is of infinite measure (μ_{N-m}), E^a_{N-m} is of infinite measure (μ_{N-m}). We now partition the set E^a_{N-m} into (not necessarily) disjoint subsets indexed by the samples $s \in \bar{S}$. Let for a specified sample s, $L^{a,s}_{N-m}$ be the subset consisting of all those points $x \in E^a_{N-m}$ for which $h(s,x) \neq o$ i.e.

$$x \in L^{a,s}_{N-m} \text{ if, and only if, } x \in E^a_{N-m} \text{ and } h(s,x) \neq 0.$$

Then from the definition of E^a_{N-m} it follows that

(60) $$E^a_{N-m} = \bigcup_{s \in \bar{S}} L^{a,s}_{N-m}$$

Since E^a_{N-m} has infinite measure (μ_{N-m}) there must be at least one non-null (μ_{N-m}) set in the r.h.s. of (60).

Now there are two possibilities:

(A) For every non-null (μ_{N-m}) set $L^{a,s}_{N-m}$ in the r.h.s. of (60), the sample $s \neq s_o$, contains all the units $u_1, u_2, \ldots, u_m$ with in addition some other units;

(B) There exists at least one non-null (μ_{N-m}) set $L^{a,s}_{N-m}$ in the r.h.s. of (60) for which the sample s, contains only some k, $(0 \leq k < m)$ out of the first m units.

We shall first consider that (B) holds and there exist one or more non-null sets $L_{N-m}^{a,s}$ in the r.h.s. of (60) for which the sample s does not contain all the first m units; if there are more than one such set we select one of them arbitrarily. Let L_{N-m}^{a,s_1} be the set selected and suppose the sample s_1 is of sample size m_1 and that it contains k, $(0 \leq k < m)$ out of the first m units, the remaining (m_1-k) units being from the last (N-m) units $u_{m+1}, u_{m+2}, \ldots, u_N$. Then take any point $a' \in L_{N-m}^{a,s_1}$. Since $a' \in P_{N-m}^{a}$, we have

$$(61) \qquad a' = \{a_1, a_2, \ldots, a_m, a'_{m+1}, a'_{m+2} \ldots, a'_N\}$$

Then for the point a' we define as in (53) a $(N-m_1)$ dimensional hyperplane $P_{N-m_1}^{a'}$ by

$$(62) \qquad \begin{aligned} x_i &= a_i \quad \text{for } i \in s_1, \quad i \leq m \\ &= a'_i \quad \text{for } i \in s_1, \quad i > m \end{aligned}$$

Next putting

$$\bar{a}_1 = \frac{1}{m_1}\left[\sum_{\substack{i \epsilon s_1 \\ i \leq m}} a_i + \sum_{\substack{i \epsilon s_1 \\ i > m}} a'_i \right]$$

we define as in (56) a subset $Q_{N-m_1}^{a'} \subset P_{N-m_1}^{a'}$ by

$$(63) \qquad x \in Q_{N-m_1}^{a'} \text{ if, and only if, } x \epsilon P_{N-m_1}^{a'}$$

and, if $h = h(s_1, a') > 0$

$$\bar{a}_1(1 - \frac{m_1}{N}) + v(s_1, a') < \frac{1}{N} \sum_{i \notin s_1} x_i \leq \bar{a}_1(1 - \frac{m_1}{N}) + v(s_1, a') + h_1$$

and, if $h_1 < 0$,

$$\bar{a}_1(1 - \frac{m_1}{N}) + v(s_1, a') + h_1 \leq \frac{1}{N} \sum_{i \notin s_1} x_i < \bar{a}_1(1 - \frac{m_1}{N}) + v(s_1, a')$$

Next in (63), assign to all co-ordinates x_i for $i \notin s_1$ and $i > m$ fixed values equal to the corresponding co-ordinates of the point a', i.e. for i satisfying $i \notin s_1$, $i > m$, $x_i = a_i'$. Thus in the inequalities in (63), we replace the middle term by

(64) $$\frac{1}{N}\left[\sum_{\substack{i \in s_1 \\ i > m}} a_i' + \sum_{\substack{i \in s_1 \\ i \leq m}} x_i \right]$$

and since s_1 contains k out of the first units, there are (m-k) values of i for which $i \notin s_1$, and $i \leq m$. Hence the inequalities define a (m-k) dimensional subset $Q_{m-k}^{a'}$.

Clearly $Q_{m-k}^{a'}$ is of infinite measure (μ_{m-k}) in the hyperplane $P_{m-k}^{a'}$, i.e.

$$Q_{m-k}^{a'} \subset P_{m-k}^{a'}$$

the hyperplane $P_{m-k}^{a'}$ being defined by

(65) $x_i = a_i$ for $i \in s_1$, $i \leq m$

$x_i = a_i'$ for $i > m$.

The hyperplane $P_{m-k}^{a'}$ is wholly orthogonal to P_{N-m}^{a} and hence the set L_{N-m}^{a,s_1} with the set $Q_{m-k}^{a'}$ defined for each $a' \in L_{N-m}^{a,s_1}$, by substituting (64) for the middle term in(63), determine a set

$$D_{N-k}^{a} \subset P_{N-k}^{a}$$

P_{N-k}^{a} being the hyperplane defined by $x_i = a_i$ for each i, such that $i \in s_1$, $i \leq m$. Combining (63), (65) and (66), the explicit definition of the set D_{N-k}^{a} is given below

$x = (x_1, x_2, \ldots, x_n) \in D_{N-k}^{a}$ if, and only if,

$x \in Q_{m-k}^{a'}$ for some $a' \in L_{N-m}^{a,s_1}$.

