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FOR GROUPED DATA*

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Summary. The object of the present investigation is to extend the findings of Hájek [5] on asymptotically most powerful rank order tests (AMPROT) to grouped data where the underlying distributions are essentially continuous but the observable random variables correspond to a finite or countable set of contiguous class intervals. In this context, the two sample problem for grouped data is considered and various efficiency results are also studied.

1. Introduction. Let us consider a sequence of random vectors $\underline{X}_v = (X_{v1}, \dots, X_{vN_v})$ consisting of N_v independent random variables, where X_{vi} has a continuous cumulative distribution function (cdf) $F_{vi}(x)$, for $i=1, \dots, N_v$, $1 \leq v < \infty$. As in Hájek [5], we consider the model

$$(1.1) \quad F_{vi}(x) = F(\sigma^{-1}[x - \alpha - \beta c_{vi}]), \quad i=1, \dots, N_v, \quad 1 \leq v < \infty$$

where α , β and $\sigma(>0)$ are real parameters, $(c_{v1}, \dots, c_{vN_v})$ are known quantities, concerning which we make the following assumptions:

$$(1.2) \quad \sum_{i=1}^{N_v} c_{vi} = 0, \quad \sum_{i=1}^{N_v} c_{vi}^2 = C_v^2, \quad \sup_v C_v^2 < \infty;$$

$$(1.3) \quad \max_{1 \leq i \leq N_v} c_{vi}^2 / C_v^2 = o(1).$$

Hájek [5] has considered the class of cdf's for which the square root of the

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probability density function possesses a quadratically integrable derivative i. e.,

$$(1.4) \quad \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x) dx = A^2(F) < \infty,$$

where $f(x) = dF(x)/dx$ and $f'(x) = df(x)/dx$. Throughout this paper, we shall also stick to the assumptions (1.1) through (1.4). In Hájek's case, $X_{\mathbf{v}}$ is observable, while in our case, we have a finite or countable set of class intervals

$$(1.5) \quad I_j: a_j < x \leq a_{j+1}, \quad j=0, 1, \dots, \infty \text{ (without any loss of generality),}$$

[where $-\infty = a_0 < a_1 < a_2 < \dots < \infty$ is any (finite or countable) set of ordered points on the real line $(-\infty, \infty)$,] and the observable stochastic vector is $\tilde{X}_{\mathbf{v}}^* = (X_{\mathbf{v}1}^*, \dots, X_{\mathbf{v}N_{\mathbf{v}}}^*)$, where

$$(1.6) \quad X_{\mathbf{v}i}^* = \sum_{j=0}^{\infty} I_j Z_{ij},$$

$$(1.7) \quad Z_{ij} = \begin{cases} 1, & \text{if } X_{\mathbf{v}i} \in I_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } i=1, \dots, N_{\mathbf{v}}, \quad j=0, \dots, \infty.$$

Thus, having observed $\tilde{X}_{\mathbf{v}}^*$, we want to test the null hypothesis

$$(1.8) \quad H_0: \beta=0 \text{ i. e., no regression,}$$

against the set of alternatives that $\beta > 0$ (but infinitely close to 0).

It may be noted that if actual practice, even if the parent cdf's are continuous, the process of data collection mostly introduces such a set of class

intervals on which the data are recorded. This results in so called grouped data, where the usual nonparametric methods (for continuous variables) are not strictly applicable, and the present author is not aware of any optimum non-parametric procedures for such data. The object of the present investigation is to consider some permutationally distribution free tests for regression for grouped data and by a generalization of Hajek's [5] ideas, to show that these are AMPROT for the same problem. In this context, the two sample problem for grouped data is also studied. Two different types of efficiency factors are studied here. First, how these AMPROT compared with the tests by Chernoff and Savage [2], Hajek [5] and Gastwirth [3]? Secondly, how the various bounds for the efficiency factors available in the continuous case compared with the parallel bounds for grouped data? Finally, in the two sample case, it is also indicated how Capon's [1] results can also be generalized to grouped data.

2. Asymptotically most powerful parametric test. This test is considered in brief, as it will be essentially required in the sequel. Let us define

$$(2.1) \quad F_j = F([a_j - \alpha]/\sigma), \quad P_j = F_{j+1} - F_j \quad \text{for } j=0, 1, \dots, \infty;$$

$$(2.2) \quad \Delta_j = [f(F^{-1}(F_j)) - f(F^{-1}(F_{j+1}))]/P_j, \quad j=0, \dots, \infty,$$

$$(2.3) \quad A^2(F, \{I_j\}) = \sum_{j=0}^{\infty} \Delta_j^2 P_j.$$

Now, Δ_j can be written as

$$(2.4) \quad \int_{F_j}^{F_{j+1}} \phi(u) du / \int_{F_j}^{F_{j+1}} du, \quad \text{where } \phi(u) = -f'(F^{-1}(u))/f(F^{-1}(u)), \quad 0 < u < 1,$$

and hence,

$$(2.5) \quad A(F, \{I_j\}) = \sum_{j=0}^{\infty} \left\{ \frac{\int_{F_j}^{F_{j+1}} \phi(u) du}{F_j} \right\}^2 \int_{F_j}^{F_{j+1}} du$$

$$\leq \sum_{j=0}^{\infty} \int_{F_j}^{F_{j+1}} \phi^2(u) du = A^2(F),$$

uniformly in $\{I_j\}$ i.e., for all possible $-\infty = a_0 < a_1 < a_2 < \dots < \infty$.

