ON A CLASS OF NONPARAMETRIC TESTS FOR MANOVA IN TWO WAY LAYOUTS*

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ON A CLASS OF NONPARAMETRIC TESTS FOR MANOVA IN TWO WAY LAYOUTS*

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SUMMARY. The object of the present investigation is to propose and study a class of nonparametric tests for the multivariate analysis of variance (MANOVA) problem relating to complete two way layouts. In this context, the concept of rank-permutations for multidimensional interchangeability is developed, and the same is incorporated in the formulation of a class of genuinely distribution-free rank order tests. Asymptotic properties of the class of proposed tests are studied and compared with those of the standard parametric ones.

1. INTRODUCTION

Let us consider a complete two way layout comprising of n complete blocks (replicates), each block containing $r(\geq 2)$ plots where r different treatments are applied. The yield (response) is a p variate quantitative (stochastic) vector, and we denote by $X_{ij}^{(k)}$ the k-th response for the jth treatment placed in the ith block for $i = 1, \ldots, n$, $j = 1, \ldots, r$, $k = 1, \ldots, p$. In the sequel, it will be assumed that $n, r, p \geq 2$. Let then

$$\underset{\sim}{X}_{ij}' = (X_{ij}^{(1)}, \ldots, X_{ij}^{(p)}),\ i = 1, \ldots, n,\ j = 1, \ldots, r; \tag{1.1}$$

$$\underset{\sim}{\mu}' = (\mu^{(1)}, \ldots, \mu^{(p)}); \tag{1.2}$$

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$$\underset{\sim}{\alpha}_i' = (\alpha_i^{(1)}, \ldots, \alpha_i^{(p)}),\ i = 1, \ldots, n; \tag{1.3}$$

$$\underset{\sim}{\tau}_j' = (\tau_j^{(1)}, \ldots, \tau_j^{(p)}),\ j = 1, \ldots, r; \tag{1.4}$$

and $$\underset{\sim}{e}_{ij}' = (e_{ij}^{(1)}, \ldots, e_{ij}^{(p)}),\ j = 1, \ldots, r,\ i = 1, \ldots, n. \tag{1.5}$$

We adopt the usual linear model as

$$\underset{\sim}{X}_{ij} = \underset{\sim}{\mu} + \underset{\sim}{\alpha}_i + \underset{\sim}{\tau}_j + \underset{\sim}{e}_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, r, \tag{1.6}$$

where $\underset{\sim}{\mu}$ is the vector of mean effects, $\underset{\sim}{\alpha}_i$ the block effects ($i = 1, \ldots, n$), $\underset{\sim}{\tau}_j$ the treatment effects ($j = 1, \ldots, r$), and $\underset{\sim}{e}_{ij}$ the residual error vectors ($i = 1, \ldots, n$, $j = 1, \ldots, r$). These component vectors are assumed to be mutually independent. Our problem is to have a comprehensive test for the hypothesis of no treatment effects i.e.,

$$H_o:\ \underset{\sim}{\tau}_1 = \ldots = \underset{\sim}{\tau}_r. \tag{1.7}$$

In the parametric case, it is usually assumed that $\underset{\sim}{e}_{ij}$ ($i = 1, \ldots, n$, $j = 1, \ldots, r$) are $N(= nr)$ independent and identically distributed stochastic vectors distributed according to a multinormal distribution with a null mean vector and a dispersion matrix (positive definite) $\underset{\sim}{\Sigma} = ((\sigma_{kq}))$, where σ_{kq} is the covariance of $(e_{ij}^{(k)}, e_{ij}^{(q)})$, for $k, q = 1, \ldots, p$. The parametric MANOVA tests are either based on the likelihood ratio criterion or on the characteristic roots of some determinental equations. The likelihood ratio criterion reduces to the ratio of two generalized variances and can be expressed as the product of several (p) independent beta variables (cf. Anderson (1958, Chapter 8)). Alternatively, one may work with the smallest characteristic root of the determinental equation involving the same generalized variances. Occasionally, some symmetric function of the roots are also used. For details, the reader is referred to Rao (1965, chapter 8). The parametric tests thus appear to be deterministic, but they are not very simple, especially for $p > 2$. Further, in this procedure the assumptions of independence and multinormality of the error vectors play an indispensible role. Unlike the univariate case, very little has been investigated about the effects of departure from these two basic assumptions on the performance characteristics of the parametric MANOVA tests. On the otherhand, the assymption of multinormality of the error vectors is often found to be dubious, especially in many biometric problems. Further, in many problems, there appears to be sufficient evidence on the stochastic dependence of the error vectors within the same block. For example, in agricultural experiments, the presence of <u>spatial correlation</u> may distort the stochastic independnece of the error vectors within the same block. Similar dependence may be due to <u>genetic effects</u> in many animal feeding experiments. The object of the present investigation is to relax both the assymptions of multinormality as well as independence of the error components. In fact, for the tests proposed here, we require only that

(i) the joint distribution function $F(\underset{\sim}{e}_{i1}, \ldots, \underset{\sim}{e}_{ir})$ of $\underset{\sim}{e}_{i1}, \ldots, \underset{\sim}{e}_{ir}$ is continuous and independent of $i = 1, \ldots, n$, and

(ii) $F(\underset{\sim}{e}_{i1}, \ldots, \underset{\sim}{e}_{ir})$ is a symmetric function of its r arguments (vectors) $\underset{\sim}{e}_{i1}, \ldots, e_{ir}$ i.e., F remains invariant under any permutation of the r vectors among themselves, or in other words, $\underset{\sim}{e}_{i1}, \ldots, \underset{\sim}{e}_{ir}$ are symmetric dependent stochastic vectors.

Evidently, both the assumptions (i) and (ii) are much less restrictive than the usual assumptions of independence and multinormality. Thus, the proposed method appears to have a comparatively wider scope of applicability.

In the nonparametric case, practically no work has been done on this line. For completely randomized layouts, very recently some nonparametric MANOVA tests have been offered by Chatterjee and Sen (1964, 1966), Sen (1965, 1966a), Puri and Sen (1966), and Anderson (1965), among few others. Bhapkar (1965) has also presented some asymptotically distribution-free test for the same problem. The present author (1966 b) has considered some rank methods for combination of independent experiments in MANOVA. The same procedure is applicable in our situation here, but it fails to be suitable in some respects. This problem may also be regarded as the multivariate generalization of the nonparametric ANOVA tests relating to two way layouts. Such ANOVA tests have been considered by Friedman (1937), Durbin (1951), Brown and Mood (1951), Benard and Elteren (1953), and others. These are all based on intra-block rankings, and the same method can be generalized to the MANOVA problem. The present author (1966 c) has considered a modified approach to nonparametric ANOVA tests for two way layouts. Extending an idea of Hodges and Lehmann (1962), he has considered the rankings after alignment, and under a suitable permutation model, has obtained a class of genuinely distribution-free tests based on these modified rankings. This results, in most of the cases, in an increased (at least asymptotically) efficiency of the proposed test. The object of this paper is to generalize the method of rankings after alignment to the MANOVA problem and to offer some suitable non-parametric tests for the same. For this purpose, the concept of <u>multidimensional interchangeability</u> is developed and certain rank permutational ideas are formulated. With the aid of this a class of properly distribution-free rank order tests for the hypothesis in (1.7) is developed. Further, the celebrated Chernoff-Savage (1958) theorem on the asymptotic normality and power-efficiency of a class of univariate nonparametric test-statistics, as extended to the multivariate case by Puri and Sen (1966) and to the problem of compound symmetry of multivariate distributions by Sen (1966 c), is extended further to take care of the problem of multidimensional interchangeability, to be considered here. With the aid of this, the asymptotic power and power-efficiency of the proposed class of tests are studied.

