THE COVARIANCE ANALYSIS FOR DEPENDENT DATA

by

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1. Introduction and notations.

In this note we consider the covariance analysis model. In the classical theory this model is studied on the hypothesis that the data are independent and normally distributed, say $N(\mu, \alpha I)$, where I is the nxn identity matrix, and in order to test the hypothesis of equal influence of the treatments, two independent statistics are considered, say $(t_1 - t_2)$ and t_2 (functions of the data), whose ratio is distributed as the random variable $F'(n_1, n_2, \lambda)$ (Sne decor's F' random variable with n_1 and n_2 degrees of freedom and λ non-contrality parameter).

In this note we want to remove the hypothesis of independence of the data, and find the set of all the matrices V such that, the data being normally but not independently distributed, say $N(\mu, V)$ with V p. d. (positive definite), the two statistics (t_1-t_2) and t_2 are independent and their ratio has the same distribution $F'(n_1, n_2, \lambda)$ as the classical case, i.e. when $V = \alpha I$.

In order to do this, we suppose that we have a set of data, say Y, such that it is possible to express them in a linear model.

In particular we suppose that we have one factor with t levels or treatments and that each observed quantity can be written as:

$$y_{ij} = \mu + \tau_i + x_{ij}\beta + e_{ij} \qquad i = 1, \ldots, t; \quad j = 1, \ldots, r_i$$

where μ is a constant; τ_i is the i-th treatment constant; β is the regression coefficient (unknown); x_{ij} are observed fixed quantities and the vector $e = [e_{11}, \ldots, \ell_{t}, r_t]'$ is a normal r.v.(random variable) with mean vector $\underline{0}$ and variance-covariance matrix p.d., say V, so that the data Y are normally distributed with mean vector μ and variance-covariance matrix V. Now, in the classical analysis of covariance $V = \alpha I$ because the data are supposed to be independent, and to test the hypothesis $H_o: \tau_1 = \tau_2 = \ldots = \tau_t$, the statistics used are:

$$t_1(y_{11}, \ldots, y_t, r_t) = \sum_{ij} y_{ij}^2 - \frac{1}{n} Y_{..}^2 - \frac{[\Sigma_{ij} x_{ij} y_{ij} - \frac{X_{..} Y_{..}}{n}]^2}{\Sigma_{ij} x_{ij}^2 - \frac{X_{..}^2}{n}}$$

$$t_2(y_{1,1}, \ldots, y_{t,r_t}) = \sum_{ij} y_{ij}^2 - \sum_{i} \frac{Y_{i.}^2}{r_i} - \frac{[\Sigma_{ij} x_{ij} y_{ij} - \frac{X_{i.} Y_{i.}}{r_i}]^2}{\Sigma_{ij} x_{ij}^2 - \frac{X_{i.}^2}{r_i}}$$

where $Y_{..} = \Sigma_{ij} y_{ij}$; $X_{..} = \Sigma_{ij} x_{ij}$; $Y_{i.} = \Sigma_{j} y_{ij}$; $X_{i.} = \Sigma_{j} x_{ij}$ $n = \Sigma_{i} r_i$;

$n_i = \sum_{j=1}^{i} r_i$

We denote by : U the n x n matrix which has all elements equal to unify; U_i the n x n matrix which has elements $u_{pq}^{(i)}$, $p = 1, \ldots, n$, $q = 1, \ldots, n$, all equal zero except for the r_i^2 elements $u_{pq}^{(i)}$ with $p = n_{i-1} + 1, \ldots, n_i$, $q = n_{i-1} + 1, \ldots, n_i$; $Y' = [y_{11}, y_{12}, \ldots, y_{t,r_t}] = [y_{11}, \ldots, y_n]$ the n x 1 vector of the data; $X' = [x_{11}, x_{12}, \ldots, x_{t,r_t}] = [x_1, \ldots, x_n]$ the n x 1 vector of the observed fixed quantities x. We suppose also that $x_i \neq 0$, $i = 1, \ldots, n$, and we use the following notations: $Z' = [x_1^{-1}, \ldots, x_n^{-1}]$; $S = [I - U n^{-1}]$; $D = \frac{S XX' S}{X' S X}$; $A = S - D$;

$$C = \frac{(I - \Sigma U_i r_i^{-1}) X X'(I - \Sigma U_i r_i^{-1})}{X'(I - \Sigma U_i r_i^{-1})X}; \quad B = I - \Sigma U_i r_i^{-1} - C;$$

$$\underset{n \times n}{Q} = \begin{pmatrix} q_1 & \cdots & q_1 \\ \vdots & & \vdots \\ q_n & \cdots & q_n \end{pmatrix}$$, q_i arbitrary for all i; and

$$\Lambda_{n \times n} = \begin{pmatrix} \lambda_1 & 0 \dots 0 \\ 0 & \lambda_2 \dots 0 \\ \vdots & \\ 0 & \dots \lambda_n \end{pmatrix}, \quad \lambda_i \text{ orbitrary for all } i.$$