Again for β sufficiently close to zero, we have

$$(2.6) \quad F_{v_i}(a_{j+1}) - F_{v_i}(a_j) = P_j \{1 + (\beta/\sigma) c_{v_i} \Delta_j + o(\beta)\},$$

uniformly in $j(=0, \dots, \infty)$, (by virtue of (1.2) and (1.3)). Thus, for β sufficiently close to zero, the likelihood function is

$$(2.7) \quad L(X_v^* | \beta) = \prod_{i=1}^{N_v} \{ \sum_{j=0}^{\infty} Z_{ij} P_j [1 + (\beta/\sigma) c_{v_i} \Delta_j + o(\beta)] \}.$$

Consequently,

$$(2.8) \quad L(X_v^* | \beta) / L(X_v^* | \beta=0) = 1 + (\beta/\sigma) \sum_{i=1}^{N_v} c_{v_i} \sum_{j=0}^{\infty} \Delta_j Z_{ij} + o(\beta).$$

Let us denote by T_v

$$(2.9) \quad T_v = \sum_{i=1}^{N_v} c_{v_i} \sum_{j=0}^{\infty} \Delta_j Z_{ij}.$$

Hence, from Neyman-Pearson's fundamental lemma (cf. Lehmann [6, p. 65]), it readily follows that for testing $H_0: \beta=0$ against $\beta > 0$, the asymptotically most powerful test function is

$$(2.10) \quad \psi_1(X_{\mathbf{v}}^*) = \begin{cases} 1, & \text{if } T_{\mathbf{v}} > T_{\mathbf{v},\epsilon} \\ \gamma_{\epsilon}, & \text{if } T_{\mathbf{v}} = T_{\mathbf{v},\epsilon} \\ 0, & \text{if } T_{\mathbf{v}} < T_{\mathbf{v},\epsilon} \end{cases}$$

where $T_{\mathbf{v},\epsilon}$ and γ_{ϵ} are so chosen that $E\{\psi_1(X_{\mathbf{v}}^*) | H_0\} = \epsilon$, $0 < \epsilon < 1$, ϵ being the desired level of significance of the test.

[It may be noted that in actual practice both α and σ in (1.1) are mostly unknown, and as a result, Δ_j 's are also so. However, as in (3.1) - (3.4) of Hájek [5], we may estimate α and σ and work with the estimated Δ_j 's.]

Now $\sum_{j=0}^{\infty} Z_{ij} \Delta_j$, $i=1, \dots, N_{\mathbf{v}}$ are independent random variables, and hence $T_{\mathbf{v}}$ is a linear function of $N_{\mathbf{v}}$ independent random variables, where (1.3) guarantees the condition for the central limit theorem to be satisfied by the coefficients. Consequently, we get on using the classical central limit theorem and avoiding the details of derivation, the following.

THEOREM 2.1 For β infinitely close to zero

$$\mathcal{L}([T_{\mathbf{v}} - (\beta/\sigma)C_{\mathbf{v}}^2 A^2(F, \{I_j\})] / C_{\mathbf{v}} A(F, \{I_j\})) \rightarrow N(0, 1),$$

where $C_{\mathbf{v}}^2$ and $A^2(F, \{I_j\})$ are defined in (1.2) and (2.3), and $\mathcal{L}(Z) \rightarrow N(0, 1)$ indicates that Z converges in law to a normal distribution with zero mean and unit variance.

Thus, from (2.10) and theorem 2.1 we get that the asymptotic power of the test (2.10) is given by

$$(2.11) \quad 1 - \Phi(\tau_{\epsilon} - (\beta/\sigma)C_{\mathbf{v}} A(F, \{I_j\})),$$

where $\Phi(x)$ is the standardized normal cdf and $\Phi(\tau_\epsilon) = 1 - \epsilon$.

3. Asymptotically most powerful rank order test. Let us define

$$(3.1) \quad \sum_{i=1}^{N_{\mathbf{v}}} Z_{ij} = N_{\mathbf{v}j}, \text{ for } j=0, \dots, \infty, \text{ so that } N_{\mathbf{v}} = \sum_{j=0}^{\infty} N_{\mathbf{v}j};$$

$$(3.2) \quad F_{N_{\mathbf{v}},0} = 0, \quad F_{N_{\mathbf{v}},j+1} = \sum_{\ell=0}^j N_{\mathbf{v}\ell} / N_{\mathbf{v}} \text{ for } j=0, 1, \dots, \infty.$$