2. SOME PRELIMINARY NOTIONS.

Let us define a set of r^2 real quantities by

$$c_{\ell j} = \delta_{\ell j} - 1/r \text{ for } j, \ell = 1, \ldots, r, \tag{2.1}$$

where $\delta_{\ell j}$ is the usual Kronecker delta. Thus, $\Sigma_{j=1}^{r} c_{\ell j} = 0$ for all $\ell = 1, \ldots, r$. Let us then consider the r intra-block contrasts

$$\underset{\sim}{Y}_{i\ell} = \Sigma_{j=1}^{r} c_{\ell j} \underset{\sim}{X}_{ij}, \ \ell = 1, \ldots, r. \tag{2.2}$$

From (1.6) and (2.2), we have

$$\underset{\sim}{Y}_{i\ell} = (\underset{\sim}{\tau}_{\ell} - \frac{1}{r} \Sigma_{j=1}^{r} \underset{\sim}{\tau}_{j}) + (\underset{\sim}{e}_{i\ell} - \frac{1}{r} \Sigma_{j=1}^{r} \underset{\sim}{e}_{ij}), \tag{2.3}$$

where the first factor on the right hand side of (2.3) vanishes when H_o in (1.7) holds. Further, by assumption (ii) of section 1, we get with some simple reasonings that the joint distribution of $[(\underset{\sim}{e}_{i\ell} - \frac{1}{r} \Sigma_{j=1}^{r} \underset{\sim}{e}_{ij}), \ \ell = 1, \ldots r]$ is a symmetric function of the r (vector) arguments. Consequently, from (2.3), we get that under H_o in (1.7), the joint distribution of $(\underset{\sim}{Y}_{i}, \ldots, \underset{\sim}{Y}_{ir})$ will be a symmetric function of the r vectors $\underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}_{ir}$. On the otherhand, if H_o in (1.7) does not hold, the joint distribution of $(\underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}_{ir})$ will be a symmetric function of its (vector) arguments only when each one of them is adjusted by appropriate location vectors. Thus, if instead of the observed responses $\underset{\sim}{X}_{ij}$'s, we work with the block-adjusted yields $\underset{\sim}{Y}_{ij}$'s, our problem of testing H_o in (1.7) reduces to that of testing the hypothesis of interchangeability of the vectors $\underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}_{ir}$ (for all $i = 1, \ldots, n$), against translation type of alternatives. This is termed the problem of multidimensional interchangeability, and a formulation of an appropriate rank permutation model for the same, will be considered in the next section. The necessary rank order statistics will be defined now.

Let us pool the N(= nr) observations $\{Y_{ij}^{(k)}, j = 1, \ldots, r, i = 1, \ldots, n\}$ into a combined set and denote the ordered observations by

$$Y_{(1)}^{(k)} < \ldots < Y_{(N)}^{(k)}, \tag{2.4}$$

where by virtue of the assumed continuity of the distribution of the error vectors, the possibility of ties in (2.4) may be neglected, in probability. Let then C(u) be the usual sign-function viz.,

$$c(u) = \begin{cases} 1, \text{ if } u > 0 \\ 0, \text{ if } u \leq 0, \end{cases} \tag{2.5}$$

and let

$$R_{ij}^{(k)} = 1 + \sum_{\alpha=1}^{N} c(Y_{ij}^{(k)} - Y_{(\alpha)}^{(k)}), \tag{2.6}$$

for i = 1, ..., n, j = 1, ..., r.

Thus $R_{ij}^{(k)}$ stands for the rank of $Y_{ij}^{(k)}$ within the set (2.4). This ranking procedure is employed separately for each k = 1, ..., p. Consequently, any vector $\underset{\sim}{Y}_{ij}$ having p elements is made to correspond to a rank p-vector

$$\underset{\sim}{R}'_{ij} = (R_{ij}^{(1)}, \ldots, R_{ij}^{(p)}), \tag{2.7}$$

for i = 1, ..., n, j = 1, ..., r. The composite collection is a p x N matrix

$$\underset{\sim}{R}_N^{pxN} = (\underset{\sim}{R}_{11}, \ldots, \underset{\sim}{R}_{1r}, \ldots, \underset{\sim}{R}_{n1}, \ldots, \underset{\sim}{R}_{nr}), \tag{2.8}$$

$\underset{\sim}{R}_N$ will be termed a <u>collection (rank) matrix.</u> Each row of $\underset{\sim}{R}_N$ is a permutation of the numbers 1, ..., N. For any positive integer N(= nr, n = 1, 2, ...) we define p sequences of real numbers by

$$\underset{\sim}{E}_N^{(k)} = (E_{N,1}^{(k)}, \ldots, E_{N,N}^{(k)}), \; k = 1, \ldots, p. \tag{2.9}$$

$E_{N,\alpha}^{(k)}$'s are all real quantities and are explicit functions of $(\frac{\alpha}{N+1})$. We adopt the coventional Chernoff-Savage (1958) form and write

$$E_{N,\alpha}^{(k)} = J_N^{(k)}(\frac{\alpha}{N+1}), \; \alpha = 1, \ldots, N, \; k = 1, \ldots, p, \tag{2.10}$$

where the function $J_N^{(k)}$ need be defined only at $\frac{\alpha}{N+1}$, $\alpha = 1, \ldots, N$. However, we shall find it more convenient to extend its domain of definition to (0, 1) according to the Chernoff-Savage convention. Also, we define rp requences of indicator functions $\{Z_{N,\alpha}^{(j,k)}, \alpha = 1, \ldots, N\}$, for $j = 1, \ldots, r$, $k = 1, \ldots, p$ by

$$Z_{N,\alpha}^{(j,k)} = \begin{cases} 1, \text{ of } Y_{(\alpha)}^{(k)} \text{ is some } Y_{ij}^{(k)} (i = 1, \ldots, n), \\ 0, \text{ otherwise}, \end{cases} \tag{2.11}$$

for $\alpha = 1, \ldots, N$. Then we define rp rnak order statistics

$$T_{N,j}^{(k)} = \frac{1}{n} \sum_{\alpha=1}^{N} E_{N,\alpha}^{(k)} Z_{N,\alpha}^{(j,k)}, \; j = 1, \ldots, r, \; k = 1, \ldots, p. \tag{2.12}$$

It may be noted that

$$\frac{1}{r} \sum_{j=1}^{r} T_{N,j}^{(k)} = \frac{1}{N} \sum_{\alpha=1}^{N} E_{N,\alpha}^{(k)} = \bar{E}_N^{(k)} (\text{say},), \; k = 1, \ldots, p; \tag{2.13}$$

where $\bar{E}_N^{(1)}, \ldots, \bar{E}_N^{(p)}$ are all known constants (depending on N). Thus, at most $(r - 1)p$ of the rp variables in (2.12) are linearly independent. Our proposed test is based on the set of random variables in (2.12). To develop strictly distribution-free tests for the hypothesis (1.7), we shall consider in the next section some permutation model. But, before that it may be worth writing a point of clarification. The class of statistics in (2.12) has some similarity with that of a similar class of statistics considered by Puri and Sen (1966). However, in the later case, we have a one way classification with N independent p-variate observations, while in this case, we have a two way classification with n independent pr-variate observations. This makes the situation somewhat more complicated, and requires a more specialized attention for both the permutation as well as asymptotic test theory.

3. RANK PERMUTATIONS FOR MULTIDIMENSIONAL INTERCHANGEABILITY.

The collection matrix $\mathbf{R}_N^{p\times N}$, given by (2.8), is now expressed in terms of n submatrices $\mathbf{R}_1^{p\times r}, \ldots, \mathbf{R}_n^{p\times r}$, where $\mathbf{R}_i^{p\times r}$ is the matrix of the r rank p-tuplets corresponding to $(\mathbf{Y}_{i1}, \ldots, \mathbf{Y}_{ir})$, for $i = 1, \ldots, n$. Thus, we have

$$\mathbf{R}_N^{p\times N} = (\mathbf{R}_1^{p\times r}, \ldots, \mathbf{R}_n^{p\times r}). \qquad (3.1)$$

Now under the null hypothesis (1.7), the joint distribution function $G(\mathbf{Y}_{i1},\ldots,\mathbf{Y}_{ir})$ is a symmetric function of $\mathbf{Y}_{i1}, \ldots, \mathbf{Y}_{ir}$, and hence, the same remains invariant under any permutation of the r vectors in the r positions of G. Since, there are r! possible permutations of the r vectors among themselves, the permutational probability (i.e., conditional probability) mass associated with each of the r! possible permutations is equal to $(r!)^{-1}$, (under H_o in (1.7),) for all $i = 1, \ldots, n$. Since, $(\mathbf{Y}_{i1}, \ldots, \mathbf{Y}_{ir})$ is distributed (jointly) independently of $(\mathbf{Y}_{i'1}, \ldots, \mathbf{Y}_{i'r})$ for all $i \neq i' = 1, \ldots, n$, the joint distribution of

$$Y_N = (Y_{11}, \ldots, Y_{ir}, \ldots, Y_{n1}, \ldots, Y_{nr}) \qquad (3.2)$$

remains invariant under the following finite group $\mathcal{G}_n$ of transformations $\{ g_n \}$ which maps the sample space of $\underset{\sim}{Y}_N$ onto itself. The number of elements of $\mathcal{G}_n$ is equal to $(r!)^n$, and typically a transformation g_n is such that

