ON SOME OPTIMUM NONPARAMETRIC PROCEDURES IN TWO-WAY LAYOUT*

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1) After August 31, 1966 at the Courant Institute of Mathematical Sciences, New York University, New York.

2) On leave of absence from Calcutta University.

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For the estimation and testing of contrasts in two-way layout, some optimum nonparametric procedures based on Chernoff-Savage [2] type of rank order statistics are considered here. The asymptotic properties of the proposed methods are studied and compared with those of the least square method.

1. INTRODUCTION

In a two-way layout with one observation per cell, the observable random variables $X_{i\alpha}(i=1,\ldots,c,\ \alpha=1,\ldots,N)$ are of the form

$$X_{i\alpha} = \mu+\beta_\alpha + \tau_i + e_{i\alpha};\ \sum_{\alpha=1}^{N} \beta_\alpha = 0,\ \sum_{i=1}^{c} \tau_i = 0 \tag{1.1}$$

where μ is the <u>mean-effect</u>, β's are the <u>block effects</u>, τ's are the <u>treatment effects</u> and $e_{i\alpha}$'s are <u>the resudual error components</u>. It is assumed that $e_{i\alpha}$ $(i=1,\ldots,c,\ \alpha=1,\ldots,N)$ are independent and identically distributed random variables (i.i.d.r.v.) having a continuous cumulative distribution function (cdf) $F(e)$. Let $\theta = \sum_{i=1}^{c} \ell_i\tau_i$ (where $\sum_{i=1}^{c} \ell_i = 0$) be any contrast in τ's and let

$$V_\alpha = \sum_{i=1}^{c} \ell_i X_{i\alpha} = \theta + \sum_{i=1}^{c} \ell_i e_{i\alpha},\ \alpha=1,\ldots,N. \tag{1.2}$$

We denote the cdf of $\sum_{i=1}^{c} \ell_i e_{i\alpha}$ by $G_c(e)$ and assume it to be symmetric about $e=0$. Then $V_1,\ldots,V_N$ are N i.i.d.r.v. having the cdf $G_c(x-\theta)$. The least square

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(ℓ.s.) estimate of θ is given by

$$\hat{\theta}_N = \frac{1}{N} \sum_{\alpha=1}^{N} V_\alpha = \bar{V}_N \text{ (say,)} \tag{1.3}$$

and if σ_c^2 be the variance of G_c, the variance of $\hat{\theta}_N$ is equal to σ_c^2/N. Further, if F (or equivalently G_c) is normal, $\hat{\theta}_N$ is the minimum variance unbiased (MVU) estimator of θ. The object of the present investigation is to consider some nonparametric estimators of θ and to compare their performances with that of $\hat{\theta}_N$. These estimates are based on a celebrated class of rank order statistics due to Chernoff and Savage [2] which we may pose as follows. Let $Z_{N,\alpha} = 1$ if the α-th smallest observation among $|V_1|, \ldots, |V_N|$ is from a positive V, and otherwise let $Z_{N,\alpha} = 0$ for $\alpha=1,\ldots,N$. Then the desired rank order statistic may be expressed as

$$h_N(V_1,\ldots,V_N) = \frac{1}{N} \sum_{\alpha=1}^{N} E_{N,\alpha} Z_{N,\alpha}, \tag{1.4}$$

where $E_{N,\alpha}$ is the expected value of the α-th order statistic of a sample of size N drawn from a distribution

$$\psi^*(x) = \begin{cases} \psi(x) - \psi(-x) & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{1.5}$$

$\psi(x)$ being symmetric about $x=0$ and it is assumed that ψ satisfies all the regularity conditions of Theorem 1 of Chernoff and Savage [2]. In particular, if $\psi(x)$ is the standardized normal cdf, h_N in (1.4) will be termed a <u>normal score statistic</u>. Along the same fashion as in [3], these rank order statistics will be used to derive suitable translation invariant robust estimates of θ. Further, it will be shown that the use of the normal score statistic will lead to an estimator which is (asymptotically) at least as efficient as the ℓ.s. estimate, for all G_c (and hence F). In this sense, the proposed method can be regarded as an optimum one.

