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ON THE WEAK LAW OF LARGE NUMBERS AND THE
GENERALIZED ELEMENTARY RENEWAL THEOREM

by

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Institute of Statistics Mimeo Series No. 490

September 1966

This research was supported by the Office of
Naval Research Contract No. NONR 855 (09).

DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA

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SUMMARY

$\{X_n\}$ is a sequence of independent, non-negative, random variables and $G_n(x) = P\{X_1 + \dots + X_n \leq x\}$. $\{a_n\}$ is a sequence of non-negative constants such that, for some $\alpha > 0$, $\gamma > 0$, and function of slow growth $L(x)$,

$$\sum_{r=1}^n a_r \sim \frac{\alpha N^\gamma L(N)}{\Gamma(1+\gamma)}, \text{ as } N \rightarrow \infty.$$

A Generalized Elementary Renewal Theorem (GERT) gives conditions such that, for some $\mu > 0$,

$$(*) \quad \Psi(x) \equiv \sum_{r=1}^n a_r G_r(x) \sim \frac{\alpha L(x)}{\Gamma(1+\gamma)} \left(\frac{x}{\mu}\right)^\gamma, \text{ as } x \rightarrow \infty.$$

The Weak Law of Large Numbers (WLLN) states that $(X_1 + \dots + X_n)/n \rightarrow \mu$, as $n \rightarrow \infty$, in probability. Theorem 1 proves that WLLN implies (*). Theorem 3 proves that (*) implies WLLN, if additionally, it is given that $\sum_{j=1}^n P\{X_j > n\epsilon\} \rightarrow 0$ as $n \rightarrow \infty$, for every small $\epsilon > 0$. Theorem 2 supposes the $\{X_n\}$ to have finite expectations and proves (*) implies WLLN if, additionally, it is given that

$$\limsup_{n \rightarrow \infty} \frac{\epsilon X_1 + \epsilon X_2 + \dots + \epsilon X_n}{n} \leq \mu,$$

in which case $(\epsilon X_1 + \dots + \epsilon X_n)/n$ necessarily tends to μ as $n \rightarrow \infty$.

Finally, an example shows that (*) can hold while the WLLN fails to hold.

Much use is made of the fact that a necessary and sufficient condition for the WLLN is that, for all small $\epsilon > 0$,

$$\frac{1}{n} \int_0^{n\epsilon} \sum_{j=1}^n P\{X_j > x\} dx \rightarrow \mu, \text{ as } n \rightarrow \infty.$$

1. Introduction. Let $\{X_n\}$, $n = 1, 2, \dots$, be a sequence of independent, non-negative, random variables; write $F_n(x) \equiv P\{X_n \leq x\}$; $S_n = X_1 + X_2 + \dots + X_n$; $G_n(x) = P\{S_n \leq x\}$; when the first moments exist, write $\mu_n = EX_n$. Let $\{a_n\}$ be a sequence of non-negative constants such that, for some constants $\alpha > 0$, $\gamma > 0$, and some function of slow growth $L(x)$,

$$(1.1) \quad \sum_{n=1}^N a_n \sim \frac{\alpha N^\gamma L(N)}{\Gamma(1 + \gamma)}, \text{ as } N \rightarrow \infty. \dagger$$

By a Generalized Elementary Renewal Theorem (GERT) we shall mean a theorem that establishes conditions such that

$$(1.2) \quad \sum_{n=1}^{\infty} a_n G_n(x) \sim \frac{\alpha L(x)}{\Gamma(1 + \gamma)} \left(\frac{x}{\mu}\right)^\gamma, \text{ as } x \rightarrow \infty.$$

for some constant $\mu > 0$. The special case of (1.2) when $a_n = 1$ for all n is the Elementary Renewal Theorem (ERT).

By a Weak Law of Large Numbers (WLLN) we shall mean a theorem that establishes conditions such that

$$(1.3) \quad \frac{S_n}{n} \rightarrow \mu, \text{ as } n \rightarrow \infty,$$

in probability.

This paper extends and leans heavily upon an earlier one (Smith, 1963) which we shall henceforth refer to simply as S. It will be concerned with weakening conditions of S for a GERT to hold for non-negative random variables (specifically, we drop the assumption of finite means) and with investigating

† We carry the factor $\Gamma(1 + \gamma)$ to simplify comparisons with S.

to what extent a GERT implies the WLLN and vice versa.

Two conditions play an important role in our work. They are

(A) For every small $\epsilon > 0$,

$$\sum_{j=1}^n \{1 - F_j(n\epsilon)\} \rightarrow 0, \text{ as } n \rightarrow \infty ;$$

(B) For every small $\epsilon > 0$,

$$\frac{1}{n} \int_0^{n\epsilon} \sum_{j=1}^n \{1 - F_j(x)\} dx \rightarrow \mu, \text{ as } n \rightarrow \infty .$$

It is an easy exercise to show that (B) implies (A) ; all we can infer from (A), concerning (B) , is that the upper and lower limits, as $n \rightarrow \infty$, of

$$\frac{1}{n} \int_0^{n\epsilon} \sum_{j=1}^n \{1 - F_j(x)\} dx$$

are independent of the small $\epsilon > 0$.