$$g_n \underset{\sim}{Y}_N = \underset{\sim}{Y}_N^* = (\underset{\sim}{Y}_{11}^*, \ldots, \underset{\sim}{Y}_{1r}^*, \ldots, \underset{\sim}{Y}_{n1}^*, \ldots, \underset{\sim}{Y}_{nr}^*), \tag{3.3}$$

where $(\underset{\sim}{Y}_{i1}^*, \ldots, \underset{\sim}{Y}_{ir}^*)$ is any permutation of $(\underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}_{ir})$, $i = 1, \ldots, n$. Let $\underset{\sim}{\mathcal{Y}}_N$ be the Np-dimensional sample space of $\underset{\sim}{Y}_N$, (and we take it to be the Np-dimensional Euclidean space). Evidently, the sample space of $\underset{\sim}{Y}_N^*$ is the same as that of $\underset{\sim}{Y}_N$, and moreover, under H_o in (1.7), the joint distribution of $\underset{\sim}{Y}_N$ remains invariant under the group of transformations $\mathcal{G}_n$. Let now $S(\underset{\sim}{Y}_N)$ be a (real or vector valued) function on $\underset{\sim}{Y}_N$. Then, for any $\underset{\sim}{Y}_N \varepsilon \underset{\sim}{\mathcal{Y}}_N$, we will have a set of $(r!)^n$ values of $S(\underset{\sim}{Y}_N)$, obtained under the group of transformations $\mathcal{G}_n$, and this set is denoted by $\Sigma(\underset{\sim}{Y}_N)$. Then, under the null hypothesis (1.7), the conditional distribution of $S(\underset{\sim}{Y}_N)$ over the set $\Sigma(\underset{\sim}{Y}_N)$ will be uniform. Let us define $T_{N,j}^{(k)}$ as in (2.2), and let

$$\underset{\sim}{T}_N^{r\times p} = ((T_{N,j}^{(k)}))_{j = 1, \ldots, r,\ k = 1, \ldots, p} \tag{3.4}$$

Then, it follows that $\underset{\sim}{T}_N$ is a stochastic matrix, which under the group of transformations $\mathcal{G}_n$ can have only $(r!)^n$ possible realizations. Since $\underset{\sim}{T}_N$ is an explicit function of the N rank p-tuplets $\underset{\sim}{R}_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, r$, it will be more convenient for us to review the above invariance argument in terms of the following rank-invariance argument.

The way in which we have defined $\underset{\sim}{R}_N$ in (2.8) and (3.1), it follows that for any $\underset{\sim}{Y}_N \varepsilon \underset{\sim}{\mathcal{Y}}_N$ there will be a corresponding collection matrix $\underset{\sim}{R}_N$. On examining the group of transformations $\mathcal{G}_n$, it will be clear that the transformation g_n on $\underset{\sim}{Y}_N$, given by (3.3), gives rise to another collection matrix $\mathbf{R}_N^*$, which is obtained by applying the same transformation g_n on the original collection matrix $\mathbf{R}_N$. Thus, under the group of transformations $\mathcal{G}_n$ of $\{g_n\}$, the rank collection matrix $\mathbf{R}_N$ (corresponding to $\mathbf{Y}_N \varepsilon \mathcal{Y}_N$) gives rise to a set of $(r!)^n$ rank collection matrices (obtained by applying the same transformations $\{g_n\}$,) and this set is denoted by $\Sigma(\mathbf{R}_N)$. If $\mathbf{R}_N^*$ is any member of $\Sigma(\mathbf{R}_N)$, we note that $\mathbf{R}_N^*$ is really derived from $\mathbf{R}_N$ by a finite number of inversions of the columns of the later. Thus we may write

$$\mathbf{R}_N^* \sim \mathbf{R}_N \ (\text{mod } \mathcal{G}_n) \text{ for all } \mathbf{R}_N^* \varepsilon \Sigma(\mathbf{R}_N). \tag{3.5}$$

Hence, the set $\Sigma(\mathbf{R}_N)$ contains $(r!)^n$ rank-matrices which are permutationally (under inversions of intra-block columns) equivalent (under $\mathcal{G}_n$) to $\mathbf{R}_N$. Thus, we term $\Sigma(\mathbf{R}_N)$ as the <u>permutation set</u> (mod $\mathcal{G}_n$) of $\mathbf{R}_N$. $\mathbf{R}_N$ like $\mathbf{Y}_N$ is a stochastic variable, and each row of $\mathbf{R}_N$ is a permutation of 1, ..., N. Thus, $\mathbf{R}_N$ can have $(N!)^p$ possible realizations, and this set of all possible realizations of $\mathbf{R}_N$ is denoted by $\mathcal{R}_N$, so that

$$\mathbf{R}_N \ \varepsilon \ \Sigma(\mathbf{R}_N) \subset \mathcal{R}_N. \tag{3.6}$$

The probability distribution of $\mathbf{R}_N$ on $\mathcal{R}_N$ (defined on an additive class of subsets A_N of $\mathcal{R}_N$,) will depend on the unknown joint distributions $G\ (\mathbf{Y}_{i1}, \ldots, \mathbf{Y}_{ir})$, $i = 1, \ldots, n$, even under H_o in (1.7). Thus, unlike the case of univariate one way classified data, the use of the unconditional distribution of $\mathbf{R}_N$ will fail to provide a distribution-free test. However, from what has been discussed before, it follows that

$$P\ \{\mathbf{R}_N = \mathbf{R}_N^* \ \big|\ \Sigma(\mathbf{R}_N),\ H_o\} = (r!)^{-n}, \tag{3.7}$$

for all $\mathbf{R}_N^* \varepsilon \Sigma(\mathbf{R}_N)$, independently of $G(\mathbf{Y}_{i1}, \ldots, \mathbf{Y}_{ir})$, $i = 1, \ldots, n$. Now, the way in which $\underset{\sim}{E}_N^{(k)}$, k = 1, ..., p, are defined by (2.9), (2.10), it follows that $\underset{\sim}{T}_N$ in [(2.12), (3.4)] is an explicit function of $\underset{\sim}{R}_N$. Thus, the set $\Sigma(\underset{\sim}{R}_N)$ will give rise to a set of $(n!)^n$ realizations of $\underset{\sim}{T}_N$, and this set is denoted by $\Sigma(\underset{\sim}{T}_N)$. Hence, under the permutational probability measure (3.7), we will have a completely specified permutational distribution of $\underset{\sim}{T}_N$, and the corresponding permutational probability measure is denoted by $\mathcal{P}_n$. Let us then consider a test function $\phi(\underset{\sim}{Y}_N)(0 \leq \phi \leq 1)$, which to each $\underset{\sim}{Y}_N \in \underset{\sim}{\mathcal{Y}}_N$ associates a probability of rejecting H_o in (1.7), with the aid of $\mathcal{P}_n$. It follows that we can always select $\phi(\underset{\sim}{Y}_N)$ in such a manner that

$$\sum_{Y_N^* \in \Sigma(Y_N)} \phi \quad (\underset{\sim}{Y}_N^*) = (r!)^n \cdot \varepsilon, \tag{3.8}$$

where $\varepsilon(0 < \varepsilon < 1)$ is the preassigned level of significance of the test. Consequently, $\phi(\underset{\sim}{Y}_N)$ has the $S(\varepsilon)$ - structure of tests [cf. Lehmann and Stein (1949)], and is a similar size ε test for the null hypothesis (1.7).

Now, in actual practice, we prefer to use some single-valued function of $\underset{\sim}{T}_N$ as a test-statistic. There seems to be no definite suggestions regarding the structure of this test-statistics, and an optimum choice naturally may depend appreciably on the particular class of alternatives we have in mind. However, it may be suitable (though not necessarily optimum) to consider the following test-statistic which is the quadratic-form associated with the asymptotic permutation distribution of $\underset{\sim}{T}_N$. For this, let us consider first the permutational moments of $\underset{\sim}{T}_N$. It readily follows that

$$E\ \{T_{N,j}^{(k)}\ |\mathcal{P}_n\} = \bar{E}_N^{(k)}, \text{ for } k = 1, \ldots, p,\ j = 1, \ldots, r. \tag{3.9}$$

Let us define

$$\bar{E}^{(k)}_{NR_{i.}}(k) = \frac{1}{r}\ \Sigma^{r}_{j=1}\ E^{(k)}_{N,R_{ij}}(k)\ , \quad i = 1, \ldots, n,\ k = 1, \ldots, p, \tag{3.10}$$

as the intra-block averages. Also let

$$v_{kq}\ (\underset{\sim}{R}_N) = \frac{1}{n(r-1)} \sum_{i=1}^{n} \sum_{j=1}^{r} \left\{ E^{(k)}_{N,R_{ij}}(k) - \bar{E}^{(k)}_{N,R_{i.}}(k) \right\} \left\{ \bar{E}^{(q)}_{N,R_{i}}(q) - \bar{E}^{(q)}_{N,R_{i.}}(q) \right\}, \tag{3.11}$$

for k, q = 1, ..., p;

$$\underset{\sim}{V}_N(\underset{\sim}{R}_N) = ((v_{kq}(\underset{\sim}{R}_N)))_{k,\ q\ =\ 1,\ \ldots,\ p} \tag{3.12}$$