2. RAW AND ADJUSTED ESTIMATORS OF CONTRASTS IN τ's

By definition, $h_N(V_1-t,\ldots,V_N-t)$ is $\downarrow$ in t, and further it follows from (1.2), (1.4) and (1.5) that the distribution of $h_N(V_1-\theta,\ldots,V_N-\theta)$ is symmetric about some known origin, which we denote by μ_N. Let us then denote (following [3])

$$\begin{aligned} \theta_N^* &= \operatorname{Sup}\{\theta:\ h_N(V_1-\theta,\ldots,V_N-\theta) > \mu_N\} \\ \theta_N^{**} &= \operatorname{Inf.}\{\theta:\ h_N(V_1-\theta,\ldots,V_N-\theta) < \mu_N\} \end{aligned} \tag{2.1}$$

and let

$$\tilde{\theta}_N = \tfrac{1}{2}(\theta_N^* + \theta_N^{**}). \tag{2.2}$$

Proceeding then precisely on the same line as in [3], it follows that $\tilde{\theta}_N$ is a translation invariant robust estimator of θ and its distribution is symmetric about θ. Also, the <u>asymptotic relative efficiency</u> (A.R.E.) of $\tilde{\theta}_N$ with respect to $\hat{\theta}_N$ is

$$e_{\tilde{\theta}_N,\hat{\theta}_N} = \sigma_c^2\Big(\int_{-\infty}^{\infty} \frac{d}{dx} J[G_c(x)]\ dG_c(x)\Big)^2 \Big/ \int_0^1 J^2(u)\ du, \tag{2.3}$$

where $J(u) = \psi^{-1}(u)$ is the inverse of the cdf $\psi(x)$: $0 \le u \le 1$. It is well known (cf. [2]) that when $\psi(x)$ is the standardized normal cdf, (2.3) is always greater than or equal to one (uniformly in G_c). Thus, as in the case of one-way layout the use of normal scores leads to asymptotically optimum estimators of contrasts. For reasons to be explained below, we shall term $\tilde{\theta}_N$ as the <u>raw or</u>

unadjusted estimator of θ.

Now the estimators in (2.2) for different contrasts are incompatible in the sense that they do not satisfy the linear relations satisfied by the contrasts they estimate. So as in ([1], [4]) it might be desirable to replace them by a mutually compatible system. With this end in view, we first obtain the estimates of $\tau_i - \tau_j$ for $i \neq j=1,\ldots,c$. Let us denote

$$X^*_{ij,\alpha} = X_{i\alpha} - X_{j\alpha},\ \Delta_{ij} = \tau_i - \tau_j \text{ and } e^*_{ij,\alpha} = e_{i\alpha} - e_{j\alpha}, \tag{2.4}$$

for $\alpha=1,\ldots,N$. We denote the cdf of $e^*_{ij,\alpha}$ by $G(e)$ and note that as $e_{i\alpha}$'s are i.i.d.r.v., $G(e)$ is symmetric about zero for all $F(e)$. Thus from (1.1) and (2.4), the cdf of $X^*_{ij,\alpha}$ becomes equal to $G(x-\Delta_{ij})$, for $i \neq j=1,\ldots,c$. Thus, on defining $\underset{\sim}{X}^*_{ij} = (X^*_{ij,1},\ldots,X^*_{ij,N})$, we may use the same rank order statistic h_N, defined in (1.4), and proceeding as in (2.1) and (2.2) arrive at the estimate

$$Y_{ij} = \tfrac{1}{2}(\Delta^*_{N,ij} + \Delta^{**}_{N,ij}) \text{ of } \Delta_{ij},\ i < j=1,\ldots,c. \tag{2.5}$$

Now, as in [1] and [4], we define

$$Y_{i\cdot} = \frac{1}{c}\sum_{j=1}^{c} Y_{ij},\ Y_{ii} = 0, \text{ for } i=1,\ldots,c, \tag{2.6}$$

and define the compatible or adjusted estimator as

$$Z_{ij} = Y_{i\cdot} - Y_{j\cdot} \text{ for } i \neq j=1,\ldots,c. \tag{2.7}$$

These estimators satisfy the same linear relations which are satisfied by Δ_{ij}'s. We can then define $\theta = \sum_{i=1}^{c} \ell_i \tau_i = \sum_{i=1}^{c}\sum_{j=1}^{c} d_{ij}\Delta_{ij}$, and as a compatible estimator of it we may consider

$$\sum_{i=1}^{c}\sum_{j=1}^{c} d_{ij} Z_{ij} = \widetilde{\theta}_{cN} \text{ (say)}. \tag{2.8}$$

It may be noted that similar studies have been made by Lehmann [4], using, in particular, the Wilcoxon's signed rank statistic, and our proposed methods generalize his findings to a more wider class of rank order statistics. To make the large sample comparison of the properties of $\hat{\theta}_N$, $\tilde{\theta}_N$ and $\tilde{\theta}_{cN}$, we shall require certain limit theorems which we shall consider in the next section.

3. ASYMPTOTIC NORMALITY OF $\tilde{\theta}_N$ and $\tilde{\theta}_{cN}$

As a basis for this study, we first consider the following.