We also use the further notations: $\chi^2(a, \lambda)$ for the chi-square r. v. with a degrees of freedom and λ non-centrality parameter; $N(j, P)$ for the normal distribution with mean vector j and variance-covariance matrix P(p.d). We will also write $< a > \iff < k >$ meaning that the statement a is equivalent to the statement k, and $D(P) = D(R)$ meaning that the r.v. P is distributed like the r.v. R. Of course $t_1(y_1, \dots, y_n)$ and $t_2(y_1, \dots, y_n)$ can now be written in the matricial form: $t_1(y_1, \dots, y_n) = Y' A Y$, $t_2(y_1, \dots, y_n) = Y'BY$

Now we suppose that the data are dependent, i.e. that V is a general symmetric p.d. matrix and we want to find all the matrices V such that, the data being dependent, we have $Y'(A-B)Y$ and $Y'B Y$ independent and

$$\frac{Y'(A-B)Y}{Y'BY} \; \frac{n-t-1}{t-1} \text{ distributed like } F'(t-1, n-t-1, \frac{\mu' A \mu}{2}) \text{ as in the}$$

classical case. We will write T for the matrices of this set and we show that T is a symmetric p.d. matrix of the form: $T = \alpha I + Q + Q' + \frac{1}{2}\{X Z' \Lambda + \Lambda Z X'\}$. (1)

2. The structure of the matrix T.

Theorem 1. If Y is $N(\mu, V)$, (i.e. if Y is normally distributed with mean vector μ and variance-covariance matrix V , where V is p.d. and α is a positive constant, then $< D(Y' A Y) = D(\chi^2(n-2, \frac{\mu' A \mu}{2})) > \iff < V = T >$ where T has the structure (1)

First we prove that if K is a n x n matrix, then

$$< K = K', \; S K S = 0 > \iff < K = Q + Q' >$$

Proof of the implication: $\Rightarrow$

$(I-U n^{-1}) K(I-U n^{-1}) = 0$ implies $K - U K n^{-1} - K U n^{-1} + U K U (n^2)^{-1} = 0$. Let k_{ij} be the term of the i-th row and j-th column of the matrix K. The condition

$k_{ij} - \frac{k_{i.}}{n} - \frac{k_{.j}}{n} + \frac{k_{..}}{n^2} = 0$, where $k_{i.} = \sum_j k_{ij}$, $k_{.j} = \sum_i k_{ij}$, $i = 1, \ldots, n$, $k_{..} = \sum_{ij} k_{ij}$

is satisfied. In particular we have $k_{ii} = \frac{2\, k_{i.}}{n} - \frac{k_{..}}{n^2}$ so that $k_{i.} = \frac{n}{2}(k_{ii} + \frac{k_{..}}{n^2})$

$i = 1, \ldots, n$, and similarly $k_{.j} = \frac{n}{2}(k_{jj} + \frac{k_{..}}{n^2})$ so that $k_{ij} = \frac{k_{ii} + k_{jj}}{2}$.

Now if we put $q_i = \frac{k_{ii}}{2}$, $q_j = \frac{k_{jj}}{2}$ we have $K = Q + Q'$.

Proof of the implication: $\Leftarrow$

It is evident that $K = K'$ also $S\,K\,S = 0$ may be written as $(I - U\,n^{-1})$.

$Q(I - U\,n^{-1}) + (I - U\,n^{-1})\,Q'(I - U\,n^{-1}) = 0$ and we have $(I - U\,n^{-1})Q(I - U\,n^{-1}) = 0$

because $n^{-1}\,K\,U = K$, $n^{-1}\,U\,K = n^{-1}\,\Sigma\,k_i\,U$, $(n^2)^{-1}\,U\,K\,U = n^{-1}\,\Sigma\,k_i\,U$. But

$(I - U\,n^{-1})\,Q'(I - U\,n^{-1}) = [(I - U\,n^{-1})Q(I - U\,n^{-1})]'$ and the proof is completed.