It is then easy to verify that

$$\text{Cov}\ \{T^{(k)}_{N,j},\ T^{(q)}_{N,j'} \mid {}_{n}\} = \frac{1}{nr}\ (\delta_{jj'}r - 1)v_{kq}(\underset{\sim}{R}_N), \tag{3.13}$$

for k, q = 1, ..., p, j, j' = 1, ..., r, where $\delta_{jj'}$ is the usual Kronecker delta. For the time being, let us assume that $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$, given by (3.12), is positive definite, and denote its reciprocal matrix by

$$\underset{\sim}{V}_N^{-1}(\underset{\sim}{R}_N) = ((v^{kq}(\underset{\sim}{R}_N)))_{k,\ q\ =\ 1,\ \ldots,\ p} \tag{3.14}$$

Our proposed test-statistic S_N can then be expressed as

$$S_N = n \sum_{k=1}^{p} \sum_{q=1}^{p} v^{kq}(\underset{\sim}{R}_N) \sum_{j=1}^{r} [T^{(k)}_{N,j} - \bar{E}^{(k)}_N][T^{(q)}_{N,j} - \bar{E}^{(q)}_N]\ , \tag{3.15}$$

and it may be noted that S_N is essentially a non-negative stochastic variable. We shall see later on that under certain regularity conditions on $G(\underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}_{ir})$, $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$ is positive definite with a very high probability, (precisely, in probability). However, if $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$ fails to be non-singular, we may work with the highest order principal minor of $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$ which is positive definite, and proceed similarly only with the responses pertaining to this minor. Thus, for convenience, we may assume $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$ to be positive definite. Now,

$$E(S_N | \mathcal{P}_n) = p(r - 1), \tag{3.16}$$

and S_N measures the distance of $\underset{\sim}{T}_N$, in (3.4), from the permutational centre of gravity of the same. If H_o in (1.7) does not hold, it can be shown that for at least one $k = 1, \ldots, p$ and one $j = 1, \ldots, r$, $T_{N,j}^{(k)}$ will converge to a point (stochastically) other than $\bar{E}_N^{(k)}$, and hence, by (3.15), S_N will be stochastically larger. Thus, we may propose the following test function:

$$\phi(\underset{\sim}{Y}_N) = \begin{cases} 1, \text{ if } S_N > S_{N,\varepsilon}(\underset{\sim}{R}_N), \\ \gamma(\underset{\sim}{R}_N), \text{ if } S_N = S_{N,\varepsilon}(\underset{\sim}{R}_N), \\ 0, \text{ if } S_N < S_{N,\varepsilon}(\underset{\sim}{R}_N), \end{cases} \tag{3.17}$$

where the constants $S_{N,\varepsilon}(\underset{\sim}{R}_N)$ and $\gamma(\underset{\sim}{R}_N)$ may usually depend on $\underset{\sim}{R}_N$ and are so chosen that

$$E\{\phi(\underset{\sim}{Y}_N) | \mathcal{P}_n\} = \varepsilon: \quad 0 < \varepsilon < 1. \tag{3.18}$$

(3.18) implies that $E\{\phi(\underset{\sim}{Y}_N) | H_o\} = \varepsilon$. For small values of n(and r), one may venture to evaluate the exact values of $S_{N,\varepsilon}(\underset{\sim}{R}_N)$ and $\gamma(\underset{\sim}{R}_N)$ with the aid of (3.7). However, the labor of this process of evaluation increases considerably with the increase in n(or r), and hence, as in other permutation tests, we are faced with the problem of finding out the asymptotic form of the permutation distribution of S_N. This is done in the next section.

4. ASYMPTOTIC PERMUTATION DISTRIBUTION OF S_N.

We shall impose certain regularity conditions on the p sequences $\{\underset{\sim}{E}_N^{(k)}\}$, $k = 1, \ldots, p$, defined by (2.9) and (2.10), as well as on the joint distribution function $G(\underset{\sim}{Y}_{i1}, \ldots, \underset{\sim}{Y}_{ir})$. Let us define

$$F_{N[j]}^{(k)}(x) = \frac{1}{n}\left[\text{Number of } Y_{ij}^{(k)} \le x\right], \; k = 1, \ldots, p, \; j = 1, \ldots, r; \quad (4.1)$$

$$H_N^{(k)}(x) = \frac{1}{n}\Sigma_{j=1}^{r} F_{N[j]}^{(k)}(x), \; k = 1, \ldots, p; \quad (4.2)$$

$$F_{N[j,\ell]}^{(k,q)}(x, y) = \frac{1}{n}\left[\text{Number of } (Y_{ij}^{(k)}, Y_{i\ell}^{(q)}) \le (x, y)\right], \quad (4.3)$$

for $k, q = 1, \ldots, p$, $j, \ell = 1, \ldots, r$ with either $j \neq \ell$ or $k \neq q$ or both. Now, corresponding to the joint cdf G, let us denote the marginal cdf of $Y_{ij}^{(k)}$ and of $(Y_{ij}^{(k)}, Y_{i\ell}^{(q)})$ by $F_{[j]}^{(k)}(x)$ and $F_{[j,\ell]}^{(k,q)}(x, y)$, respectively, for $j, \ell = 1, \ldots, r$, $k, q = 1, \ldots, p$, with at least one of $j \neq \ell$, $k \neq q$ being true, and let

$$H^{(k)}(x) = \frac{1}{r}\; \Sigma_{j=1}^{r}\; F_{[j]}^{(k)}(x), \text{ for } k = 1, \ldots, p. \quad (4.4)$$

With the definition of $E_{N,\alpha}^{(k)}$'s as in (2.10), we make the following assumptions concerning $J_N^{(k)}$'s.

<u>ASSUMPTION 1.</u> $\lim_{n=\infty} J_N^{(k)}(H) = J^{(k)}(H)$ <u>exists for all</u> $0 < H < 1$ <u>and is not a constant.</u> Since, we shall be interested here in translation type of alternatives, we shall further assume that

$$J^{(k)}(H) \text{ is } \uparrow \text{ in } H\text{:} \; 0 < H < 1 \text{ for all } k = 1, \ldots, p. \quad (4.5)$$

<u>ASSUMPTION 2.</u> $\frac{1}{N}\Sigma_{\alpha=1}^{N}\left|J_N^{(k)}\left(\frac{\alpha}{N+1}\right)-J^{(k)}\left(\frac{\alpha}{N+1}\right)\right| = o(N^{-\frac{1}{2}})$, (4.6)

<u>for</u> $k = 1, \ldots, p$, <u>and</u>

$$\int_{-\infty}^{\infty}\left[J_N^{(k)}\left(\frac{N}{N+1}H_N^{(k)}(x)\right)-J^{(k)}\left(\frac{N}{N+1}H_N^{(k)}(x)\right)\right]dF_{N[j]}^{(k)}(x) = o_p(N^{-\frac{1}{2}}), \quad (4.7)$$

<u>for all</u> $k = 1, \ldots, p$, $j = 1, \ldots, r$.

<u>ASSUMPTION 3.</u> $J^{(k)}(H)$ <u>is absolutely continuous in H:</u> $0 < H < 1$, <u>and</u>

$$\left[\frac{d^r}{dH^r}J^{(k)}(H)\right] \leq K\left[H(1-H)\right]^{-r-\frac{1}{2}+\delta}, \quad (4.8)$$

<u>for</u> $r = 0, 1$, <u>and some</u> $\delta > 0$, <u>where K is a finite positive constant.</u>

Also for the positive definiteness and asymptotic convergence of the covariance matrix $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$, given by (3.12), we require two more mild regularity conditions.