If now $N_{\mathbf{v}j} > 0$, we define

$$(3.3) \quad \hat{\Delta}_{\mathbf{v}j} = [f(F^{-1}(F_{N_{\mathbf{v}},j})) - f(F^{-1}(F_{N_{\mathbf{v}},j+1}))] / [F_{N_{\mathbf{v}},j+1} - F_{N_{\mathbf{v}},j}]$$

$$= \int_{F_{N_{\mathbf{v}},j}}^{F_{N_{\mathbf{v}},j+1}} \phi(u) du / \int_{F_{N_{\mathbf{v}},j}}^{F_{N_{\mathbf{v}},j+1}} du,$$

where F^{-1} is the inverse of $F(x)$ and $\phi(u)$ is defined by (2.4). If $N_{\mathbf{v},j} = 0$, we conventionally let

$$(3.4) \quad \hat{\Delta}_{\mathbf{v}j} = \phi(F_{N_{\mathbf{v}},j}).$$

Our proposed test-statistic is then

$$(3.5) \quad S_{\mathbf{v}} = \sum_{i=1}^{N_{\mathbf{v}}} c_{\mathbf{v}i} \sum_{j=0}^{\infty} \hat{\Delta}_{\mathbf{v}j} Z_{ij},$$

and we shall see later on that $S_{\mathbf{v}}$ provides an AMPROT for the hypothesis $\beta=0$ against $\beta > 0$.

3.1. Null distribution of $S_{\mathbf{v}}$. Since, we are dealing with grouped data, even under H_0 in (1.8), the distribution of $S_{\mathbf{v}}$ will depend on the unknown $\Delta_j (j=0, \dots, \infty)$. However, under a very simple permutation model, $S_{\mathbf{v}}$ will provide a distribution

free test. Now, under H_0 in (1.8), $X_{v_i}^*$ ($i=1, \dots, N_v$) are independent and identically distributed random variables (i.i.d.r.v.), and hence \tilde{X}_v^* has a joint distribution which remains invariant under any permutation of its N_v arguments. Thus, in the N_v -dimensional real space R^{N_v} , we have a set of $N_v!$ permutationally equiprobable points and the permutational (conditional) probability measure defined on this set is denoted by \mathcal{P}_v . Hence, under \mathcal{P}_v , all the $N_v!$ equally likely realizations have the common probability $1/N_v!$. Now, N_{vj} in (3.1) and $\hat{\Delta}_{vj}$ in (3.3) are unaffected by the permutations of the coordinates of \tilde{X}_v^* i.e., they are permutation-invariant. Hence, by some simple reasonings it follows that

$$(3.6) \quad E_{\mathcal{P}_v} \{Z_{ij}\} = N_{vj}/N_v \text{ for all } i=1, \dots, N_v, j=0, \dots, \infty;$$

$$(3.7) \quad E_{\mathcal{P}_v} \{Z_{ij} \cdot Z_{i'j'}\} = 0 \text{ for all } i=1, \dots, N_v, j \neq j'=0, \dots, \infty;$$

$$(3.8) \quad E_{\mathcal{P}_v} \{Z_{ij} \cdot Z_{i'j'}\} = N_{vj} (N_{vj'} - \delta_{jj'}) / N_v (N_v - 1)$$

for all $i \neq i'=1, \dots, N_v$ and $j, j'=0, \dots, \infty$; where $\delta_{jj'}$ is the usual Kronecker delta.

By analogy with (2.3), we define

$$(3.9) \quad A^2(F_{N_v}, \{I_j\}) = \sum_{j=0}^{\infty} \hat{\Delta}_{vj}^2 N_{vj} / N_v,$$

and it is easy to see that (3.9) also satisfies (2.5), uniformly in $\{I_j\}$ and for all $\{F_{N_v}\}$. From (3.5) through (3.9), we have

$$(3.10) \quad E_{\mathcal{P}_v} \{S_v\} = 0 \quad \text{and} \quad E_{\mathcal{P}_v} \{S_v^2\} = [N_v / (N_v - 1)] C_v^2 A^2(F_{N_v}, \{I_j\}),$$

where C_v^2 is defined by (1.2). Now, under \mathcal{P}_v , S_v can only assume $N_v!$ possible

values (actually, there are $N_{\nu}! / \prod_{j=0}^{\infty} N_{\nu j}!$ distinct equally likely permutations of X_{ν}^*), and hence, for small values of N_{ν} , the upper tail of the permutation distribution of S_{ν} , can be evaluated to formulate the test function:

$$(3.11) \quad \psi_2(X_{\nu}^*) = \begin{cases} 1, & \text{if } S_{\nu} > S_{\nu, \epsilon} \\ \delta_{\epsilon}, & \text{if } S_{\nu} = S_{\nu, \epsilon} \\ 0, & \text{if } S_{\nu} < S_{\nu, \epsilon} \end{cases}$$

where $S_{\nu, \epsilon}$ and δ_{ϵ} are so chosen that $E\{\psi_2(X_{\nu}^*) | \mathcal{P}_{\nu}\} = \epsilon$, the level of significance. This implies that $E\{\psi_2(X_{\nu}^*) | H_0\} = \epsilon$, i.e., $\psi_2(X_{\nu}^*)$ is a similar size ϵ test.