THEOREM 3.1. The joint limiting distribution of the random variables $N^{\frac{1}{2}}(Y_{ij}-\Delta_{ij})$, $1 \le i < j \le c$, is a $\binom{c}{2}$-variate normal distribution with zero means and a covariance matrix $\underset{\sim}{\Sigma} = ((\sigma_{ij,rs}))$, where

$$\sigma_{ij,rs} = \begin{cases} A^2/B^2, & \text{if } i=r,\ j=s,\ i\neq j \\ \lambda_J(G)/B^2, & \text{if } i=r,\ j\neq s,\ i\neq j \\ -\lambda_J(G)/B^2, & \text{if } i=s,\ j\neq r,\ i\neq j \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

where

$$A^2 = \int_0^1 J^2(u)du, \quad B = \int_{-\infty}^{\infty} \frac{d}{dx} J[G(x)]\ dG(x), \tag{3.2}$$

and

$$\lambda_J(G) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J[G(x)]\ J[G(y)]\ dG^*(x,y), \tag{3.3}$$

and where $J(u) = \psi^{-1}(u)$ $(0 \le u \le 1)$, $\psi(x)$ being symmetric about zero, and $G^*(x,y)$ is the joint cdf of $e^*_{ij,\alpha}$ and $e^*_{i\ell,\alpha}$ $(j\neq\ell)$, whose marginal cdf's are $G(x)$ and $G(y)$, respectively.

The proof of the theorem rests on the following.

LEMMA 3.2. Suppose that the random variables $X^*_{ij,\alpha}$ have the distribution specified by the cdf $G(x + N^{-\frac{1}{2}}a_{ij})$, $1 \leq i < j \leq c$ (i.e., G is fixed but Δ_{ij}, defined by (2.4), is equal to $-N^{-\frac{1}{2}}a_{ij}$, where a_{ij}'s are real and finite), and let $h_{N,ij}$ be defined as in (1.4) (with $X^*_{ij,\alpha}$, $\alpha=1,\ldots,N$) and $\psi(x)$ in (1.5) satisfying the assumptions of theorem 1 of Chernoff and Savage [2]. Then the random variables $[N^{\frac{1}{2}}(h_{N,ij} - \alpha_{ij}),\ 1 \leq i < j \leq c]$ where α_{ij} is defined in (5.4) have a $\binom{c}{2}$-variate limiting normal distribution (as $N\to\infty$) with a null mean vector and a covariance matrix $\underset{\sim}{\Gamma} = ((\tau_{ij,rs}))$, where

$$\tau_{ij,rs} = \begin{cases} \frac{1}{4}A^2 & \text{if } i=r,\ j=s,\ i\neq j \\ \frac{1}{4}\lambda_J(G), & \text{if } i=r,\ j\neq s,\ i\neq j,\ r\neq s, \\ -\frac{1}{4}\lambda_J(G), & \text{if } i=s,\ j\neq r,\ i\neq j,\ r\neq s. \\ 0\ , & \text{otherwise}. \end{cases} \tag{3.4}$$

The proof of this lemma is given in the appendix.

Proof of theorem 3.1. By (9.2) of [3],

$$\begin{aligned} &\lim_{N\to\infty} P\{N^{\frac{1}{2}}(Y_{ij}-\Delta_{ij}) \leq a_{ij} \text{ for all } 1 \leq i < j \leq c\} \\ &= \lim_{N\to\infty} P_N\{N^{\frac{1}{2}}(h_{N,ij} - \mu_N) \leq 0 \text{ for all } 1 \leq i < j \leq c\} \end{aligned} \tag{3.5}$$

where

$$\mu_N = \int_0^\infty \psi^{*-1}[G(x) - G(-x)]\, dG(x), \tag{3.6}$$

ψ^* being defined in (1.5) and P_N indicating the probability which is computed for the sequence of shifts $\Delta_{ij} = N^{-\frac{1}{2}} a_{ij}$, $1 \leq i < j \leq c$. Furthermore, for these sequences, it is easy to show that as $N\to\infty$

$$N^{\frac{1}{2}}(\mu_N - \alpha_{ij}) \to \tfrac{1}{2} a_{ij} B, \text{ for } 1 \leq i < j \leq c, \tag{3.7}$$

where B is defined in (3.2). Hence, it follows that

$$\lim_{N\to\infty} P\{N^{\frac{1}{2}}(Y_{ij}-\Delta_{ij}) \leq a_{ij} \text{ for all } 1 \leq i < j \leq c\}$$

$$= \lim_{N=\infty} P_N\{N^{\frac{1}{2}}(h_{N,ij}-\mu_N) \leq \tfrac{1}{2}B\, a_{ij} \text{ for } 1 \leq i < j \leq c\} \tag{3.8}$$

Now, by lemma 3.2, the right hand side of (3.8) is equal to $Q(\frac{1}{2}a_{12}B, \ldots, \frac{1}{2}a_{c-1,c}B)$, where $Q(\underset{\sim}{x})$ is the $\binom{c}{2}$-variate multinormal cdf having a null mean vector and a covariance matrix $\underset{\sim}{\Gamma}$, defined by (3.4). The rest of the proof of the theorem is straight forward and is omitted.