It is known from the work of Bobrov (1939), described by Gnedenko and Kolmogorov (1954; see especially page 141), that condition (B) is necessary and sufficient for the WLLN (1.3) to hold. Thus the WLLN implies (A) and (B). It is interesting, therefore, that we are able to prove (in section 2) : -

Theorem 1 A sufficient condition for the Generalized Elementary Renewal Theorem (1.2) is that (B) shall hold. Thus the WLLN implies the GERT.†

† Consequently it is impossible to have (1.2) and (1.3) holding simultaneously, but for different values of μ .

The question naturally arises, does the holding of a GERT (1.2) imply the truth of the WLLN? After some efforts we have discovered that the answer to this question is in the negative. In section 5 we shall present the necessary counter-example for which (1.2) is true and (1.3) false. However, it transpires that the validity of a GERT does imply the WLLN if a weak supplemental condition is given to be satisfied. The situation is perhaps comparable to that arising with Abelian and Tauberian theorems in analysis; the Abelian theorem usually holds "in general," the converse Tauberian theorem usually requires the satisfaction of an extra "Tauberian Condition." We shall prove, in sections 3 and 4 the following two theorems relevant to this paragraph.

Theorem 2 If the GERT (1.2) holds, if all the means $\mu_n = EX_n$ are finite, and if

$$\limsup_{n \rightarrow \infty} \frac{\mu_1 + \mu_2 + \dots + \mu_n}{n} \leq \mu,$$

then the WLLN (1.3) holds and

$$\frac{\mu_1 + \mu_2 + \dots + \mu_n}{n} \longrightarrow \mu \text{ as } n \longrightarrow \infty.$$

Theorem 3 If the GERT (1.2) holds together with condition (A), then the WLLN (1.3) holds.

As we have said, this paper leans heavily on S; one result buried in S turns out to be especially important. The argument of section 5 of S (pp. 689 - 698) essentially (if not ostensibly) proves the following:

Fundamental Lemma Suppose there exists a $\delta > 0$, such that, for every $\epsilon > 0$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \int_0^{\epsilon n} \sum_{j=1}^n \{1 - F_j(x)\} dx \geq \delta.$$

Then for any small $\eta > 0$ we can find a large $C(\eta)$ such that

$$\sum_{n \geq Cx}^{\infty} a_n G_n(x) < \eta x^\gamma L(x)$$

for all large x .

One further comment is called for. If $\gamma = 0$ in (1.2) the constant μ disappears from the right-hand side and we have a simpler relation

$$(1.4) \quad \sum_{n=1}^{\infty} a_n G_n(x) \sim \alpha L(x), \text{ as } x \rightarrow \infty.$$

It seems that this special case needs special treatment, and that (1.4) will hold under considerably more general conditions than Theorem 1 suggests; we hope to study (1.4) elsewhere, and throughout this present paper take $\gamma > 0$.

2. Proof of Theorem 1

To begin with we shall establish (leaning heavily on arguments in S) the following.

Lemma 2.1 Under conditions (A) and (B) there is an unbounded non-decreasing function $\ell(n)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \int_0^{r/\ell(r)} \{1 - F_r(x)\} dx = \mu.$$

Proof From (A) and (B) we can evidently find an unbounded non-decreasing $\lambda(n)$ such that, as $n \rightarrow \infty$,

$$(2.1) \quad \sum_{j=1}^n \left\{ 1 - F_j \left(\frac{n}{\lambda(n)} \right) \right\} \rightarrow 0,$$

$$(2.2) \quad \frac{1}{n} \int_0^{n/\lambda(n)} \sum_{j=1}^n \{ 1 - F_j(x) \} dx \rightarrow \mu.$$

Lemma 9 of S then shows that we can find another unbounded non-decreasing $w(n) \leq \lambda(n)$ and such that $w(n)/n$ decreases to zero as $n \rightarrow \infty$. But

$$\begin{aligned} \frac{1}{n} \int_{n/\lambda(n)}^{n/w(n)} \sum_{j=1}^n \{ 1 - F_j(x) \} dx \\ \leq \frac{1}{w(n)} \sum_{j=1}^n \left\{ 1 - F_j \left(\frac{n}{\lambda(n)} \right) \right\}. \end{aligned}$$

Since the right member tends to zero as $n \rightarrow \infty$, we can infer from (2.1) and (2.2) that, as $n \rightarrow \infty$.

$$(2.3) \quad \sum_{j=1}^n \left\{ 1 - F_j \left(\frac{n}{w(n)} \right) \right\} \rightarrow 0,$$

$$(2.4) \quad \frac{1}{n} \int_0^{n/w(n)} \sum_1^n \left\{ 1 - F_j(x) \right\} dx \longrightarrow \mu .$$

Let $\ell(n)$ increase much more slowly than $w(n)$; we make the function $\ell(n)$ more precise later. Suppose $t(n)$ is the least integer such that $r/\ell(r) \geq n/w(n)$ for all $r \geq t(n)$. Then, for any $\epsilon > 0$,

$$(2.5) \quad \frac{n}{w(n) t(n)} \leq \frac{1}{\ell(t(n))} \leq \epsilon$$

for all large n .