Since L_{N-m}^{a,s_1} is of positive measure (μ_{N-m}) and for each $a' \in L_{N-m}^{a,s_1}$ the set $Q_{m-k}^{a'}$

has infinite measure (μ_{m-k}), the set D^a_{N-k} has infinite measure (μ_{N-k}). Here $0 \leq k < m$.

Now let $E^a_{N-k} \subset P^a_{N-k}$ be the set consisting of all those points $x \in P^a_{N-k}$ for which $h(s,x) \neq 0$ for at least one $s \in \overline{S}$. Then

(66) $$D^a_{N-k} \subset E^a_{N-k}$$

and hence the set E^a_{N-k} is of infinite measure (μ_{N-k}). We now again partition the set E^a_{N-k} into subsets by

(67) $$E^a_{N-k} = \bigcup_{s \in \overline{S}} L^{a,s}_{N-k}$$

the subsets $L^{a,s}_{N-k}$ being defined for each specified $s \in \overline{S}$ by

$$x \in L^{a,s}_{N-k}, \text{ if and only if } x \in E^a_{N-k} \text{ and } h(s,x) \neq 0$$

Again at least one of the subsets in the r.h.s. of (68) must be non-null (μ_{N-k}). If there is only one non-null (μ_{N-k}) set $L^{a,s}_{N-k}$, such that s does not include each of the k units u_i with $i \in s_1$, $i \leq m$, we select it; if there are more than one such subset, we select one of them arbitrarily. Let L^{a,s_2}_{N-k} be the subset selected and suppose the sample s_2 contains j, $(0 \leq j < k)$ out of the k units u_i given by $i \in s_1$, $i \leq m$. Then again proceeding as from (63) to (67) we reach a set E^a_{N-j} of infinite measure (μ_{N-j}) such that for every $x \in E^a_{N-j}$, $h(s,x) = 0$ for at least one $s \in \overline{S}$.

Clearly the process can end only when we (A') either reach a set $E_N \subset R_N$ such that E_N has infinite measure (μ_N) and for every $x \in E_N$, $h(s,x) \neq 0$ for some $s \in \overline{S}$; or

(B') we reach a hyperplane P^a_{N-j} defined by some j, $(0 < j \leq m)$ out of the first m variates having fixed values equal to the corresponding co-ordinates of the point a, i.e. $x_r = a_r$, for $r = i_1, i_2, \ldots, i_j$ where $i_1, i_2, \ldots, i_j \leq m$ and a set $E^a_{N-j} \subset P^a_{N-j}$ such that E^a_{N-j} has infinite measure (μ_{N-j}) and for every $x \in E^a_{N-j}$, $h(s,x) \neq 0$ for some $s \in \overline{S}$, and further such that for any sample $s \in \overline{S}$, which does not include each of the j units $u_{i_1}, \ldots, u_{i_j}$, $h(s,x) = 0$ for almost all (μ_{N-j}) $x \in P^a_{N-j}$.

Here we note that the possibility (A) below (60) is included in the case (B'), the corresponding value of j being = m.

Now (A') is not possible because $E_N = E$ and E is a null (μ_N) set by Theorem 3.1. (B') also leads to a contradiction. For let $\bar{S}_j$ be the subset of $\bar{S}$, $\bar{S} \subset \bar{S}_j$ consisting of all those samples $s \in \bar{S}$, which include each of the units $u_{i_1}, u_{i_2}, \ldots, u_{i_j}$. Then for every $s \in (\bar{S} - \bar{S}_j)$, the set of points $x \in P^a_{N-j}$ for which $h(s,x) \neq 0$ forms a null (μ_{N-j}) set. Consequently introducing intersection sets as previously in (35), we have,

$$(68) \qquad \sum_{s \in (\bar{S} - \bar{S}_j) \cdot \bar{S}_{e_1'', e_2'', x}} p(s) = \sum_{s \in (\bar{S} - \bar{S}_j) \cdot S_{e_1, e_2, x}} p(s)$$

for almost all (μ_{N-j}) $x \in P^a_{N-j}$. (68) combined with (58) gives then,

$$\sum_{s \in \bar{S}_j \cdot \bar{S}_{e_1'', e_2'', x}} p(s) \geq \sum_{s \in \bar{S}_j \cdot \bar{S}_{e_1, e_2, x}} p(s)$$

for almost all (μ_{N-j}), $x \in P^a_{N-j}$

But then, by Theorem 4.1 already proven, the set of points $x \in P^a_{N-j}$ for which $h(s,x) \neq 0$ must be a null (μ_{N-j}) set, while according to (B') E_{N-j} is of infinite measure (μ_{N-j}).

Thus neither (A') nor (B') is possible. Hence no point $a \in E$ exists such that $h(s,a) \neq 0$ for some $s \in \bar{S}$, and thus the set E is empty.

Consequently the sign of strict inequality in (58) and consequently in (57) does not hold for any point $x \in R_N$. We have thus proved

Theorem 5.1 The set of confidence intervals $\{e_1(s,x),\ e_2(s,x)\}$ in (7) is strictly admissible.

6. Remarks.

The theorem proved here gives an optimum property of the sample mean as an estimate of the population mean. The previous result of the author (1965, II) also defines another optimum property of admissibility of the sample mean and it is of interest to compare these results. The previous result is based on taking the squared Error as the loss function. Now the squared error is one of several possible loss functions, and there may exist other reasonable loss functions, for which the sample mean is not always an admissible estimate. From this aspect, the property proved here, which is independent of any loss function may be said to have a deeper significance.

The previous result was not subject to the restriction of measurability. The present obviously holds in the class of confidence intervals determined by measurable estimates. But as discussed previously by the author (1966, IV) the restriction of measurability is of no practical significance.

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