<u>ASSUMPTION 4.</u> $\frac{1}{N}\Sigma_{\alpha=1}^{N}\left|\{J_N^{(k)}\left(\frac{\alpha}{N+1}\right)\}^2-\{J^{(k)}\left(\frac{\alpha}{N+1}\right)\}^2\right| = o(1)$, (4.9)

<u>for</u> $k = 1, \ldots, p$, <u>and</u>

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[J_N^{(k)}\left(\frac{N}{N+1}H_N^{(k)}(x)\right)J_N^{(q)}\left(\frac{N}{N+1}H_N^{(q)}(y)\right)-J^{(k)}\left(\frac{N}{N+1}H_N^{(k)}(x)\right)J^{(q)}\left(\frac{N}{N+1}H_N^{(q)}(y)\right)\right]dF_{N[j,\ell]}^{(k,q)}(x,y)$$

$$= o_p(1) \text{ for all } j, \ell = 1, \ldots, p,\ k, q = 1, \ldots, p, \quad (4.10)$$

<u>where either</u> $k \neq q$ <u>or</u> $j \neq \ell$ <u>or both. Let us also define</u>

$$Z_{ij}^{(k)} = J^{(k)}(H^{(k)}(Y_{ij}^{(k)})),\ k = 1, \ldots, p,\ j = 1, \ldots, r; \quad (4.11)$$

$$Z'_{ij} = (Z^{(1)}_{ij},\ldots,Z^{(p)}_{ij}),\ j = 1,\ldots,r; \tag{4.12}$$

$$a_{kq\cdot j\ell} = E\{Z^{(k)}_{ij}\cdot Z^{(q)}_{i\ell}\}\ \text{for } k,\ q = 1,\ldots,p,\ j,\ \ell = 1,\ldots,r; \tag{4.13}$$

$$A_{j\ell} = ((a_{kq\cdot j\ell}))_{k,q=1,\ldots,p},\ \text{for } j,\ \ell = 1,\ldots,r; \tag{4.14}$$

$$\nu_{kq} = \frac{1}{r}\sum_{j=1}^{r} a_{kq\cdot jj} - \frac{1}{r^2}\sum_{j=1}^{r}\sum_{\ell=1}^{r} a_{kq\cdot j\ell},\ \text{for } k,\ q=1,\ldots,p \tag{4.15}$$

$$\underset{\sim}{\nu} = ((\nu_{kq}))_{k,\ q=1,\ldots,p}. \tag{4.16}$$

<u>ASSUMPTION 5</u>. <u>$\underset{\sim}{\nu}$ is positive definite</u> (4.17)

Before we present the main theorems of this section, let us consider the conditions under which assumption 5 holds. Using (4.14), let us define

$$\underset{\sim}{A}_{(j,\ell)} = [\underset{\sim}{A}_{jj} + \underset{\sim}{A}_{\ell\ell} - 2\,\underset{\sim}{A}_{j\ell}],\quad j\neq\ell = 1,\ldots,r. \tag{4.18}$$

<u>THEOREM 4.1</u> <u>Assumption 5 holds if</u>

$$\max_{j\neq\ell=1,\ldots,r}[\text{Rank of } \underset{\sim}{A}_{(j,\ell)}] = p \tag{4.19}$$

<u>PROOF</u>. Let $\underset{\sim}{\ell}'=(\ell_1,\ldots,\ell_p)$ be any real and <u>non-null</u> p-vector, and let

$$t_j = \underset{\sim}{\ell}'\underset{\sim}{Z}_{ij},\ j=1,\ldots,r,\qquad t_{\cdot} = \frac{1}{r}\sum_{j=1}^{r} t_j, \tag{4.20}$$

where $\underset{\sim}{Z}_{ij}$'s are defined by (4.12). It is then easily seen that

$$\underset{\sim}{\ell}'\underset{\sim}{\nu}\underset{\sim}{\ell} = \frac{1}{r}\sum_{j=1}^{r} E(t_j^2) - E(t_{\cdot}^2) \geq 0. \tag{4.21}$$

Thus, we require only to show that for any non-null $\underset{\sim}{\ell}$, (4.21) is strictly positive. Using essentially the proof of lemma 4.1 of Sen (1966), it can be shown that $\frac{1}{r}\sum_{j=1}^{r} E(t_j^2) - E(t_{\cdot}^2)$ will be strictly positive unless

$$E(t_j t_\ell) = E(t_j^2) = \text{constant, for all } j,\ell=1,\ldots,r. \tag{4.22}$$

Now, using (4.18) and (4.19), we get that

$$E(t_j - t_\ell)^2 = \underset{\sim}{\ell}\ \underset{\sim}{A}_{(j,\ell)}\underset{\sim}{\ell} > 0, \tag{4.23}$$

for at least one pair (j,ℓ), $j \neq \ell = 1,\ldots,r$. As $E(t_j - t_\ell)^2 \leq 2[E(t_j^2) + E(t_\ell^2)]$, (4.23) implies that $E(t_j^2) > 0$ for at least one $j=1,\ldots,r$. Again, for the specific (j,ℓ) for which (4.23) holds, we may assume without any loss of generality that $E(t_\ell^2) \leq E(t_j^2)$, $E(t_j^2) > 0$, and thus, we require only to show that $E(t_j t_\ell) < E(t_j^2)$. If $E(t_\ell^2) = 0$, the proof is evident, while, if $E(t_\ell^2) > 0$, we have from (4.23) $2E(t_j t_\ell) < E(t_j^2) + E(t_\ell^2) < 2E(t_j^2)$. Hence, (4.22) can not hold for all $j,\ell = 1,\ldots r$, if (4.19) holds. Consequently, (4.21) is strictly positive.

Hence, the theorem.

It may be noted that (4.19) really implies that the vector $(\underset{\sim}{Z}_{ij} - \underset{\sim}{Z}_{i\ell})$ is of full rank for at least one $j \neq \ell = 1,\ldots,r$.

THEOREM 4.2. Under the assumptions 1 to 5, $\underset{\sim}{V}_N(\underset{\sim}{R}_N)$, defined by (3.12), converges in probability to $\underset{\sim}{\nu}$, defined by (4.16), and hence, is positive definite, in probability.

PROOF. The proof of this theorem follows as a more or less straightforward generalization of theorem 4.2 of Puri and Sen (1966) and of theorem 4.2 of Sen (1966c). Hence, for the intended brevity of the paper, it is not considered in detail.

THEOREM 4.3. Under the assumptions 1 to 5, the permutation distribution of the statistic S_N, defined by (3.15), converges asymptotically, in probability, to a chi square distribution with $p(r-1)$ degrees of freedom (d.f.).

PROOF. We shall first prove that under the permutation model considered in Section 3, $[n^{\frac{1}{2}}(T_{N,j}^{(k)} - \bar{E}_N^{(k)}),\ j=1,\ldots,r-1,\ k=1,\ldots,p]$ has asymptotically a $p(r-1)$ multinormal distribution. This would be done by proving that any arbitrary linear function of these $p(r-1)$ statistics has asymptotically a normal distribution under the permutation model of section 3. Such a linear compound can be equivalently written as (by virtue of (2.13),)

$$W_n = n^{\frac{1}{2}} \Sigma_{j=1}^{r} \Sigma_{k=1}^{p} d_{jk} T_{N,j}^{(k)} \text{ where } \Sigma_{j=1}^{r} d_{jk} = 0, \ k=1,\ldots,p. \quad (4.24)$$

Under assumption 2, (4.24) can be rewritten as

$$n^{\frac{1}{2}} \sum_{i=1}^{n} \{ \sum_{j=1}^{r} \sum_{k=1}^{p} d_{jk} J^{(k)} (\frac{R_{ij}^{(k)}}{N+1})\} + o_p(1). \quad (4.25)$$

Let us then write

$$U_{N,i}(\underset{\sim}{R}_N) = \sum_{j=1}^{r} \sum_{k=1}^{p} d_{jk} J^{(k)}(\frac{R_{ij}^{(k)}}{N+1}), \quad i=1,2,\ldots,n. \quad (4.26)$$

The random variable $U_{N,i}(\underset{\sim}{R}_N)$ can have only $r!$ possible equally likely values under our permutation model. These values are obtained by permuting the r vectors $\underset{\sim}{R}_{ij}$, $j=1,\ldots,r$ (defined by (2.7),) among themselves. Thus,

$$E\{U_{N,i}(\underset{\sim}{R}_N) | \mathcal{P}_n\} = \sum_{k=1}^{p} \{\frac{1}{r} \sum_{j=1}^{r} J^{(k)}(\frac{R_{ij}^{(k)}}{N+1})\} \sum_{j=1}^{r} d_{jk} = 0, \quad (4.27)$$

for $i=1,\ldots,n$. Similarly,

$$E\{U_{N,i}^2(\underset{\sim}{R}_N) | \mathcal{P}_n\} = \sum_{k=1}^{p} \sum_{q=1}^{p} \sum_{j=1}^{r} \sum_{\ell=1}^{r} d_{jk} d_{\ell q} E\{J^{(k)}(\frac{R_{ij}^{(k)}}{N+1}) J^{(q)}(\frac{R_{i\ell}^{(q)}}{N+1}) | \mathcal{P}_n\}$$

$$= \sum_{k=1}^{p} \sum_{q=1}^{p} (\sum_{j=1}^{r} d_{jk} d_{jq}) \{\frac{1}{r-1} \sum_{j=1}^{r} [J^{(k)}(\frac{R_{ij}^{(k)}}{N+1}) J^{(q)}(\frac{R_{ij}^{(q)}}{N+1}) - \frac{1}{r^2}(\sum_{j=1}^{r} J^{(k)}(\frac{R_{ij}^{(k)}}{N+1})) \quad (4.28)$$

$$(\sum_{j=1}^{r} J^{(q)}(\frac{R_{ij}^{(q)}}{N+1}))]\} \qquad \text{for } i=1,\ldots,n.$$