Since $w(n) \uparrow$ we have, for large n ,

$$w(n) \geq w\left(\frac{n \log w(n)}{w(n)}\right).$$

Thus, if $s(n) = \frac{n \log w(n)}{w(n)}$, we have

$$\frac{s(n)}{\log w(s(n))} \geq \frac{n}{w(n)} .$$

But we may assume $n/w(n) \uparrow$ with n , and hence $n/(\log w(n)) \uparrow$ also.

Thus, if $\ell(n) = \log w(n)$ we have $r/\ell(r) \geq n/w(n)$ for all $r \geq s(n)$. Hence $t(n) \leq s(n)$. But $s(n)/n \rightarrow 0$; thus $t(n)/n \rightarrow 0$.

If we set

$$T_1(n) = \frac{1}{n} \sum_1^{t(n)} \int_0^{n/w(n)} \left\{ 1 - F_j(x) \right\} dx$$

then, by (2.5),

$$T_1(n) \leq \left(\frac{t(n)}{n}\right) \left\{ \frac{1}{t(n)} \sum_1^{t(n)} \int_0^{et(n)} \left\{ 1 - F_j(x) \right\} dx \right\}.$$

Thus $T_1(n) \rightarrow 0$ as $n \rightarrow \infty$.

If we set

$$T_2(n) = \frac{1}{n} \sum_{j=t(n)+1}^n \int_0^{j/l(j)} (1 - F_j(x)) dx$$

then it is clear that $(\mu - \epsilon) \leq T_1(n) + T_2(n)$ for large n , and hence that $T_2(n) > \mu - 2\epsilon$ for large n . Therefore

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \int_0^{j/l(j)} (1 - F_j(x)) dx \geq \mu.$$

But

$$\frac{1}{n} \sum_{j=1}^n \int_0^{j/l(j)} (1 - F_j(x)) dx \leq \frac{1}{n} \sum_{j=1}^n \int_0^{n/l(n)} (1 - F_j(x)) dx,$$

since $j/l(j) \uparrow$, and this, in turn, is smaller than

$$\frac{1}{n} \sum_{j=1}^n \int_0^{\epsilon n} (1 - F_j(x)) dx,$$

for large n , since $l(n) \rightarrow \infty$. Thus

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \int_0^{j/l(j)} (1 - F_j(x)) dx \leq \mu$$

and the lemma is proved.

To prove the theorem we begin by setting

$$\begin{aligned} Y_r &= X_r && \text{if } X_r \leq r/l(r) \\ &= r/l(r) && \text{otherwise.} \end{aligned}$$

For $\epsilon > 0$ we write

$$L_n(\epsilon) = \frac{1}{n} \int_0^{\sum_{j=1}^n Y_j} P\{Y_j > x\} dx.$$

But if $r \leq n$, for all large n , $r/\ell(r) \leq n/\ell(n) \leq \epsilon n$, so that

$$L_n(\epsilon) = \frac{1}{n} \sum_{r=1}^n \int_0^{r/\ell(r)} (1 - F_r(x)) dx$$

and hence, by Lemma 2.1, $L_n(\epsilon) \rightarrow \mu$ as $n \rightarrow \infty$.

The sequence $\{Y_n\}$ satisfies the conditions of Theorem 1 of S. Thus

$$(2.6) \quad \sum_{r=1}^{\infty} a_r P\{Y_r \leq x\} \sim \frac{\alpha x^\gamma L(x)}{\Gamma(1+\gamma) \mu^\gamma},$$

as $x \rightarrow \infty$.

Furthermore both $\{Y_n\}$ and $\{X_n\}$ satisfy conditions (A) and (B).

Thus, by the result of Bobrov already quoted, as $n \rightarrow \infty$,

$$\frac{Y_1 + \dots + Y_n}{n} \rightarrow \mu, \text{ in probability,}$$

$$\frac{X_1 + \dots + X_n}{n} \rightarrow \mu, \text{ in probability.}$$

Hence, if $Z_n = X_n - Y_n$,

$$\frac{Z_1 + \dots + Z_n}{n} \rightarrow 0, \text{ in probability.}$$

Plainly $G_n(x) \equiv P\{X_1 + \dots + X_n \leq x\} \leq P\{Y_1 + \dots + Y_n \leq x\}$.