Let us now define

$$(3.12) \quad W_{\nu i} = \sum_{j=0}^{\infty} \hat{\Delta}_{\nu j} Z_{ij} \quad \text{for } i=1, \dots, N_{\nu}.$$

Under \mathcal{P}_{ν} , $\hat{\Delta}_{\nu j}$'s are all invariant while Z_{ij} 's are stochastic. Thus, it follows that $N_{\nu j}$ of $W_{\nu i}$'s are equal to $\hat{\Delta}_{\nu j}$, for $j=0, \dots, \infty$. We now impose the nondegeneracy condition on $F(x)$ as

$$(3.13) \quad \sup_j [F_{j+1} - F_j] < 1 \Rightarrow \sup_j [N_{\nu j} / N_{\nu}] < 1, \text{ with probability one.}$$

Thus, writing S_{ν} equivalently as $\sum_{i=1}^{N_{\nu}} c_{\nu i} W_{\nu i}$, it follows from the well-known permutational central limit theorem by Wald-Wolfowitz-Noether-Hoeffding-Hájek (cf. [4]) that under (1.2), (1.3), (3.9) and (3.13)

$$(3.14) \quad \mathcal{L}_{\mathcal{P}_{\nu}}(S_{\nu} / C_{\nu} A(F_{N_{\nu}}, \{I_j\})) \rightarrow N(0, 1), \text{ in probability.}$$

Consequently, from (3.11) and (3.14), we have

$$(3.15) \quad S_{\nu, \epsilon} \rightarrow \tau_{\epsilon} C_{\nu} A(F_{N_{\nu}}, \{I_j\}), \quad \delta_{\epsilon} \rightarrow 0, \text{ in probability,}$$

where τ_{ϵ} is defined by (2.11). [It may be noted that \mathcal{P}_{ν} being a conditional probability measure, (given $(N_{\nu j}, j=0, \dots, \infty)$), (3.14) and (3.15) hold in probability i.e., for almost all $(N_{\nu j}, j=0, \dots, \infty)$.]

3.2. Asymptotic optimality of $\psi_2(X_{\nu}^*)$. The main contention of this paper is to establish the asymptotic equivalence of $\psi_1(X_{\nu}^*)$ and $\psi_2(X_{\nu}^*)$, in (2.10) and (3.11), respectively. For this, let us first consider the following lemmas.

LEMMA 3.1. Under (1.1) through (1.4) and for β infinitely close to zero (being of the order $N_{\nu}^{-\frac{1}{2}}$), $A^2(F_{N_{\nu}}, \{I_j\})$ converges in probability to $A^2(F, \{I_j\})$, uniformly in $\{I_j\}$.

PROOF. We shall prove the lemma only for $\beta=0$ as the rest of the proof will follow by the contiguity argument of Hájek [5]. Let us select a sequence of real and positive numbers $\{\eta_{\nu}\}$ in such a manner that

$$(3.16) \quad \lim_{\nu \rightarrow \infty} \eta_{\nu} = 0 \text{ but } \lim_{\nu \rightarrow \infty} N_{\nu}^{\frac{1}{4}} \eta_{\nu} = \infty.$$

For any given N_{ν} , the set of class intervals $\{I_j\}$ is divided into two subsets $G_{\nu}^{(1)}$ and $G_{\nu}^{(2)}$, where

$$(3.17) \quad G_{\nu}^{(1)} = \{I_j: P_j \geq \eta_{\nu}\}, \quad G_{\nu}^{(2)} = \{I_j: P_j < \eta_{\nu}\}.$$

Let $F_{N_{\nu}}(x) = N_{\nu}^{-1}$ [Number of $X_{\nu i} \leq x$] be the empirical cdf of X_{ν} . Then by making use of the fact that $\text{Sup}_j |F_{N_{\nu j}} - F_j| \leq \text{Sup}_x |F_{N_{\nu}}(x) - F(x)|$ and the well-known result on Kolmogorov-Smirnoff statistics, we readily get that

$$(3.18) \quad \text{Sup}_j \{N_{\nu}^{-\frac{1}{2}} |F_{N_{\nu}, j} - F_j|\} \text{ is bounded in probability.}$$

Further, it is easily seen that

$$(3.19) \quad \begin{aligned} & \text{Sup}_j |f(F^{-1}(F_{N_{\nu}, j})) - f(F^{-1}(F_j))| = \text{Sup}_j \left| \int_{F_{N_{\nu}, j}}^{F_j} \phi(u) du \right| \\ & \leq \text{Sup}_j \left| \int_{F_{N_{\nu}, j}}^{F_j} \phi^2(u) du \right|^{\frac{1}{2}} |F_{N_{\nu}, j} - F_j|^{\frac{1}{2}} \leq A^2(F) \text{Sup}_j |F_{N_{\nu}, j} - F_j|^{\frac{1}{2}} \end{aligned}$$

Hence, from (2.2), (3.3), (3.18) and (3.19), we have for all $I_j \in G_{\nu}^{(1)}$,

$$(3.20) \quad \hat{\Delta}_{\nu j}^2 N_{\nu j} / N_{\nu} = P_j [\Delta_j + R_{\nu j}^{(1)}]^2 [1 + R_{\nu j}^{(2)}],$$

where

$$(3.21) \quad \text{Sup}_j |R_{\nu j}^{(2)}| = o_p([N_{\nu}^{\frac{1}{2}} \eta_{\nu}]^{-1}), \quad \text{Sup}_j |R_{\nu j}^{(1)}| = o_p(N_{\nu}^{-\frac{1}{4}}).$$