Hence, the theorem.

THEOREM 3.3. The joint distribution of $[N^{\frac{1}{2}}(Z_{ic}-\Delta_{ic}),\ i=1,\ldots,c-1]$ is asymptotically normal with a null mean vector and a covariance matrix $\underset{\sim}{\Gamma} = ((\gamma_{ij}))_{i,j=1,\ldots,c-1}$, where

$$\gamma_{ii} = 2\sigma_o^2(1-\rho), \quad \gamma_{ij} = \sigma_o^2(1-\rho), \quad i\neq j=1,\ldots,c-1, \tag{3.9}$$

and where

$$\sigma_o^2 = [(c-1)/c^2]\{A^2 + (c-2)\lambda_J(G)\}/B^2, \quad \rho=-1/(c-1), \tag{3.10}$$

A^2, B and $\lambda_J(G)$ being defined in (3.2) and (3.3), respectively.

Proof. It follows from (2.5), (2.6) and (2.7) that Z_{ij}'s are linear functions of Y_{ij}'s. Further, from theorem 3.1, it follows that $\mathrm{Var}\{N^{\frac{1}{2}}(Y_{i\cdot}-\Delta_{i\cdot})\} = \sigma_o^2$ and and $\mathrm{Cov}\{N^{\frac{1}{2}}(Y_{i\cdot}-\Delta_{i\cdot}),\ N^{\frac{1}{2}}(Y_{j\cdot}-\Delta_{j\cdot})\} = \rho\sigma_o^2$, where σ_o^2 and ρ are defined by (3.10). The rest of the proof of the theorem follows directly from theorem 3.1.

Hence the theorem.

It follows from theorem 3.1 and theorem 3.3 that the A.R.E. of Z_{ij} with respect to Y_{ij} (in the sense of reciprocal of the ratio of their asymptotic cariances) is equal to

$$e_{Z_{ij}, Y_{ij}} = \frac{cA^2}{2[A^2 + (c-2)\,\lambda_J(G)]}\,, \tag{3.11}$$

where A^2 and $\lambda_J(G)$ are defined by (3.2) and (3.3). Incidentally, (3.11) is independent of $1 \leq i < j \leq c$. We have then the following theorem.

<u>THEOREM 3.4</u>. <u>If</u> $J(u) = \psi^{-1}(u)$, $0 \leq u \leq 1$; <u>and</u> $\psi(x)$ <u>is symmetric about</u> $x=0$, <u>then under the conditions of theorem 1 of Chernoff and Savage [3],</u>

$$e_{Z_{ij}, Y_{ij}} \geq 1, \quad \text{for all } G \in \mathcal{G},\ \psi \in \mathcal{G}, \tag{3.12}$$

$\mathcal{G}$ <u>being the class of all absolutely continuous cdf's which are symmetric about</u> $x=0$.

<u>Proof</u>. It follows from (3.11) that if we can show that

$$\lambda_J(G) \leq \tfrac{1}{2}A^2 \text{ for all } G \in \mathcal{G},\ \psi \in \mathcal{G}, \tag{3.13}$$

(3.12) will follow immediately. To prove this, let us write

$$Z^*_{i\alpha} = X_{i\alpha} - X_{i+1,\alpha} \text{ for } i=1,\ldots,c-1 \text{ and } Z^*_{c\alpha} = X_{c\alpha} - X_{1\alpha}, \tag{3.14}$$

and let

$$Y^*_{i\alpha} = J[G(Z^*_{i\alpha})],\ i=1,\ldots,c, \quad Y^*_{\cdot\alpha} = \sum_{i=1}^{c} Y^*_{i\alpha}\,. \tag{3.15}$$

Then from (3.2), (3.3) and some simple algebraic manipulations it follows that

$$E[Y^*_{\cdot\alpha}]^2 = c\,A^2[1 - 2\lambda_J(G)/A^2] \geq 0. \tag{3.16}$$

(3.13) readily follows from (3.16).

Hence, the theorem.

In general, if $\theta = \sum_{i=1}^{c} \ell_i \tau_i$ be any contrast, we can rewrite it as $\sum_{i=1}^{c} \sum_{j=1}^{c} d_{ij} \triangle_{ij}$ and consider the estimates

$$\sum_{i=1}^{c} \sum_{j=1}^{c} d_{ij} Y_{ij} = \tilde{\theta}_{UN} \text{ and } \sum_{i=1}^{c} \sum_{j=1}^{c} d_{ij} Z_{ij} = \tilde{\theta}_{cN}, \tag{3.17}$$

and then with the aid of theorems 3.1 and 3.3, we readily arrive at the following.