In all that follows let us write

$$\Psi(x) = \sum_{n=1}^{\infty} a_n G_n(x)$$

Thus to prove our theorem we need only show

$$(2.7) \quad \liminf_{x \rightarrow \infty} \frac{\mathbb{V}(x)}{x^\gamma L(x)} \geq \frac{\alpha}{\mu^\gamma \Gamma(1+\gamma)}$$

It is not hard to show that $\{Y_n + \epsilon\}$ also satisfies Theorem 1 of S, with μ increased to $\mu + \epsilon$. Thus

$$\frac{1}{x^\gamma L(x)} \sum_{n=1}^{\infty} a_n P \left\{ Y_1 + Y_2 + \dots + Y_n + n\epsilon \leq x \right\} \sim \frac{\alpha}{(\mu+\epsilon)^\gamma \Gamma(1+\gamma)} .$$

Also, by the "fundamental lemma" of S1, given any $\eta > 0$ we can find $C \equiv C(\eta)$ such that

$$\frac{1}{x^\gamma L(x)} \sum_{n=Cx}^{\infty} a_n P \left\{ Y_1 + \dots + Y_n + n\epsilon \leq x \right\} < \eta$$

for all large x . Therefore

$$(2.8) \quad \liminf_{x \rightarrow \infty} \frac{1}{x^\gamma L(x)} \sum_1^{Cx} a_n P \left\{ Y_1 + \dots + Y_n + n\epsilon \leq x \right\} \geq \frac{\alpha}{(\mu+\epsilon)^\gamma \Gamma(1+\gamma)} - \eta .$$

Now

$$\begin{aligned} & P \left\{ X_1 + \dots + X_n \leq x \right\} \\ & \geq P \left\{ Y_1 + \dots + Y_n + n\epsilon \leq x \text{ \& } Z_1 + \dots + Z_n \leq n\epsilon \right\} \end{aligned}$$

$$\geq P \left\{ Y_1 + \dots + Y_n + n\epsilon \leq x \right\} \\ - P \left\{ Z_1 + \dots + Z_n > n\epsilon \right\},$$

and so

$$(2.9) \sum_{n=1}^{Cx} a_n P \left\{ X_1 + \dots + X_n \leq x \right\} \\ \geq \sum_{n=1}^{Cx} a_n P \left\{ Y_1 + \dots + Y_n + n\epsilon \leq x \right\} \\ - \sum_{n=1}^{Cx} a_n P \left\{ Z_1 + \dots + Z_n > n\epsilon \right\}.$$

But $P \left\{ Z_1 + \dots + Z_n > n\epsilon \right\} \rightarrow 0$ as $n \rightarrow \infty$, so that one can establish easily that

$$\sum_{n=1}^{Cx} a_n P \left\{ Z_1 + \dots + Z_n > n\epsilon \right\} = o(x^{\gamma} L(x)).$$

Hence, from (2.8) and (2.9) and the arbitrariness of η we finally deduce the desired (2.7).

3. Proof of Theorem 2

For this theorem we shall make considerable use of Laplace-Stieltjes transforms. If $A(x)$, say, is a function of bounded variation in every interval we write

$$A^*(s) \equiv \int_{0-}^{\infty} e^{-sx} dA(x),$$

for this transform, for those values of s which make the integral exist. We shall restrict s to real values.

To begin we need:

Lemma 3.1 As $k \rightarrow \infty$, for fixed $s > 0$, $\rho > 0$,

$$\sum_{r=1}^k a_r e^{-\frac{r\rho s}{k}} \sim \frac{\alpha k^\rho L(k)}{\rho^\gamma \Gamma(\gamma)} \int_0^\rho e^{-sx} x^{\gamma-1} dx$$

Proof (Recally > 0) Let $D_k(x)$ be the distribution function associated with atoms of probability

$$\frac{a_r}{\sum_{r=1}^k a_r}$$

at the points $r\rho/k$, for $r = 1, 2, \dots, k$. Then it is easy to see that

$$\lim_{k \rightarrow \infty} D_r(x) = \lim_{k \rightarrow \infty} \frac{\sum_1^{[xk/\rho]} a_r}{\sum_1^r a_r} = \left(\frac{x}{\rho}\right)^\gamma,$$

from (1.1) and the familiar properties of functions of slow growth. Thus, by the continuity theorem for the Laplace transform, we find that, as $k \rightarrow \infty$,

$$\int_0^{\rho} e^{-sx} dD_k(x) \rightarrow \frac{\gamma}{\rho^{\gamma}} \int_0^{\rho} e^{-sx} x^{\gamma-1} dx.$$

The lemma is an immediate consequence of this limit.