Consequently, from (3.16), (3.20), (3.21) and some simple algebraic manipulations, we have

$$(3.22) \quad \sum_{G_{\nu}^{(1)}} \hat{\Delta}_{\nu j}^2 N_{\nu j} / N_{\nu} = \sum_{G_{\nu}^{(1)}} \Delta_j^2 P_j + o_p(1).$$

Again, using (2.4) and (3.3), we have

$$(3.23) \quad \Delta_j^2 P_j = \int_{F_j}^{F_{j+1}} \phi^2(u) du - \int_{F_j}^{F_{j+1}} [\phi(u) - \Delta_j]^2 du;$$

$$(3.24) \quad \hat{\Delta}_{\nu j}^2 / N_{\nu} = \int_{F_{N_{\nu}, j}}^{F_{N_{\nu}, j+1}} \phi^2(u) du - \int_{F_{N_{\nu}, j}}^{F_{N_{\nu}, j+1}} [\phi(u) - \hat{\Delta}_{\nu j}]^2 du,$$

where Δ_j and $\hat{\Delta}_{\nu j}$ are also the conditional mean of $\phi(u)$ on $[F_j, F_{j+1}]$ and $[F_{N_{\nu}, j}, F_{N_{\nu}, j+1}]$, respectively. Now, using (3.18) and some routine analysis, it is easily seen that

$$(3.25) \quad \Sigma_{G_{\nu}^{(2)}} \int_{F_{N_{\nu}, j}}^{F_{N_{\nu}, j+1}} \phi^2(u) du = \Sigma_{G_{\nu}^{(2)}} \int_{F_j}^{F_{j+1}} \phi^2(u) du + o_p(1),$$

and we shall show that the second integral on the right hand side of (3.23) converges to zero (as $\nu \rightarrow \infty$) and the second one on the right hand side of (3.24) converges to zero, in probability. For an unessential simplification of this proof we shall assume (as in lemma 2.2 of [4]) that $\phi(u)$ is \uparrow in u . It follows from the existence of (1.4) that if $P_j < \eta_{\nu}$ for all $I_j \in G_{\nu}^{(2)}$, then

$$(3.26) \quad \lim_{\nu \rightarrow \infty} \Sigma_{G_{\nu}^{(2)}} \phi^2(F_j) P_j = \lim_{\nu \rightarrow \infty} \Sigma_{G_{\nu}^{(2)}} \phi^2(F_{j+1}) P_j = \int_{G_{\nu}^{(2)}} \phi^2(u) du.$$

Now for all $u \in [F_j, F_{j+1}]$

$$(3.27) \quad [\phi(u) - \Delta_j]^2 \leq [\phi(F_{j+1}) - \phi(F_j)]^2 \leq \phi^2(F_{j+1}) - \phi^2(F_j).$$

Therefore

$$(3.28) \quad \Sigma_{G_{\nu}^{(2)}} \int_{F_j}^{F_{j+1}} [\phi(u) - \Delta_j]^2 du \leq \Sigma_{G_{\nu}^{(2)}} [\phi^2(F_{j+1}) P_j - \phi^2(F_j) P_j] \rightarrow 0$$

as $\nu \rightarrow \infty$ (by (3.26)). Similarly, from (3.16), (3.18) we have for all $I_j \in G_{\nu}^{(2)}$, $F_{N_{\nu}, j+1} - F_{N_{\nu}, j} < \eta_{\nu} + o_p(N_{\nu}^{-\frac{1}{2}}) < 2 \eta_{\nu}$ for adequately large N_{ν} . Consequently, as in (3.26) and (3.27), we get that $\sum_{F_{N_{\nu}, j}}^{F_{N_{\nu}, j+1}} [\phi(u) - \hat{\Delta}_{\nu j}]^2 du \xrightarrow{P} 0$.

Hence,

$$(3.29) \quad \sum_{G_{\nu}^{(2)}} \hat{\Delta}_{\nu j}^2 \frac{N_{\nu j}}{N_{\nu}} = \sum_{G_{\nu}^{(1)}} \Delta_j^2 P_j + o_p(1).$$

Hence, the lemma follows from (3.22) and (3.29).

LEMMA 3.2. Under H_0 in (1.8), $E\{S_{\nu} - T_{\nu}\}^2 \rightarrow 0$ as $\nu \rightarrow \infty$, where S_{ν} and T_{ν} are defined in (3.7) and (2.9), respectively.