Since the permutations of the rank-vectors within the ith block is independent of the permutations within the i'th block for $i \neq i' = 1, \ldots, n$, under our permutation model, $\{U_{N,i}(\underset{\sim}{R}_N),\ i=1,\ldots,n\}$ are mutually independent. Hence, to prove the desired result, we may use the Berry-Essen theorem [cf. Loeve (1962, p. 288)], according to which it is sufficient to show that

$$\lim_{n=\infty} \frac{\Sigma_{i=1}^{n} E\{|U_{N,i}(\underset{\sim}{R}_N)|^s| \mathcal{O}_n\}}{[\Sigma_{i=1}^{n} E\{U_{N,i}^2(\underset{\sim}{R}_N)| \mathcal{O}_n\}]^{3/2}} = 0. \tag{4.29}$$

From (3.11), (3.12) and (4.28), we get that

$$\frac{1}{n}\Sigma_{i=1}^{n} E\{U_{N,i}^2(\underset{\sim}{R}_N)| \mathcal{O}_n\} = \sum_{j=1}^{r}\sum_{k=1}^{p}\sum_{q=1}^{p} d_{jk}d_{jq}v_{kq}(\underset{\sim}{R}_N)$$

$$= \sum_{j=1}^{r} \{\sum_{k=1}^{p}\sum_{q=1}^{p} d_{jk}d_{jq}v_{kq}(\underset{\sim}{R}_N)$$

$$\xrightarrow{P} \sum_{j=1}^{r} \{\sum_{k=1}^{p}\sum_{q=1}^{p} d_{jk}d_{jq}\nu_{kq}\}, \tag{4.30}$$

whereby theorem 4.1 and assumption 5, the right hand side of (4.30) is a (non-zero) positive constant, for any given $(d_{jk},\ j=1,\ldots,r,\ k=1,\ldots,p)$. Thus, it is sufficient to show that the numerator of the left hand side of (4.29) is $o_p(N^{3/2})$, and this readily follows from assumption 3 and (4.26). Hence, under our permutation model, the first term of (4.25) has asymptotically, in probability, a normal distribution. Once this is established, we consider the quadratic form associated with the asymptotic multinormal distribution of $\{n^{\frac{1}{2}}(T_{N,j}^{(k)} - \bar{E}_N^{(k)}),\ j=1,\ldots,r-1,\ k=1,\ldots,p\}$, and using some well-known results on the limiting distribution of continuous functions of random variables [cf. Sverdrup (1952)], it is easily seen that under our permutation model, the statistic S_N, given by (3.15), has asymptotically, in probability, a chi square distribution with $p(r-1)$ d.f.

Hence, the theorem.

It may be noted that the permutation distribution of S_N being essentially a conditional distribution, the convergence in theorem 4.3 holds, in probability, i.e., for almost all $\underset{\sim}{Y}_N$. If we now denote by $\chi^2_{t,\epsilon}$ the upper 100ϵ% point of the chi square distribution with t d.f., then from (3.17) and theorem 4.3, we arrive at the following.

THEOREM 4.4. *$S_{N,\epsilon}(\underset{\sim}{R}_N)$ and $\gamma(\underset{\sim}{R}_N)$, defined by (3.17), converge, in probability to $\chi^2_{p(r-1),\epsilon}$ and 0, respectively.*

By virtue of theorem 4.4, the *exact permutation test*, considered in (3.17), reduces asymptotically to

$$\phi(Y_N) = \begin{cases} 1, & \text{if } S_N \geq \chi^2_{p(r-1),\epsilon} \\ 0, & \text{otherwise;} \end{cases} \tag{4.31}$$

and (4.31) will be termed henceforth the *asymptotic permutation test*.

5. ASYMPTOTIC POWER OF THE PROPOSED TESTS.

In this section we shall study the asymptotic power and power-efficiency of our proposed class of tests. This requires first of all the study of the asymptotic (unconditional) distribution of S_N, when the null hypothesis (1.7) is not necessarily true. For this study, we also adopt the same notations as in section 4, and write

$$T^{(k)}_{N,j} = \int_{-\infty}^{\infty} J^{(k)}_N \left(\frac{N}{N+1} H^{(k)}_N(x)\right) dF^{(k)}_{N[j]}(x), \tag{5.1}$$

for $j=1,\ldots,r$, $k=1,\ldots,p$. The statistics in (5.1) has some analogy with a class of similar statistics considered by Puri and Sen (1966). However, in this case of two way layout we are faced with n independent pr-variate observations, while in the earlier case, Puri and Sen were faced with the oneway layout involving $N(=nr)$ p-variate observations. This makes the situation somewhat more complicated in our case, and the necessary modifications will be studied here. Let us define

$$\mu_j^{(k)} = \int_{-\infty}^{\infty} J^{(k)}(H^{(k)}(x))\, dF_{[j]}^{(k)}(x), \tag{5.2}$$

for $j=1,\ldots,r$, $k=1,\ldots,p$. Also let

$$\beta_{jj'\cdot\ell\ell'}^{(k,q)} = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} [F_{[j,j']}^{(k,q)}(x,y) - F_{[j]}^{(k)}(x)F_{[j']}^{(q)}(y)]\; J'^{(k)}(H^{(k)}(x))\; J'^{(q)}(H^{(q)}(y))\cdot dF_{[\ell]}^{(k)}(x)\; dF_{[\ell']}^{(q)}(y), \tag{5.3}$$

for $j,j',\ell,\ell' = 1,\ldots,r$, $k,\ q=1,\ldots,p$, with either $j\neq j'$ or $k\neq q$ or both, while

$$\beta_{jj\cdot\ell\ell'}^{(k,k)} = \iint_{-\infty<x<y<\infty} F_{[j]}^{(k)}(x)[1-F_{[j]}^{(k)}(y)]\; J'^{(k)}(H^{(k)}(x))\; J'^{(k)}(H^{(k)}(y))\; dF_{[\ell]}^{(k)}(x)dF_{[\ell']}^{(k)}(y)$$

$$+ \iint_{-\infty<x<y<\infty} F_{[j]}^{(k)}(x)[1-F_{[j]}^{(k)}(y)]\; J'^{(k)}(H^{(k)}(x))\; J'^{(k)}(H^{(k)}(y))\; dF_{[\ell']}^{(k)}(x)dF_{[\ell]}^{(k)}(y),$$

$$\text{for } j=1,\ldots,r,\ k=1,\ldots,p,\ \ell,\ell' = 1,\ldots,r. \tag{5.4}$$

Finally, let

$$\beta_{jj'\cdot}^{(k,q)} = \frac{1}{r}\left\{\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\left[\beta_{jj'\cdot\ell\ell'}^{(k,q)} + \beta_{\ell\ell'\cdot jj'}^{(k,q)} - \beta_{\ell j'\cdot j\ell}^{(k,q)} - \beta_{j\ell'\cdot\ell j'}^{(k,q)}\right]\right\} \tag{5.5}$$

for $k,q=1,\ldots,p$; $j,j' = 1,\ldots,r$.

<u>THEOREM 5.1</u>. <u>If the assumptions 1,2 and 3 of section 4 hold, then for arbitrarily</u>

continuous $G(Y_{i1},\dots,Y_{ir})$, the random variables $[N^{\frac{1}{2}}(T_{N,j}^{(k)} - \mu_j^{(k)}),\ j=1,\dots,r,$ $k=1,\dots,p]$ has asymptotically a multinormal distribution with a null mean vector and a dispersion matrix with elements $\beta_{jj'}^{(k,q)}$, defined by (5.5).

(It may be noted that by virtue of (2.13), (4.4) and (5.2), the above multinormal distribution will be essentially singular with a rank less than or equal to $p(r-1)$.)