Lemma 3.2 As $k \rightarrow \infty$, for fixed $s > 0$, $\rho > 0$,

$$\sum_{k+1}^{\infty} a_r e^{-\frac{rps}{k}} \sim \frac{\alpha L(k)}{\Gamma(\gamma)} \left(\frac{k}{\rho}\right)^{\gamma} \int_{\rho}^{\infty} e^{-sx} x^{\gamma-1} dx.$$

Proof We have that

$$\sum_{k+1}^{\infty} a_r e^{-\frac{rps}{k}} = \sum_{r=1}^{\infty} a_r e^{-\frac{rps}{k}} - \sum_{r=1}^k a_r e^{-\frac{rps}{k}}$$

Lemma 4 of § (p. 682) shows that

$$\sum_{r=1}^{\infty} a_r e^{-\frac{rps}{k}} \sim \frac{\alpha k^{\gamma}}{\rho^{\gamma} s^{\gamma}} I\left(\frac{k}{s}\right)$$

as $k \rightarrow \infty$. This relation and Lemma 3.1 now establish Lemma 3.2, if we use the relation

$$\int_0^{\infty} e^{-sx} x^{\gamma-1} dx = \frac{\Gamma(\gamma)}{s^{\gamma}}, \quad \gamma > 0.$$

Let us now turn to the proof of the theorem. Note that $G_n(x)$ is the d.f. of $X_1 + X_2 + \dots + X_n$ and thus is the d.f. of a non-negative random variable with mean $\mu_1 + \mu_2 + \dots + \mu_n$. Thus, by Jensen's inequality, for real $s > 0$,

$$G_n^*(s) \geq e^{-(\mu_1 + \dots + \mu_n)s}$$

Further, if we set

$$\lambda = \limsup_{n \rightarrow \infty} \frac{\mu_1 + \dots + \mu_n}{n},$$

it follows that, for an arbitrary $\epsilon > 0$,

$$(3.1) \quad G_n^*(s) \geq e^{-n(\lambda+\epsilon)s}$$

for all $s > 0$ and all sufficiently large n .

Now, from Lemma 3.1 and (3.1) we can conclude that

$$(3.2) \quad \liminf_{k \rightarrow \infty} \frac{1}{k L(k)} \sum_{r=1}^k a_r G_r^* \left(\frac{s}{k} \right) \\ \geq \frac{\alpha}{(\lambda+\epsilon)^\gamma \Gamma(\gamma)} \int_0^{(\lambda+\epsilon)} e^{-sx} x^{\gamma-1} dx.$$

Evidently,

$$G_n^*(s) = F_1^*(s) F_2^*(s) \dots F_n^*(s).$$

If $r > k$, $F_{(k+1)}^*(s) \dots F_r^*(s)$ is the generating function of $X_{(k+1)} + \dots + X_r$. Thus, for all large k , all $s > 0$,

$$e^{-(\mu_1 + \dots + \mu_k)s} F_{(k+1)}^*(s) \dots F_r^*(s) \geq e^{-r(\lambda+\epsilon)s}.$$

Let us set

$$v_k = \frac{\mu_1 + \dots + \mu_k}{k}.$$

Then the last inequality and Lemma 3.2 show that, as $k \rightarrow \infty$,

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{e^{-\nu k s}}{k^\gamma L(k)} \sum_{r=(k+1)}^{\infty} a_r F_{(k+1)}^* \left(\frac{s}{k} \right) \cdots F_r^* \left(\frac{s}{k} \right) \\ \geq \frac{\alpha}{(\lambda+\epsilon)^\gamma \Gamma(\gamma)} \int_{(\lambda+\epsilon)}^{\infty} e^{-sx} x^{\gamma-1} dx . \end{aligned}$$

Let

$$A_k(s) = \frac{1}{k^\gamma L(k)} \sum_{r=1}^k a_r G_r^* \left(\frac{s}{k} \right) ,$$

$$B_k(s) = \frac{e^{-\nu k s}}{k^\gamma L(k)} \sum_{r=(k+1)}^{\infty} a_r F_{(k+1)}^* \left(\frac{s}{k} \right) \cdots F_r^* \left(\frac{s}{k} \right) .$$

Then (3.2) and the last inequality prove, in view of the arbitrariness of ϵ ,

$$(3.3) \quad \liminf_{k \rightarrow \infty} A_k(s) \geq \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_0^\lambda e^{-sx} x^{\gamma-1} dx$$

$$(3.4) \quad \liminf_{k \rightarrow \infty} B_k(s) \geq \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_\lambda^\infty e^{-sx} x^{\gamma-1} dx .$$

But, if we recall that

$$\Psi(x) = \sum_{n=1}^{\infty} a_n G_n(x) ,$$

then

$$\frac{\Psi^* \left(\frac{s}{k} \right)}{k^\gamma L(k)} = A_k(s) + e^{v_k s} G_k^* \left(\frac{s}{k} \right) B_k(s) .$$