<u>THEOREM 3.4</u>. <u>The A.R.E. of $\tilde{\theta}_{cN}$ with respect to $\tilde{\theta}_{UN}$ is given by (3.11) and (3.12) for all</u> $\{(\ell_1,\ldots,\ell_c): \sum_1^c \ell_i = 0\}$.

Let us next compare the compatible estimate $\tilde{\theta}_{cN}$ and the least square estimate $\hat{\theta}_N$ in (1.3). Rewriting $\tilde{\theta}_{cN}$ as $\sum_{i=1}^{c} \ell_i Y_{i\cdot}$, we get from theorems 3.1 and 3.3 that the asymptotic variance of $N^{\frac{1}{2}}(\tilde{\theta}_{cN}-\theta)$ is equal to

$$\left(\sum_{i=1}^{c} \ell_i^2\right) \sigma_o^2 \, (1-\rho) \tag{3.18}$$

where σ_o^2 and ρ are defined in (3.10). Also, σ_c^2, defined in (2.3), is nothing but $\left(\sum_{i=1}^{c} \ell_i^2\right) \sigma^2$, where σ^2 is the variance of $_{i\alpha}$, in (1.1). Thus,

$$\begin{aligned} \text{A.R.R.}_{\tilde{\theta}_{cN}, \hat{\theta}_N} &= \sigma^2/\sigma_o^2 \, (1-\rho) \\ &= \frac{2B^2\sigma^2}{A^2} \cdot \frac{c}{2[1 + (c-2)\lambda_J(G)/A^2]} \, . \end{aligned} \tag{3.19}$$

Now, the variance of the cdf G(x) of $X_{i\alpha}-X_{j\alpha}$ is nothing byt $2\sigma^2$, and hence, the first factor on the right hand side of (3.19) is the usual efficiency-factor of the allied one sample rank order test with respect to student's t-test, while the second factor is by theorem 3.3, is always at least as large as 1. Hence,

$$\text{A.R.E.}_{\hat{\theta}_{cN},\hat{\theta}_N} \geq 2B^2\sigma^2/A^2. \tag{3.20}$$

In particular, for normal scores, (3.20) is greater than or equal to one for all G, and hence, the compatible normal score estimator $\hat{\theta}_{cN}$ (i.e., with J(u) as the inverse of the normal cdf) is asymptotically at least as efficient as the least square estimator $\hat{\theta}_N$, for arbitrary $\ell_1,\ldots,\ell_c$, $\sum_1^c \ell_i=0$. Finally, from (2.3) and (3.19),

$$\text{A.R.E.}_{\hat{\theta}_{cN},\tilde{\theta}_N} = \frac{2B^2\sigma^2}{\sigma_c^2\left[\int_{-\infty}^{\infty} \frac{d}{dx} J[G_c(x)]\,dG_c(x)\right]^2} \cdot \frac{c}{2[1+(c-2)\lambda_J(G)/A^2]}, \tag{3.21}$$

where G_c is the cdf of V_α, defined by (1.2), whose variance is σ_c^2. Now, $(2\sigma^2B^2/A^2)$ is the A.R.E. of the test based on h_N in (1.4) with respect to the student's t-test when the parent cdf is G(x), while $\sigma_c^2[\int_{-\infty}^{\infty} \frac{d}{dx} J[G_c(x)]\, dG_c(x)]^2/A^2$ is the same A.R.E. when the parent cdf is $G_c(x)$. Thus, (3.21) will in general depend on both the cdf's G(x) and $G_c(x)$, unless the two are identical. However, if the parent cdf F(x) of $e_{i\alpha}$ is normal, then both G(x) and $G_c(x)$ will also be normal, and hence (3.21) will be equal to

$$c/2[1 + (c-2)\lambda_J(G)/A^2] \geq 1. \tag{3.22}$$

4. CONFIDENCE INTERVAL AND TESTS FOR CONTRASTS

We have so far considered the problem of estimation of contrasts using the rank order statistics of the type (1.4). The corresponding problems of testing and confidence intervals for contrasts are briefly sketched below.