PROOF.
$$E\{S_{\nu} - T_{\nu}\}^2 = E_{\nu}\{E_{\nu} [S_{\nu} - T_{\nu}]^2\}$$

$$= E_{\nu} \{E_{P_{\nu}} [\sum_{j=0}^{\infty} \sum_{i=1}^{N_{\nu}} c_{\nu i} z_{ij} (\hat{\Delta}_{\nu j} - \Delta_j)]^2\}$$

(3.30)

$$= \frac{N_{\nu}}{N_{\nu}-1} \sum_{i=1}^{N_{\nu}} c_{\nu i}^2 E_{\nu} [\sum_{j=0}^{\infty} (\hat{\Delta}_{\nu j} - \Delta_j)^2 \frac{N_{\nu j}}{N_{\nu}} - \{ \sum_{j=0}^{\infty} (\hat{\Delta}_{\nu j} - \Delta_j) \frac{N_{\nu j}}{N_{\nu}} \}^2],$$

where $E_{P_{\nu}}$ and E_{ν} stand for the expectation over the permutation distribution and the distribution of the order statistic associated with X_{ν}^* , respectively.

Thus, it follows from (3.30), that we are only to show that

$$(3.31) \quad E_{\nu} \{ \sum_{j=0}^{\infty} (\hat{\Delta}_{\nu j} - \Delta_j)^2 \frac{N_{\nu j}}{N_{\nu}} \} \rightarrow 0 \text{ as } \nu \rightarrow \infty.$$

We rewrite the expression in (3.31) as

$$(3.32) \quad E_{\nu} \{ \sum_{j=0}^{\infty} \hat{\Delta}_{\nu j}^2 \frac{N_{\nu j}}{N_{\nu}} \} - 2 \sum_{j=0}^{\infty} \Delta_j E_{\nu} \{ \hat{\Delta}_{\nu j} \frac{N_{\nu j}}{N_{\nu}} \} + \sum_{j=0}^{\infty} \Delta_j^2 P_j.$$

Now, $\sum_{j=0}^{\infty} \hat{\Delta}_{\nu j}^2 N_{\nu j} / N_{\nu}$ is nothing but $A^2(F_{N_{\nu}}, \{I_j\})$, defined by (3.9), and hence, it is bounded by $A^2(F)$, defined by (1.4), uniformly in all $(N_{\nu j}, j=0, 1, \dots, \infty)$. Further, by lemma 3.1, it converges to $\sum_{j=0}^{\infty} \Delta_j^2 P_j$, in probability. Since, for a bounded valued random variable convergence, in probability, to a constant implies the convergence of the expectation to the same constant, we readily get

$$(3.33) \quad E_{\nu} \left\{ \sum_{j=0}^{\infty} \hat{\Delta}_{\nu j}^2 N_{\nu j} / N_{\nu} \right\} \rightarrow \sum_{j=0}^{\infty} \Delta_j^2 P_j \text{ as } \nu \rightarrow \infty.$$

Precisely, on the same line, it follows that

$$(3.34) \quad E_{\nu} \left\{ \sum_{j=0}^{\infty} \hat{\Delta}_{\nu j} \Delta_j N_{\nu j} / N_{\nu} \right\} \rightarrow \sum_{j=0}^{\infty} \Delta_j^2 P_j \text{ as } \nu \rightarrow \infty.$$

(3.32), (3.33) and (3.34) imply (3.31), which in turn asserts the truth of the lemma by (3.30).

Hence, the lemma.

LEMMA 3.3. Under H_0 in (1.8), (T_{ν}, S_{ν}) converges in law to a bivariate normal distribution which degenerates on the line $T_{\nu} = S_{\nu}$.

PROOF. By virtue of lemma 3.2, any linear function $aS_{\nu} + bT_{\nu}$ converges in mean square to $(a+b) T_{\nu}$, and hence, from theorem 2.1, has asymptotically a normal distribution with zero mean (under H_0) and variance $(a+b)^2 C_{\nu}^2 A^2(F, \{I_j\})$. The rest of the proof follows from theorem 2.1, lemma 3.1 and lemma 3.2, and hence is omitted.

THEOREM 3.4. Under the sequence of alternatives in (1.1) (i.e., with β tending to zero at the rate $N_{\nu}^{-\frac{1}{2}}$),

$$\Delta_{\nu} \left([S_{\nu} - (\beta/\sigma) C_{\nu}^2 A^2(F, \{I_j\})] / C_{\nu} A(F, \{I_j\}) \right) \rightarrow N(0, 1).$$

The proof is an immediate consequence of lemma 4.2 of Hájek [5] and our lemma 3.3, and hence is not reproduced again.

Consequent of theorem 3.4, the test $\psi_2(X_{\nu}^*)$ in (3.11) has asymptotically the power function

$$(3.35) \quad 1 - \Phi(\tau_{\epsilon} - (\beta/\sigma)C_{\nu}A(F, \{I_j\})),$$

which agrees with (2.11). Thus, S_{ν} provides the AMPROT for $H_0: \beta=0$ against $\beta>0$.

This result also applies in particular to the two sample problem, where

$$(3.36) \quad C_{\nu i} = \begin{cases} \delta_{\nu}/m_{\nu} & \text{for } i=1, \dots, m_{\nu} \\ -\delta_{\nu}/n_{\nu} & \text{for } i=m_{\nu}+1, \dots, N_{\nu}, \quad m_{\nu}+n_{\nu} = N_{\nu}, \end{cases}$$

where δ_{ν} is real. Further, the results derived here can also be extended to the problems of symmetry and scalar alternatives. For that one will have to work with Capon's [1] technique and use his $\phi(u)$ (defined by (iv) on p. 89 of [1]) instead of the $\phi(u)$ in (1.4). The rest of the procedure will be very similar to the one considered here, and hence is omitted. Finally, the impact of these findings on AMPROT for truncated/censored two sample problem will be considered in the next section.