However, our assumption that the GERT holds implies (see S):

$$\Psi^*(s) \sim \frac{\alpha L(1/s)}{\mu^\gamma s^\gamma} \quad \text{as } s \downarrow 0,$$

so that

$$(3.5) \quad \frac{\Psi^* \left(\frac{s}{k} \right)}{k^\gamma L(k)} \rightarrow \frac{\alpha}{\mu^\gamma s^\gamma}, \quad \text{as } k \rightarrow \infty .$$

Thus

$$\begin{aligned} & \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_0^\lambda e^{-sx} x^{\gamma-1} dx \\ & + \left\{ \limsup_{k \rightarrow \infty} e^{v_k s} G_k^* \left(\frac{s}{k} \right) \right\} \frac{\alpha}{\lambda^\gamma \Gamma(\gamma)} \int_\lambda^\infty e^{-sx} x^{\gamma-1} dx \\ & \leq \frac{\alpha}{\mu^\gamma s^\gamma}, \end{aligned}$$

which implies

$$\begin{aligned} & \limsup_{k \rightarrow \infty} e^{v_k s} G_k^* \left(\frac{s}{k} \right) \\ & \leq 1 + \frac{\left(\frac{\lambda^\gamma - \mu^\gamma}{\mu^\gamma} \right) \frac{\Gamma(\gamma)}{s^\gamma}}{\int_\lambda^\infty e^{-sx} x^{\gamma-1} dx} \end{aligned}$$

$$= 1 + \frac{\Gamma(\gamma)(\lambda/\mu)^{\gamma-1}}{\int_{s\lambda}^{\infty} e^{-u} u^{\gamma-1} du} .$$

But $e^{v_k s} G_k^*(s/k) \geq 1$. Thus we must have $(\lambda/\mu)^{\gamma} - 1 \geq 0$. Since we are given that $\lambda \leq \mu$ we are forced to conclude that $\lambda = \mu$ and that

$$(3.6) \quad e^{v_k s} G_k^*\left(\frac{s}{k}\right) \rightarrow 1, \text{ as } k \rightarrow \infty .$$

From (3.6) we can infer that

$$(3.7) \quad \frac{S_k}{k} - v_k \rightarrow 0, \text{ as } k \rightarrow \infty ,$$

in probability. Thus, given any small $\epsilon > 0$, we have, for all large n ,

$$P \{S_n \leq n(v_n + \epsilon)\} > 1 - \epsilon .$$

Therefore

$$\begin{aligned} \Psi(n \overline{v_n + \epsilon}) &\geq \sum_{r=1}^n a_r P\{S_r \leq n \overline{v_n + \epsilon}\} \\ &\geq \left\{ \sum_{r=1}^n a_r \right\} P\{S_n \leq n \overline{v_n + \epsilon}\} \\ (3.8) \quad &\geq (1 - \epsilon) \left\{ \sum_{r=1}^n a_r \right\} \end{aligned}$$

If we set $x_n = n(v_n + \epsilon)$ then $x_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\frac{\Psi(x_n)}{x_n^\gamma L(x_n)} \longrightarrow \frac{\alpha}{\mu^\gamma \Gamma(1+\gamma)} .$$

Thus, from (3.8) ,

$$(1 - \epsilon) \limsup_{n \rightarrow \infty} \frac{\sum_{r=1}^n a_r}{x_n^\gamma L(x_n)} \leq \frac{\alpha}{\mu^\gamma \Gamma(1+\gamma)} .$$

From this inequality and well-known properties of functions of slow growth (plus the fact that $(v_n + \epsilon)$ lies in a bounded interval not containing the origin) we infer

$$(1 - \epsilon) \limsup_{n \rightarrow \infty} \frac{1}{(v_n + \epsilon)^\gamma} \leq \frac{1}{\mu^\gamma} .$$

Since ϵ is arbitrary we may deduce that

$$\liminf_{n \rightarrow \infty} v_n \geq \mu$$

and that $v_n \rightarrow \mu$ as $n \rightarrow \infty$ (since we are given $\limsup v_n \leq \mu$). This proves part of the theorem. However, from (3.7) we deduce the W.L.L.N. : —

$$\frac{S_n}{n} \longrightarrow \mu , \text{ as } n \longrightarrow \infty ,$$

in probability. This completes the proof.

4. Proof of Theorem 3

The Laplace transform argument of the last section does not seem to carry over to the case when means are infinite. We are forced to the following quite different approach.

Let us choose $\epsilon > 0$ and set

$$v_n(\epsilon) = \frac{1}{n} \sum_{j=1}^n \int_0^{n\epsilon} \{1 - F_j(x)\} dx.$$

Lemma 4.1 If $v_n(\epsilon)$ is bounded, then for any $\eta > 0$,

$$P\{S_n \leq n[v_n(\epsilon) - \eta]\} \leq \rho(\eta, \epsilon)$$

uniformly in n , where $\rho(\eta, \epsilon)$ can be made arbitrarily small by choosing ϵ small enough.