We will use, also, the following-known theorem (see [1]): If the random vector Y is $N(\mu, V)$ with V p.d., and if the $n \times n$ matrix W is real and symmetric (rank $W = w$), then

$$< D\{Y'\,W\,Y\} = D\{\chi'^2(w, \frac{\mu'\,W\,\mu}{2})\} > \Longleftrightarrow < W\,V\,W = W >$$

So to prove Theorem 1 we have to prove:

$$< A\,V\,A = \alpha\,A > \Longleftrightarrow < V = T >$$

(for put $W = A\,\alpha^{-1}$).

Proof of the implication: $\Leftarrow$

For $A\,T\,A = (S-D)\,T\,(S-D)$ we have $A\,T\,A = S\,T\,S - S\,T\,D - D\,T\,S + D\,T\,D$,

but substituting T and recalling the previous results, we have: $S\,T\,S = \alpha\,S + \frac{1}{2}\,S\,X\,Z'\Lambda\,S + \frac{1}{2}\,S\,\Lambda\,Z\,X'\,S$; $S\,T\,D = \alpha\,D + \frac{1}{2}\,S\,X\,Z'\Lambda\,D + \frac{1}{2}\,S\,\Lambda\,Z\,X'\,S$;

$D\,T\,S = \alpha\,D + \frac{1}{2}\,S\,X\,Z'\,\Lambda\,S + D\,\Lambda\,Z\,X'\,S$; $D\,T\,D = \alpha\,D + \frac{1}{2}\,S\,X\,Z'\,\Lambda\,D + \frac{1}{2}\,D\,\Lambda\,Z\,X'\,S$

and thus $A\,T\,A = \alpha\,A$.

Proof of the implication: $\Rightarrow$

Let us suppose that we may have a matrix $V^* = T + H$ for which $A\,V^*\,A = \alpha\,A$

and we show that H must have the same structure as T. In fact $A V^* A = \alpha A$ is $A T A + A H A = \alpha A$ and because $A T A = \alpha A$ we have to study the equation $A H A = 0$. By developing the left hand side of $A H A = 0$ we have $S(H - H S X X' - X X' S H + X X' S H S X X') S = 0$ and by our previous result $H - H S X X' - X X' S H + X X' SHSXX' = R + R'$ where R is a matrix of the structure of Q and the r_i, $i = 1, \ldots, n$, are arbitrary.

Now, let h_{ij} be an element of the matrix H, then $h_{ij} = x_j \sum_r h_{i,r}(x_r - \Sigma x_i) + x_i \sum_k h_{kj} \quad \Sigma(x_k - \Sigma x_i) - x_i x_j \sum_k (x_k - \Sigma x_i) \sum_r h_{kr} (x_r + - \Sigma x_i) + r_i + r_j$.

Putting $\sum_r h_{ir}(x_r - \Sigma x_i) = t_i$, $\sum_k h_{kj}(x_k - \Sigma x_i) = t_j$, $\sum_k (x_k - \Sigma x_i) \sum_r h_{kr}(x_r - \Sigma x_i) = q$

and solving the system of equations

$$\begin{cases} h_{ij} = x_j t_i + x_i t_j - x_i x_j q + r_i r_j \\ h_{ii} = 2 x_i t_i - x_i^2 q + 2 r_i \\ h_{jj} = 2 x_j t_j - x_j^2 q + 2 r_j \end{cases}$$

we have $h_{ij} = \frac{1}{2} \{ \frac{x_i}{x_j} h_{jj} + \frac{x_j}{x_i} h_{ii} - (1 - \frac{x_i}{x_j}) r_j - (1 - \frac{x_j}{x_i}) r_i \}$. Hence $H = \frac{1}{2}\{ X Z' (\Sigma - \Delta) + (\Sigma - \Delta) Z X' + U \Delta + \Delta' U\}$ i.e. $H = F + F' + \frac{1}{2} \{X Z'\Gamma + \Gamma Z X'\}$, where Γ is a diagonal matrix, F is a matrix of the same structure as Q. Then H has the same structure as $T - \alpha S$, and thus the structure of $V = T + H$ is the some as that of T and the Theorem is proved.

3. Analysis of covariance

Theorem 2 If the random vector Y is $N(\mu, T)$, then

$$< D\{Y' A Y\} = D\{\alpha \chi^2(n-2, \frac{\mu' A \mu}{2})\} > \iff$$

$$< D\{Y' (A - B)Y = D\{\alpha \chi^2(t - 1, \frac{\mu'(A-B)\mu}{2})\}, D\{Y' B Y\} = D\{\alpha \chi^2 (n-t-1)\},$$