4. Asymptotic efficiency. Suppose now for the AMPROT we work with the assumed density function $f(x)$ instead of the true density function $g(x)$ ($=G'(x)$), where

$$(4.1) \quad A^2(G) = \int_{-\infty}^{\infty} [g'(x)/g(x)]^2 dG(x) < \infty.$$

We define G_j and P_j^* as in (2.1) with F replaced by $G(x)$, and $F_{N_{\nu}, j}$ as in (3.2).

Also, we let

$$(4.2) \quad \phi(u) = f'(F^{-1}(u))/f(F^{-1}(u)) \text{ and}$$

$$\phi^*(u) = g'(G^{-1}(u))/g(G^{-1}(u)), \quad 0 < u < 1,$$

$$(4.3) \quad \Delta_j^* = \int_{G_j}^{G_{j+1}} \phi^*(u) du / P_j^*, \quad \Delta_j^{**} = \int_{G_j}^{G_{j+1}} \phi(u) du / P_j^*,$$

$$(4.4) \quad A^2(G, \{I_j\}) = \sum_{j=0}^{\infty} \Delta_j^{*2} P_j^*, \quad B^2(F, \{I_j\}) = \sum_{j=0}^{\infty} (\Delta_j^{**})^2 P_j^*,$$

$$(4.5) \quad C(F, G, \{I_j\}) = \sum_{j=0}^{\infty} \Delta_j^* \Delta_j^{**} P_j^*;$$

$$(4.6) \quad \rho(\{I_j\}) = C(F, G, \{I_j\}) / [A(G, \{I_j\}) \cdot B(F, \{I_j\})].$$

THEOREM 4.1. Under (1.1) - (1.4) and (4.1), the asymptotic power of the test (3.11) is equal to

$$(4.7) \quad 1 - \Phi(\tau_{\epsilon} - \rho(\{I_j\})(\beta/\sigma)C_{\nu} A(G, \{I_j\})).$$

PROOF. Defining

$$(4.8) \quad T_{\nu}^* = \sum_{i=1}^N c_{\nu i} \sum_{j=0}^{\infty} \Delta_j^* Z_{ij}, \quad T_{\nu}^{**} = \sum_{i=1}^N c_{\nu i} \sum_{j=0}^{\infty} \Delta_j^{**} Z_{ij},$$

and following the same approach as in sections 2 and 3, we get that

$$(4.9) \quad (i) \quad E\left[\frac{S_{\nu} - T_{\nu}^{**}}{\nu} \middle| H_0\right] \rightarrow 0 \text{ as } \nu \rightarrow \infty,$$

$$(4.10) \quad (ii) \quad \Delta\left(\frac{[T_{\nu}^* - (\beta/\sigma)C_{\nu}^2 A^2(G, \{I_j\})]}{C_{\nu} A(G, \{I_j\})}\right) \rightarrow N(0, 1),$$

$$(4.11) \quad (iii) \quad \Delta\left(\frac{T_{\nu}^{**}}{C_{\nu} B(F, \{I_j\})} \middle| H_0\right) \rightarrow N(0, 1),$$

$$(4.12) \quad (iv) \quad \Delta(T_{\nu}^*, T_{\nu}^{**} | H_0) \text{ tends to a bivariate normal distribution with a correlation coefficient } \rho(\{I_j\}), \text{ given by (4.6).}$$

The rest of the proof follows directly from lemma 4.2 of Hájek [5] and (4.9). Hence, the theorem.

As in Hájek [5], we can interpret $[\rho(\{I_j\})]^2$ as the efficiency factor of (3.11) with respect to the asymptotically most powerful parametric test; the interpretation for the two sample problem again being the ratio of the sample sizes needed to attain the same power.

REMARK 1. The loss in efficiency due to grouping of data, as obtained from our theorem 3.4 and theorem 1.1 of Hájek [5], is equal to

$$(4.13) \quad 1 - A^2(F, \{I_j\})/A^2(F) \\ = \sum_{j=0}^{\infty} \int_{F_j}^{F_{j+1}} [\phi(u) - \Delta_j]^2 du / \int_0^1 \phi^2(u) du,$$

where Δ_j and $\phi(u)$ are defined by (2.2) and (2.4), respectively. (4.13) can be made arbitrary small, provided the Lebesgue measures of all the class intervals $\{I_j\}$ are also arbitrarily small. Again, from our theorem 4.1 and theorem 6.1 of Hájek [5], it follows that the loss in efficiency in the case where the assumed density differs from the true density, is given by

$$(4.14) \quad 1 - [\rho(\{I_j\})/\rho] \{1 - \sum_{j=0}^{\infty} \int_{F_j}^{F_{j+1}} [\phi(u) - \Delta_j]^2 du / \int_0^1 \phi^2(u) du\},$$

where $\rho(\{I_j\})$ is defined by (4.6) and ρ by (6.3) of [5]. (4.14) may be greater than, equal to or less than (4.13), depending on ρ and $\rho(\{I_j\})$.