Proof Suppose $v_n(\epsilon) < A$ for all n (and note, by the way, that this inequality is preserved if ϵ is reduced). By a much used argument of S (see p. 679 of that paper), we get, for every $t > 0$,

$$P\{S_n \leq n[v_n(\epsilon) - \eta]\} \leq e^{Q_n(t)}, \text{ say,}$$

where

$$Q_n(t) = nt v_n(\epsilon) [1 - e^{-nt\epsilon}] - nt\eta.$$

Thus

$$Q_n(1/n\sqrt{\epsilon}) \leq \frac{A}{\sqrt{\epsilon}} [1 - e^{-\sqrt{\epsilon}}] - \frac{\eta}{\sqrt{\epsilon}}$$

and the right-hand side of this last inequality can be made as large and negative as we wish by choosing ϵ small enough. This establishes the lemma.

Lemma 4.2 If a G.E.R.T. holds, together with condition (A), then

$$\liminf_{n \rightarrow \infty} v_n(\epsilon) \geq \mu.$$

Proof. Suppose there is a number $\mu_1 < \mu$ and that $v_n(\epsilon) < \mu_1$ for infinitely many n . These inequalities are not upset if ϵ is reduced.

Define a truncation scheme as follows:

$$\begin{aligned} X_{rn} &= X_r & \text{if } X_r \leq n\epsilon, \\ &= n\epsilon & \text{if } X_r > n\epsilon, \end{aligned}$$

for $r, n = 1, 2, 3, \dots$. Set

$$T_n = X_{1n} + X_{2n} + \dots + X_{nn}.$$

The argument of Lemma 4.1 applies to T_n and shows

$$P\{T_n < n[v_n(\epsilon) - \eta]\} \leq \rho(\eta, \epsilon).$$

But $ET_n = n v_n(\epsilon)$; thus we can employ an argument already used in S, as follows.

Let η_1, η_2 be two small positive numbers.

$$\begin{aligned} v_n(\epsilon) \geq \{v_n(\epsilon) - \eta_1\} & \left[P\{T_n \leq n[v_n(\epsilon) + \eta_2]\} - P\{T_n \leq n[v_n(\epsilon) - \eta_1]\} \right] \\ & + \{v_n(\epsilon) + \eta_2\} \left[1 - P\{T_n \leq n[v_n(\epsilon) + \eta_2]\} \right] \end{aligned}$$

Let us suppose $\eta_2 > \eta_1$. Then

$$\begin{aligned} & [\eta_2 - \eta_1] P\{T_n \leq n[v_n(\epsilon) + \eta_2]\} \\ & \geq \eta_2 - [v_n(\epsilon) - \eta_1] P\{T_n \leq n[v_n(\epsilon) - \eta_1]\}, \end{aligned}$$

and so

$$\begin{aligned} & P\{T_n \leq n[v_n(\epsilon) + \eta_2]\} \\ & \geq \frac{\eta_2}{\eta_2 - \eta_1} - [v_n(\epsilon) - \eta_1] \rho(\eta_1, \epsilon). \end{aligned}$$

Suppose we choose $\eta_1 = \eta_2^2$ and assume n is such that $v_n(\epsilon) \leq \mu_1$. Then we have

$$P\{T_n \leq n[v_n(\epsilon) + \tau_2]\} \geq 1 - \delta(\eta_2, \epsilon),$$

where we can make $\delta(\eta_2, \epsilon)$ as small as we like by first choosing η_2 and then ϵ .

Put

$$\chi_n(\epsilon) = \sum_{j=1}^n P\{X_j > n\epsilon\}.$$

By condition (A) we know $\chi_n(\epsilon) \rightarrow 0$ as $n \rightarrow \infty$.

Put

$$x_n = n[v_n(\epsilon) + \tau_2].$$

Then it is not hard to see that

$$P\{S_n \leq x_n\} \geq P\{T_n \leq x_n\} - \chi_n(\epsilon).$$

Now we have, as in the previous section,

$$\begin{aligned} \Psi(x_n) &\geq P\{S_n \leq x_n\} \sum_{j=1}^n a_j \\ &\geq [1 - \delta(\eta_2, \epsilon) - \chi_n(\epsilon)] \sum_{j=1}^n a_j, \end{aligned}$$

if n is such that $v_n(\epsilon) \leq \mu_1$.

Thus

$$\liminf_{n \rightarrow \infty} \frac{[v_n(\epsilon) + \eta_2]^Y}{\mu^Y} \geq 1 - \delta(\eta_2, \epsilon).$$

$$\text{i.e.} \quad \frac{[\mu_1 + \eta_2]^Y}{\mu^Y} \geq 1 - \delta(\eta_2, \epsilon).$$

Since η_2 and ϵ can be chosen arbitrarily small we conclude $\mu_1 \geq \mu$. This proves the lemma.

Lemma 4.3 If a G.E.R.T. holds, together with condition (A), then $v_n(\epsilon)$ is a bounded function of n for fixed ϵ .