Using V_α, $\alpha=1,\ldots,N$, defined by (1.2), the problem of testing H_o: $\theta=\sum_{i=1}^{c}\ell_i\tau_i=\theta_o$, reduces to that of testing the symmetry of the distribution of $V_\alpha-\theta_o$ around zero, and hence, we may use the rank order test based on the statistic $h_N(V_1-\theta_o,\ldots,V_N-\theta_o)$, defined in (1.4). For large $N, N^{\frac{1}{2}}[h_N(V_1-\theta_o,\ldots, V_N-\theta_o) - \mu_N]$ has a normal distribution with zero mean and variance $\frac{1}{4}A^2$, where A^2 is defined by (3.2) and $\mu_N = \int_0^1 J(u)du$, $j(u) = \psi^{-1}(u)$, ψ being defined in (1.5). Thus, the test can be carried out using the standard normal tables. Again, by virtue of this result,

$$\lim_{N=\infty} P\{-\tfrac{1}{2}A\ \tau_{\alpha/2} \le N^{\frac{1}{2}}[h_N(V_1-\theta_o,\ldots,V_N-\theta_o) - \mu_N] \le \tfrac{1}{2}A\ \tau_{\alpha/2}\} = 1-\alpha \tag{4.1}$$

where $\tau_{\alpha/2}$ is the upper $100(\alpha/2)\%$ point of the standardized normal distribution. Thus, if we let

$$\begin{aligned}\theta_{L,N} &= \text{Inf}\{\theta:\ h_N(V_1-\theta,\ldots,\ V_N-\theta) = \mu_N + \tfrac{1}{2}A\tau_{\alpha/2}N^{-\frac{1}{2}}\}\\ \theta_{U,N} &= \text{Sup}\{\theta:\ h_N(V_1-\theta,\ldots,\ V_N-\theta) = \mu_N - \tfrac{1}{2}A\tau_{\alpha/2}N^{-\frac{1}{2}}\}\ ,\end{aligned} \tag{4.2}$$

then proceeding on the same line as in Sen [7], we get that

$$P\{\theta_{L,N} < \theta < \theta_{U,N}|\theta\} = 1-\alpha, \tag{4.3}$$

which is our desired confidence limit for θ.

Now the above test and confidence interval are subject to same criticisms as $\hat{\theta}_N$ in section 2. We may overcome these to some extent by the following procedure.

On defining $\hat{\theta}_{cN}$ as in (2.8), we have $N^{\frac{1}{2}}(\hat{\theta}_{cN}-\theta)$ asymptotically normally distributed with zero mean and variance $(\sum_{i=1}^{c}\ell_i^2)\ \sigma_o^2\ (1-\rho)$, (cf. theorem 3.3 and (3.18),) where σ_o^2 and ρ are defined in (3.10). Thus, the only problem remains is to estimate $\sigma_o^2(1-\rho)$ i.e., $\lambda_J(G)$ and B, defined by (3.2) and (3.3). Now, defining $X^*_{ij,\alpha}$ as in (2.4) and working with the statistic $h_N(X^*_{ij,1}-\Delta_{ij},\ldots,X^*_{ij,N}-\Delta_{ij})$ in (1.4), we get on proceeding precisely on the same line as in (4.1) through (4.3) that

$$P\{\Delta_{ij,L,N} \leq \Delta_{ij} \leq \Delta_{ij,U,N} | \Delta_{ij}\} = 1-\alpha, \tag{4.4}$$

where $\Delta_{ij,L,N}$ and $\Delta_{ij,U,N}$ are defined as in (4.2), for $1 \leq i < j \leq c$. Further, it follows from the results of Sen [7] that

$$\hat{B}_{ij,N} = 2A\ \tau_{\alpha/2}/[N^{\frac{1}{2}}(\Delta_{ij,U,N} - \Delta_{ij,L,N})]$$

is a consistent estimator of B, in (3.2), for all $i < j = 1,\ldots,c$. Consequently, we may consider the pooled estimate

$$\hat{B}_N = \binom{c}{2}^{-1} \sum_{i<j=1}^{c} \hat{B}_{ij,N}, \tag{4.5}$$

which will be a translation-invariant consistent estimator of B. Finally, the following is a convenient and consistent estimator of $\lambda_J(G)$.

Suppose for any distince (i,j,k), we consider the N paired variables $(X^*_{ij,\alpha}, X^*_{ik,\alpha})$ for $\alpha=1,\ldots,N$. Let then

$$\hat{G}_{N(i,j)}(x) = \frac{1}{N}[\text{Number of } (X^*_{ij,\alpha}-Y_{ij}) \leq x], \tag{4.6}$$

where Y_{ij} is defined by (2.5), for $i < j=1,\ldots,c$, and let

$$\hat{G}^*_{N(i:j,k)}(x,y) = \frac{1}{N}[\text{Number of } \{(X^*_{ij,\alpha}-Y_{ij}),(X^*_{ik,\alpha}-Y_{ik})\} \leq \{x,y\}], \tag{4.7}$$

for $i \neq j \neq k=1,\ldots,c$.