PROOF. We shall present only a brief sketch of the proof, as the same will follow precisely on similar lines as in theorem 5.1 of Puri and Sen (1966) and theorem 5.1 of Sen (1966c). Proceeding precisely on the same line as in the proofs of these two theorems it can be easily shown that

$$N^{\frac{1}{2}}\left|(T_{N,j}^{(k)} - \mu_j^{(k)}) - (B_{j,1N}^{(k)} + B_{j,2N}^{(k)})\right| = o_p(1), \tag{5.6}$$

for all $j=1,\dots,r,\ k=1,\dots,p,$ where

$$B_{j,1N}^{(k)} + B_{j,2N}^{(k)} = \frac{1}{r}\sum_{j'=1}^{r}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[B_{j:j'}^{(k)}(Y_{ij}^{(k)}) - B_{j':j}^{(k)}(Y_{ij'}^{(k)})\right]\right\} ; \tag{5.7}$$

$$B_{j:\ell}^{(k)}(Y_{ij}^{(k)}) = \int_{-\infty}^{\infty}\left[F_{[j](i)}^{(k)}(x) - F_{[j]}^{(k)}(x)\right] J'^{(k)}(H^{(k)}(x))\, dF_{[\ell]}^{(k)}(x) ; \tag{5.8}$$

$$F_{[j](i)}^{(k)}(x) = \begin{cases} 0, & \text{if } x < Y_{ij}^{(k)} \\ 1, & \text{if } x \geq Y_{ij}^{(k)}, \end{cases} \tag{5.9}$$

for $i=1,\dots,n,\ j,\ \ell=1,\dots,r,\ k=1,\dots,p$. It is therefore sufficient to show that for any arbitrary non-null $\underset{\sim}{\delta} = (\delta_{11},\dots,\delta_{pr})$, $N^{\frac{1}{2}}\sum_{j=1}^{r}\sum_{k=1}^{p}\delta_{jk}(B_{j,1N}^{(k)} + B_{j,2N}^{(k)})$ has asymptotically a normal distribution. By virtue of (5.7), the same can be written as $n^{-\frac{1}{2}}\sum_{i=1}^{n} B(\underset{\sim}{Y}_{i1},\dots,\underset{\sim}{Y}_{ir})$, where

$$B(\underset{\sim}{Y}_{i1},\ldots,\underset{\sim}{Y}_{ir}) = r^{-\frac{1}{2}} \sum_{j=1}^{r} \sum_{\ell=1}^{r} \sum_{k=1}^{p} \delta_{jk}[B_{j:\ell}^{(k)}(Y_{ij}^{(k)}) - B_{\ell:j}^{(k)}(Y_{i\ell}^{(k)})],\ i=1,\ldots,n. \tag{5.10}$$

Since, the random variables in (5.10) are independent and identically distributed, in order to make use of the central limit theorem under the Lindeberg's condition, it is sufficient to show that these have finite second order moments. Using (5.8), it is easily seen that $E\{B(\underset{\sim}{Y}_{i1},\ldots,\underset{\sim}{Y}_{ir})\} = 0$ for all $i=1,\ldots,n$, and by virtue of (5.10), it appears to be sufficient to show that $E\{|B_{j:\ell}^{(k)}(Y_{ij}^{(k)}|^2\} < \infty$ for all $j,\ell=1,\ldots,r$, $k=1,\ldots,p$, $i=1,\ldots,n$. Now, under the assumption 3 of section 4, it is easily seen that for any η: $0 < \eta < \delta$ (defined by (4.8),)

$$E\{|B_{j:\ell}^{(k)}(Y_{ij}^{(k)})|^{2+\eta}\} < \infty, \tag{5.11}$$

uniformly in $j,\ell=1,\ldots,r$, $k=1,\ldots,p$. Hence, the desired asymptotic normality follows readily. Again, by (5.7), (5.8) and (5.9), we have

$$E\{B_{j:\ell}^{(k)}(Y_{ij}^{(k)})\, B_{j':\ell'}^{(q)}(Y_{i'j'}^{(q)})\} = \delta_{ii'}\, \beta_{jj':\ell\ell'}^{(k,q)}, \tag{5.12}$$

where $\delta_{i\ell'}$ is the usual Kronecker delta and $\beta_{jj':\ell\ell'}^{(k,q)}$'s are defined by (5.3) and (5.4), for $j,j',\ell,\ell'=1,\ldots,r$, $k,q=1,\ldots,p$. Hence, it is easily seen that

$$N\, E\{(B_{j,1N}^{(k)} + B_{j,2N}^{(k)})(B_{\ell,1N}^{(q)} + B_{\ell,2N}^{(q)})\} = \beta_{j\ell}^{(k,q)}, \tag{5.13}$$

which is defined by (5.5), for $k,q=1,\ldots,p$, $j,\ell=1,\ldots,r$. Consequently, by (5.6), we may conclude that the dispersion matrix of the asymptotic normal distribution has elements $\beta_{j\ell}^{(k,q)}$, defined by (5.5).

Hence, the theorem.

We have already noted that the asymptotic normal distribution of theorem 5.1 is singular and of rank at most equal to p(r-1). If the null hypothesis in (1.7) is true, $G(Y_{i1},\ldots,Y_{ir})$ will be a symmetric function of the r vectors, and hence it is easily seen that (i) the marginal cdf of $Y_{ij}^{(k)}$ will be the same for all $j=1,\ldots,r$, $i=1,\ldots,n$, and is denoted by $H^{(k)}(x)$ for $k=1,\ldots,p$; (ii) the marginal cdf of $(Y_{ij}^{(k)}, Y_{ij}^{(q)})$ $(k\neq q)$ will not depend on j, and is denoted by $H_1^{(k,q)}(x,y)$ for $k\neq q=1,\ldots,p$, and (iii) the marginal cdf of $(Y_{ij}^{(k)}, Y_{i\ell}^{(q)})$ $(j\neq \ell)$ will not depend on $(j\neq\ell)$, and is denoted by $H_2^{(k,q)}(x,y)$ for $j\neq\ell=1,\ldots,r$, $k, q=1,\ldots,p$. Thus, it follows from (5.3), (5.4), (4.11) through (4.14) that in this case

$$\begin{aligned}\beta_{jj':\ell\ell'}^{(k,q)} = a_{kq\cdot jj'} &= a_{kq}^{(1)}, \quad \text{if} \quad j=j'=1,\ldots,r,\\ &= a_{kq}^{(2)} \quad \text{if} \quad j\neq j'=1,\ldots,r,\end{aligned} \tag{5.14}$$

where $a_{kq}^{(1)}$ depends only on $H_1^{(k,q)}(x,y)$ and $a_{kq}^{(2)}$ on $H_2^{(k,q)}(x,y)$, respectively. Thus, from (4.15) and (5.14), we get that in this case ν_{kq}, defined by (4.15), reduces to

$$\nu_{kq} = [(r-1)/r](a_{kq}^{(1)} - a_{kq}^{(2)}), \quad k,q=1,\ldots,p, \tag{5.15}$$

and

$$\beta_{j\ell}^{(k,q)} = (\delta_{j\ell}r-1)\ \nu_{kq}, \quad j,\ell=1,\ldots,r, \quad k,q=1,\ldots,p, \tag{5.16}$$

where $\delta_{j\ell}$ is the usual Kronecker delta. Consequently, it is easily seen that under H_o in (1.7),

$$S_N^* = n \sum_{k=1}^{p} \sum_{q=1}^{p} \nu^{kq} \sum_{j=1}^{r} (T_{N,j}^{(k)} - \mu_{.}^{(k)})(T_{N,j}^{(q)} - \mu_{.}^{(q)}) \tag{5.17}$$

(where $((\nu^{kq}))$ is the reciprocal of $((\nu_{kq}))$, and

$$\mu^{(k)} = \int_0^1 J^{(k)}(u)du, \ k=1,\ldots,p,)$$

has asymptotically a chi square distribution with p(r-1) d.f. Now, under assumption 2 of section 4

$$|N^{\frac{1}{2}}(\bar{E}_N^{(k)} - \mu^{(k)})| = o(1), \text{ for } k=1,\ldots,p, \tag{5.18}$$

and by theorem 4.2, we have under assumption 5 that

$$\underset{\sim}{V}_N(\underset{\sim}{R}_N) \xrightarrow{P} \underset{\sim}{\nu} \quad \text{i.e.,} \quad \underset{\sim}{V}_N^{-1}(\underset{\sim}{R}_N) \xrightarrow{P} \underset{\sim}{\nu}^{-1}. \tag{5.19}$$

Hence, from (3.15), (5.17), (5.18) and (5.19), we get that under H_o in (1.7)

$$S_N \overset{P}{\sim} S_N^*. \tag{5.20}$$

Hence, we arrive at the following.

THEOREM 5.2. Under H_o in (1.7) and assumptions 1 to 5 of section 4, the statistic S_N in (3.15) has asymptotically a chi square distribution with p(r-1) d.f.