$Y'(A-B)Y$ and $Y' B Y$ independent >

Proof Theorem 5 of Graybill-Marsaglia [1], says, among other things, that if: Y is $N(\mu, V)$, with V a $n \times n$ symmetric p.d. matrix, $Y' W Y = \sum_{i=1}^{k} Y' W_i Y$, rank $W_i = p_i$, rank $W = p$, then:

$$< (W V)^2 = W V, \Sigma p_i = p > \Leftrightarrow$$

$$< D\{Y' W_i Y\} = D\{\chi^2(p_i, \lambda_i)\}, \; Y' W_i Y \text{ are mutually independent} >$$

where $\lambda_i = \frac{\mu' W_i \mu}{2}$

But from Theorem 1 we have $A T A = \alpha A$ or $(\frac{A}{\alpha} T)^2 = \frac{A}{\alpha} T$. Then if we put $W_1 = \frac{A-B}{\alpha}$, $W_2 = \frac{B}{\alpha}$, $W = \frac{A}{\alpha}$ we have rank $W_1 = t-1$, rank W_2 o $n-t-1$, rank $W = n-2$, and thus we may write: $D\{Y'(A-B)Y\} = D\{\alpha \chi^2(t-1, \frac{\mu'(A-B)\mu}{2}\}$, $D\{Y'B Y\} = D\{\alpha\chi^2(n-t-1)\}$ and $A-B$ and B independent.

Theorem 3 If the random vector Y is $N(\mu, V)$, V p.d. and $Y'(A-B)Y$, $Y' B Y$ are independent, then

$$< V = T > \Leftrightarrow < D\{ \frac{Y'(A-B)Y}{Y' B Y} \frac{n-t-1}{t-1} \} = D\{F'(t-1, n-t-1, \frac{\mu' A \mu}{2})\} >$$

for all μ.

Proof of the implication: =>

This implication follows immediately from Theorems 1 and 2.

Proof of the implication: <=

Taking $\mu = \mu^* = \mu 1$ where 1 is the $n \times 1$ unify vector, we have $A \mu^* = (S-D)\mu^* = S\mu^* - D\mu^* =$

$= S \mu^* - \frac{1}{X' S X} S X X' S \mu^*$. From simple calculation we verify that $S \mu^* = 0$ so that $A \mu^* = 0$. In the same way we have $\mu^* A = 0$. But because $A = (A-B) + B$ we have that

$$< A \mu^* = 0 > \Leftrightarrow < [(A-B) \mu^* = 0 > \Rightarrow$$

$$\Rightarrow \begin{cases} < B\ A\ \mu^* = 0 > \Leftrightarrow < B\ \mu^* = 0 > \\ \text{and also} \\ <(A-B)\ A\ \mu^* = 0 > \Leftrightarrow < (A-B)\ \mu^* = 0 > \end{cases}$$

But it is known that for every V p. d. and $\mu = \mu^*$, any quadratic form is decomposable into a linear combination of independent central χ^2 r.v., i.e. $Y'(A-B)Y = \sum_{i=1}^{t-1} \lambda_i\ \chi_i^2\ (1)$ and $Y'\ B\ Y = \sum_{j=1}^{n-t-1} \alpha_j\ \chi_j^2\ (1)$

where λ_i and α_i are the **parameters** of the linear combination. Then it follows, from our hypothesis, that

$$< D \left\{ \frac{\Sigma\ \lambda_i\ \chi_i^2\ (1)}{\Sigma\ \alpha_j\ \chi_j^2\ (1)}\ \frac{n-t-1}{t-1} \right\} = D\{F'(t-1,\ n-t-1)\}\ ,$$

$Y'(A-B)Y$ independent of $Y'\ B\ Y > \Rightarrow$

$$< D\left\{\frac{\Sigma\ \lambda_i\ \chi_i^2\ (1)}{\Sigma\ \alpha_j\ \chi_j^2\ (1)}\ \frac{n-t-1}{t-1}\right\} = D\left\{\frac{\chi^2(t-1)}{\chi^2(n-t-1)}\right\},\ Y'(A-B)\ Y$$

and $Y'\ B\ Y$ independent $>$.

But from Baldessari (see [2]), this last statement is equivalent to

$< \lambda_1 = \dots = \lambda_k = \alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha > 0$, and $Y'(A-B)Y$ and $Y'\ B\ Y$ are independent $> \Leftrightarrow$

$$< D\{Y'(A-B)Y\} = D\{\alpha\ \chi^2(t-1)\},\ D\{Y'\ B\ Y\} = D\{\alpha\ \chi^2(n-t-1)\},$$

$Y'(A-B)Y$ and $Y'\ B\ Y$ independent $>$

$\Rightarrow < D(Y'\ A\ Y) = D(\alpha\ \chi^2(n-2)) > \Leftrightarrow < V = T >$

and the proof of our Theorem is completed.

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