Let also $a_{N,\alpha} = J_N(\frac{\alpha}{N})$ be the expected value of the α-th order statistic in a sample of size N drawn from the distribution $\psi(x)$, defined in (1.5), where $\psi(x)$ satisfies the conditions of theorem 1 of [2]. We then consider the statistic

$$L_{i:j,k,N} = \frac{1}{N} \sum_{\alpha=1}^{N} a_{N,R_{ij,\alpha}} \, a_{N,R_{ik\alpha}} \tag{4.8}$$

where $R_{ij,\alpha}$ is the rank of $X^*_{ij,\alpha}$ among all $(X^*_{ij,1},\ldots,X^*_{ij,N})$, for $\alpha=1,\ldots,N$ and $i \neq j=1,\ldots,c$. Since, the ranks remain invariant under change of origin, if we define $Z_{ij,\alpha} = X^*_{ij,\alpha} - \Delta_{ij}$ for $\alpha=1,\ldots,N$, the rank of $Z_{ij,\alpha}$ among all $Z_{ij,1},\ldots,Z_{ij,N}$ will also be $R_{ij,\alpha}$ for all $\alpha=1,\ldots,N$, $i \neq j=1,\ldots,c$. Let then

$$G_{N(i,j)}(x) = \frac{1}{N}[\text{Number of } Z_{ij,\alpha} \leq x], \tag{4.9}$$

$$G^*_{N(i:j,k)}(x,y) = \frac{1}{N}[\text{Number of } (Z_{ij,\alpha}, Z_{ik,\alpha}) \leq (x,y)]$$

for $i \neq j \neq k=1,\ldots,c$. We can then write (4.8) equivalently as

$$\int_{-\infty}^{\infty} \int_{\infty}^{\infty} J_N[G_{N(i,j)}(x)] J_N[G_{N(i,k)}(y)] \; dG^*_{N(i:j,k)}(x,y) \tag{4.10}$$

Now, $Z_{ij,\alpha}$ has the cdf $G(x)$, for all $i \neq j=1,\ldots,c$, and $(Z_{ij,\alpha}, Z_{ik,\alpha})$ has jointly the bivariate cdf $G^*(x,y)$, it can be shown by routine analysis that under the conditions of theorem 1 of Chernoff and Savage [2], (4.10) converges in probability to $\lambda_J(G)$, defined in (3.3). Thus, we may pool the estimators $\{L_{i:j,k,N}, i \neq j \neq k=1,\ldots,c\}$ into a single measure (by unweighted arithmatic mean), and use the same as an estimate of $\lambda_J(G)$. Once B and $\lambda_J(G)$ are estimated, we have no difficulty in providing a confidence interval to $\theta = \sum_{i=1}^{c} \ell_i \tau_i$ or to test for any hypothetical value of the same.

We have so far considered the case of single deservation per cell. Along the same line as in Lehmann [4], the results derived in this paper can be readily extended to the case of several observations per cell. For intended brevity, the details are omitted.

5. APPENDIX

Proof of lemma 3.2. Let $G_{N,ij}(x)$ be the sample cdf of N observations $X^*_{ij,1},\ldots, X^*_{ij,N}$ of which the population cdf is $G_{ij}(x) = G(x-\Delta_{ij})$. Denote

$$H_{ij}(x) = G_{ij}(x)-G_{ij}(-x) \tag{5.1}$$

$$H_{N,ij}(x) = G_{N,ij}(x)-G_{N,ij}(-x). \tag{5.2}$$

Then, proceeding as in [6], we can write

$$h_{ij} = \alpha_{ij} + B_{1N,ij} + B_{2N,ij} + \sum_{r=1}^{4} C_{rN,ij} \tag{5.3}$$

where

$$\alpha_{ij} = \int_0^\infty J^*[H_{ij}(x)]\, dG_{ij}(x) \tag{5.4}$$

$$B_{1N,ij} = \int_0^\infty J^*[H_{ij}(x)]\, d(G_{N,ij}(x)-G_{ij}(x)) \tag{5.5}$$

$$B_{2N,ij} = \int_0^\infty [H_{N,ij}(x)-H_{ij}(x)]J^{*\prime}[H_{ij}(x)]\, dG_{ij}(x) \tag{5.6}$$

and the C-terms are all $o_p(N^{-\frac{1}{2}})$ (cf. [5]).

The difference $N^{\frac{1}{2}}(h_{ij}-\mu_{ij})-N^{\frac{1}{2}}(B_{1N,ij} + B_{2N,ij})$ tends to zero, in probability, and so the vectors

$$[N^{\frac{1}{2}}(h_{ij}-\mu_{ij});\ 1\leq i<j\leq c] \text{ and } [N^{\frac{1}{2}}(B_{1N,ij}+B_{2N,ij}),\ 1\leq i<j\leq c]$$

have the same limiting distributions.