Let now $\hat{\underset{\sim}{\nu}}$ be any consistent estimator of $\underset{\sim}{\nu}$, defined by (4.15) and (5.15). If $\hat{\underset{\sim}{\nu}}$ is positive definite and we denote its reciprocal by $\hat{\underset{\sim}{\nu}}^{-1} = ((\hat{\nu}^{kq}))$, then we can have an asymptotically distribution-free test based on

$$\hat{S}_N = n \sum_{k=1}^{p} \sum_{q=1}^{p} \hat{\nu}^{kq} \sum_{j=1}^{r} (T_{N,j}^{(k)} - \bar{E}_N^{(k)})(T_{N,j}^{(q)} - \bar{E}_N^{(q)}). \tag{5.21}$$

Since, $\hat{S}_N$ can be shown to have the chi square distribution with p(r-1) d.f., when H_o in (1.7) holds, the test function may be proposed as

$$\hat{\phi}(\underset{\sim}{Y}_N) = \begin{cases} 1, & \text{if } \hat{S}_N > \chi^2_{p(r-1),\epsilon} \\ 0, & \text{otherwise.} \end{cases} \tag{5.22}$$

We shall now consider the power properties of the permutation test in (3.17) and (4.31) and the large sample test in (5.22). We shall obtain certain power-equivalence relations among these tests, and compare them with the parametric tests referred to in Section one.

By virtue of theorem 5.1, it can be shown that if the linear model (1.5) holds but the null hypothesis (1.7) is not true, then $(\mu_j^{(k)} - \bar{E}_N^{(k)})$, $j=1,\ldots,r$, $k=1,\ldots,p$, can not all converge to zero as $N\to\infty$, and hence, S_N, defined by (3.15), will be stochastically indefinitely large, as N increases. Consequently, the tests considered will be all consistent. Thus, for any given $(\underset{\sim}{\tau}_1,\ldots,\underset{\sim}{\tau}_r)$ in (1.6), (not all null), the power of the test (3.17) or (4.31) or (5.22) will be asymptotically equal to unity. Hence, forthe study of the asymptotic power properties of the tests, we shall consider a sequence of alternative hypotheses for which the power asymptotically lies in the open interval $(\epsilon,1)$. This we specify as

$$H_N: \underset{\sim}{\tau}_j = N^{-\frac{1}{2}} \underset{\sim}{\lambda}_j, \quad j=1,\ldots,r, \tag{5.23}$$

where λ_j, $j=1,\ldots,r$ are all real p-vectors, not all equal (or null). Further, for simplification of the asymptotic power function, we shall assume that the cdf $F_{[j]}^{(k)}(x)$, $F_{[j,j]}^{(k,q)}(x,y)$ and $F_{[j,\ell]}^{(k,q)}(x,y)$ are all absolutely continuous and have continuous density functions. Under $\{H_N\}$ in (5.23), we will thus have sequences of cdf's $\{F_{[j],N}^{(k)}(x)\}$ etc, defined for each N, and it is easy to verify that

$$\lim_{N=\infty} F_{[j],N}^{(k)}(x) = H^{(k)}(x) \quad \text{for all } j=1,\ldots,r, \tag{5.24}$$

$$\lim_{N=\infty} F_{[j,j],N}^{(k,q)}(x,y) = H_1^{(k,q)}(x,y) \text{ for all } j=1,\ldots,r,\ k\neq q=1,\ldots,p \tag{5.25}$$

$$\lim_{N=\infty} F_{[j,\ell],N}^{(k,q)}(x,y) = H_2^{(k,q)}(x,y) \quad \text{for } j\neq\ell=1,\ldots,r,\ k,q=1,\ldots,p. \tag{5.26}$$

Hence, in this case also (5.16) holds in the limit as $N \to \infty$. Also, if we define

$$\zeta_k = \int_{-\infty}^{\infty} \frac{d}{dx} J^{(k)}(H^{(x)}(x))\, dF^{(k)}(x), \quad k=1,\ldots,p, \tag{5.27}$$

then, it is easy to show that

$$\lim_{N=\infty} E\{N^{\frac{1}{2}}(T_{N,j}^{(k)} - \mu_{.}^{(k)}) | H_N\} = \lambda_j^{(k)} \zeta_k, \tag{5.28}$$

for all $j=1,\ldots,r$, $k=1,\ldots,p$. Hence, from the results of theorem 5.1 it follows that under $\{H_N\}$, S_N^* has asymptotically a noncentral chi square distribution with $p(r-1)$ d.f. and the noncentrality parameter

$$\Delta_S = \sum_{k=1}^{p} \sum_{q=1}^{p} \nu^{kq} \zeta_k \zeta_p \left\{\frac{1}{r} \sum_{j=1}^{r} (\lambda_j^{(k)} - \bar{\lambda}^{(k)})(\lambda_j^{(q)} - \bar{\lambda}^{(q)})\right\}, \tag{5.29}$$

where

$$\bar{\lambda}^{(q)} = \Sigma_{j=1}^{r} \lambda_j^{(q)}/r, \text{ for } q=1,\ldots,p.$$

Now, from theorem 4.2, (5.24), (5.25), (5.26) and the discussion following it, it follows that under $\{H_N\}$ also $S_N \overset{P}{\sim} S_N^*$, and hence, we have the following.

THEOREM 5.3. *Under the sequence of alternatives $\{H_N\}$ in (5.23), S_N, defined by (3.15), has asymptotically a non-central chi square distribution with $p(r-1)$ d.f. and the non-centrality parameter Δ_S, defined by (5.29), provided the conditions of theorem 5.1 hold, and in addition, the marginal cdf's corresponding to the joint cdf $G(\underset{\sim}{Y}_{i1},\ldots,\underset{\sim}{Y}_{ir})$ are all absolutely continuous and have continuous density functions.*

If we consider the large sample test, defined by (5.22), then it can be shown similarly that $\hat{S}_N \overset{P}{\sim} S_N^*$, under $\{H_N\}$, and hence, the conclusions of theorem 5.3 also applies to $\hat{S}_N$. Thus, the permutation test considered in sections 3 and 4 and the large sample test considered in (5.29), are asymptotically power equivalent for the sequence of alternatives $\{H_N\}$, in (5.23). As we have seen that the permutation tests are easy to define for small samples, we are now in a position to recommend the use of the same, for all sample sizes.

In the parametric case, the limiting distributions of various test-statistics for this problem have been studied by various workers, and the reader may be referred to Anderson (1958, Ch. 8·10), Rao [(1952, Ch. 7), (1965, Ch. 8)], and James (1960), among others. Most of the results relate to the null case, while it may be considerably difficult to formulate a general theory for the non-null cases, though some work has also been done on this line. For the likelihood ratio test, however, the asymptotic non-null distribution may be found without much difficulty, and for the sequence of alternatives in (5.23), this statistic can be shown to have asymptotically a non-central chi square distribution with $p(r-1)$ d.f. and the non-centrality parameter

$$\Delta_U = \sum_{k=1}^{p} \sum_{q=1}^{p} \sigma^{kq} \{\frac{1}{r} \sum_{j=1}^{r} (\lambda_j^{(k)} - \bar{\lambda}^{(k)})(\lambda_j^{(q)} - \bar{\lambda}^{(q)})\}, \qquad (5.30)$$

where $\lambda_j^{(k)}$ and $\bar{\lambda}^{(k)}$ are defined by (5.23) and (5.29), respectively, and $\underset{\sim}{\Sigma}^{-1} = ((\sigma^{kq})) = ((\sigma_{kq}))^{-1}$ is the reciprocal of the common dispersion matrix $\underset{\sim}{\Sigma}$. The comparison of Δ_S and Δ_U (for the purpose of studying asymptotic relative efficiency) poses the same problem as has been studied in some detail by Puri and Sen (1966). For intended brevity, this is therefore not reproduced again. The only remark that may be made here is that if we work with $\underset{\sim}{E}_N^{(k)}$'s (defined by (2.9), (2.10),) as the expected values of the order statistics in a sample of size N drawn from a standardized normal distribution and term the resulting test as Normal score MANOVA test for any two way lay out, then it is easily seen that for normal alternatives, this test is asymptotically power equivalent to the likelihood ratio test. In actual practice, the use of rank sums (i.e., $E_{N,\alpha}^{(k)} = \alpha/(N+1)$, $\alpha=1,\ldots,N$, $k=1,\ldots,p$) often results in a quite simplified procedure and at the same time does not involve any serious loss of efficiency. For details of these points, the reader may be referred to Puri and Sen (1966), the same argument being true in the two way layout case.

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