REMARK 2. If we take $I_0: x < x_0$, while I_1, \dots, I_{∞} to be all of sufficiently small Lebesgue measures, the results will relate to the AMPROT for truncated

case ($x \leq x_0$ being truncated). More than one truncation can be dealt in a similar way. Again, in the two sample problem, Gastwirth [3] has considered the censored case where only $N_v^* (< N_v)$ of the ordered variables of the combined sample are observable, and N_v^*/N_v approaches $p (0 < p < 1)$ as $v \rightarrow \infty$. In his case, N_v^* is given but the corresponding truncation point is random, while in our case, the truncation points are given and $N_{vj} (j=0, \dots, \infty)$ are random. In spite of this basic difference, the power properties can be studied by the same formulae (viz., theorems 3.1 and 3.2 of Gastwirth [3], and our theorems in section 3 and 4). However, he has only considered the optimality of his censored test and the power can be traced with the aid of our theorems 3.4 and 4.1.

REMARK 3. In the particular case of $f(x)$ being assumed to be normal, the corresponding S_v will be termed the grouped normal score statistics. The corresponding test in (3.11) will be AMPROT for normal alternatives. On the other hand, the grouped Wilcoxon's test also belongs to the class of tests considered here (namely when we work with logistic distribution). This may be written as

$$(4.15) \quad W_v = \sum_{i=1}^{N_v} c_{vi} \sum_{j=0}^{\infty} \hat{\gamma}_{vj} Z_{ij},$$

where Z_{ij} 's are defined by (1.7) and

$$(4.16) \quad \hat{\gamma}_{vj} = F_{N_v, j} + \frac{1}{2} \frac{N_{vj}}{N_v} = \frac{1}{2} (F_{N_v, j} + F_{N_v, j+1}), \quad j=0, \dots, \infty.$$

The asymptotic normality (permutationally as well as unconditionally) can be established in the same manner. The test [similar to (3.11)] based on W_v can be shown to have the asymptotic power equal to

$$(4.17) \quad 1 - \Phi(\tau_\epsilon - (\beta/\sigma)C_v \omega),$$

where

$$(4.18) \quad \omega = \sum_{j=0}^{\infty} \left\{ \frac{1}{2} (F_j + F_{j+1}) \int_{F_j}^{F_{j+1}} \phi(u) du \right\} / \left[\sum_{j=0}^{\infty} \left\{ \frac{1}{2} (F_j + F_{j+1}) \right\}^2 P_j - \frac{1}{4} \right]^{\frac{1}{2}},$$

$\phi(u)$ and P_j being defined by (2.4) and (2.1), respectively. Consequently, from theorem 3.4 and (4.17), the efficiency factor of the grouped Wilcoxon's test with respect to the grouped normal score test is equal to

$$(4.19) \quad \omega^2 / A^2(F, \{I_j\}).$$

For normal cdf, (4.19) is easily seen to be less than one, uniformly in $\{I_j\}$. But there are certain advantages of the test based on W_{ν} . First, this test is valid for all continuous cdf's, while the asymptotic optimality of the normal score on other tests, deduced in this paper, is implicitly based on the assumptions that (i) $A^2(F)$ defined by (1.4) is non-zero finite (which localizes the scope only to cdf's having continuous first and second derivatives, satisfying (1.4)), and (ii) Δ_j , defined by (2.2) is not a constant (otherwise, $T_{\nu} = 0$). For many distributions, for which the range is not extended to infinity on both sides, either of these two conditions may not hold. For example (i) for simple exponential distribution, $f'(x)/f(x) = 1$, and hence $\Delta_j = 0$ for all j , or (ii) for uniform distribution $f'(x) = 0$ a.e., and hence, $A^2(F) = 0$. In such a case, S_{ν} in (3.5) becomes equal to zero and hence, the tests are not constructable. But the so called grouped Wilcoxon's test can be used. This is AMPROT for logistic densities.

REFERENCES

- [1] CAPON, J. (1961), Asymptotic efficiency of certain locally most powerful rank tests, Ann. Math. Statist., 32, 88-100.
- [2] CHERNOFF, H. and SAVAGE, I. R. (1958), Asymptotic normality and efficiency of certain nonparametric test statistics, Ann. Math. Statist., 29, 972-994.
- [3] GASTWIRTH, J. (1965), Asymptotically most powerful rank order tests for the two-sample problem with censored data, Ann. Math. Statist., 36, 1243-1247.
- [4] HÁJEK, J. (1961), Some extensions of the Wald-Wolfowitz-Noether theorem, Ann. Math. Statist., 32, 506-523.
- [5] HÁJEK, J. (1962), Asymptotically most powerful rank order tests, Ann. Math. Statist., 33, 1124-1147.
- [6] LEHMANN, E. L. (1959), Testing Statistical Hypotheses, Wiley, New York.