Thus to prove this theorem, it suffices to show that for any real d_{ij}, $1 \leq i < j \leq c$, not all zero, $N^{\frac{1}{2}} \sum\sum_{i<j} d_{ij}(B_{1N,ij}+B_{2N,ij})$ has normal distribution in the limit. Now proceeding as in [6], we can express $N^{\frac{1}{2}} \sum\sum_{icj} d_{ij}(B_{1N,ij}+B_{2N,ij})$ as the sum of independent and identically distributed random variables having finite first two moments. The proof follows.

To compute the variance-covariance terms of $B_{1N,ij}+B_{2N,ij}$, we note, (by integrating $B_{1N,ij}$ by parts, adding $B_{2N,ij}$ to it, and using (5.1) and (5.2)) that

$$B_{1N,ij}+B_{2N,ij} = \frac{1}{N}\sum_{\alpha=1}^{N} B_{ij}(x_{ij,\alpha}) \tag{5.7}$$

where

$$\begin{aligned} B_{ij}(x) = \int_0^\infty & [G_{1,ij}(x)-G_{ij}(x)]J'^*[H_{ij}(x)]\ dG_{ij}(-x) \\ & - \int_0^\infty [G_{1,ij}(-x)-G_{ij}(-x)]J'^*[H_{ij}(x)]\ dG_{ij}(x) \end{aligned} \tag{5.8}$$

Since $E\ B_{ij}(x) = 0$, therefore

$$\mathrm{Var}(B_{1N,ij}+B_{2N,ij}) = \frac{1}{N}\ \mathrm{Var}\ B_{ij}(X_{ij})$$

$$\begin{aligned} &= \frac{2}{N} \iint_{0<x<y<\infty} G_{ij}(x)[1-G_{ij}(y)]J'^*[H_{ij}(x)]J'^*[H_{ij}(y)]\ dG_{ij}(-x)dG_{ij}(-y) \\ &+ \frac{2}{N} \iint_{0<x<y<\infty} G_{ij}(-y)[1-G_{ij}(-x)]J'^*[H_{ij}(x)]J'^*[H_{ij}(y)]dG_{ij}(x)dG_{ij}(y) \\ &- \frac{2}{N} \int_0^\infty \int_0^\infty G_{ij}(-y)[1-G_{ij}(x)]J'^*[H_{ij}(x)]J'^*[H_{ij}(y)]\ dG_{ij}(-x)dG_{ij}(y) \end{aligned} \tag{5.9}$$

Similarly

$$\mathrm{Cov}(B_{1N,ij}+B_{2N,ij},\ B_{1N,rs}+B_{2N,rs})$$

$$= \frac{1}{N^2} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} E[B_{ij}(X_{ij,\alpha})B_{rs}(X_{rs,\beta})]$$

$$= \frac{1}{N^2} \sum_{\alpha=1}^{N} E[B_{ij}(X_{ij,\alpha})B_{rs}(X_{rs,\alpha})] \tag{5.10}$$

$$= \frac{1}{N} E[B_{ij}(X_{ij,\alpha})B_{rs}(X_{rs,\alpha})]$$

The covariance matrix $\Gamma = ((\tau_{ij,rs}))$ defined in (3.4), (3.2) and (3.3), is obtained by taking limits of N $\mathrm{var}(B_{1N,ij} + B_{2N,ij})$ and N $\mathrm{Cov}(B_{1N,ij} + B_{2N,ij},\ B_{1N,rs} + B_{2N,rs})$ as $N \to \infty$.

REFERENCES

[1] Bhuchongkul, Subha and Puri, Madan Lal, "On the Estimation of Contrasts in Linear Models", The Annals of Mathematical Statistics, vol. 36, 1965, pp. 198-202.

[2] Chernoff, Herman and Savage, I. Richard, "Asymptotic Normality and Efficiency of Certain Nonparametric Tests", The Annals of Mathematical Statistics, vol. 29, 1958, pp. 972-994.

[3] Hodges, J. L., Jr., and Lehmann, E. L., "Estimates of Location Based on Rank Tests", The Annals of Mathematical Statistics, vol. 34, 1963, pp. 598-611.

[4] Lehmann, E. L., "Asymptotically Nonparametric Inference in Some Linear Models with One Observation Per Cell", The Annals of Mathematical Statistics, vol. 35, 1964, pp. 726-734.

[5] Hollander, Myles, "An Asymptotically Distribution-Free Multiple Comparison Procedure-Treatments vs. Control", The Annals of Mathematical Statistics, vol. 37, 1966, pp. 735-738.

[6] Puri, Madan L., and Sen, Pranab, K., "On the Asymptotic Normality of One Sample Chernoff-Savage Test Statistics", Journal of the American Statistical Association, (Submitted)

[7] Sen, Pranab Kumar, "On a Distribution-free Method of Estimating Asymptotic Efficiency of a Class of Nonparametric Tests", The Annals of Mathematical Statistics, vol. 37, (December issue).