Proof. By Lemma 4.2, just proved, we can appeal to the fundamental lemma (see § 1) to deduce that for any $\eta > 0$ we can find a large C such that

$$\sum_{r=Cx}^{\infty} a_r G_r(x) < \eta x^\gamma L(x)$$

for all large x .

Choose a large $M > 0$. If $v_n(\epsilon)$ is unbounded then $v_n(\epsilon) > MK$ for infinitely many n , where K is some arbitrary large number. By the much-used exponential inequality (p. 679 of S) we have, for any $t > 0$,

$$P\{S_n \leq nM\} \leq e^{nMt - nte^{-nt} v_n(\epsilon)}$$

If we assume $v_n > MK$, and set $t = 1/nM$, then we have

$$\begin{aligned} P\{S_n \leq nM\} &\leq e^{1 - Ke^{-1/M}} \\ &= \delta, \text{ say,} \end{aligned}$$

where δ can be made as small as we like by choice of large K . Then

$$\begin{aligned} \Psi(nM) &\leq \sum_{r=1}^n a_r + G_n(nM) \sum_{r=n}^{CnM} a_r \\ &\quad + \eta n^\gamma M^\gamma L(nM). \end{aligned}$$

Thus, if we let $n \rightarrow \infty$ through values such that $v_n(\epsilon) > MK$ we find

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\Psi(nM)}{n^\gamma M^\gamma L(nM)} &\leq \frac{\alpha}{M^\gamma \Gamma(1+\gamma)} + \frac{\alpha \delta}{\Gamma(1+\gamma)} \left\{ c^\gamma M^\gamma - 1 \right\} \\ &\quad + \eta. \end{aligned}$$

The left-hand member of this inequality is necessarily $\alpha \mu^{-\gamma} / \Gamma(1+\gamma)$; the right-hand member can be made arbitrarily small by choice of M , η , and then K . Thus we have a contradiction, and the lemma is proved.

We are now fully prepared for our proof of the theorem. Since a G.E.R.T. is assumed to hold, we have

$$\sum_{r=1}^{\infty} a_r G_r(x) \sim \frac{\alpha L(x)}{\Gamma(1+\gamma)} \left(\frac{x}{\mu}\right)^{\gamma}.$$

Let us fix ξ and for reasons to emerge later suppose $\xi > \mu$. Then, as $n \rightarrow \infty$,

$$\frac{\Gamma(1+\gamma)}{n^{\gamma} L(n)} \sum_{r=1}^{\infty} a_r G_r(n\xi) \rightarrow \frac{\alpha \xi^{\gamma}}{\mu^{\gamma}}.$$

In the proof of Lemma 4.3 we have mentioned that the fundamental lemma applies. Thus, if we choose a small $\eta > 0$ we can find $C \equiv C(\eta)$ such that

$$\frac{\Gamma(1+\gamma)}{n^{\gamma} L(n)} \sum_{r=Cn\xi}^{\infty} a_r G_r(n\xi) < \alpha \eta \xi^{\gamma}$$

for all large n (since $L(\xi n) / L(n) \rightarrow 1$). Thus

$$(4.1) \quad \liminf_{n \rightarrow \infty} \frac{\Gamma(1+\gamma)}{n^{\gamma} L(n)} \sum_{r=1}^{Cn\xi} a_r G_r(n\xi) \geq \alpha \xi^{\gamma} \left\{ \frac{1}{\mu^{\gamma}} - \eta \right\}.$$

At this point we are led to consider

$$\sum_{r=n+1}^{Cn\xi} a_r G_r(n\xi) \equiv \sum_1 (n), \quad \text{say.}$$

Suppose $\omega < \mu$. Set $2\eta^* = \mu - \omega$. By Lemma 4.2 we see that (whatever fixed ϵ we choose) $\nu_n(\epsilon) - \eta^* > \omega$ for all large n . Thus, by Lemma 4.1, $P\{S_n \leq n\omega\} \leq$

$\rho(\eta^*, \epsilon)$ for all large n . Since ϵ is arbitrary in this result, and since $\rho(\eta^*, \epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, we infer

$$(4.2) \quad P\{S_n \leq n\omega\} \rightarrow 0 \text{ as } n \rightarrow \infty$$

for any fixed $\omega < \mu$.

Let n_1 be the least integer such that $n_1 \omega \geq n_1 \xi$. Then

$$\begin{aligned} \sum_{n_1}^{Cn\xi} a_r G_r(n\xi) &\leq G_{n_1}(n\xi) \sum_{n_1}^{Cn\xi} a_r \\ &\leq G_{n_1}(n_1\omega) \sum_{n_1}^{Cn\xi} a_r . \end{aligned}$$

But we have just seen that $G_n(n\omega) \rightarrow 0$ for a fixed $\omega < \mu$. Thus we can conclude that

$$\sum_{n_1}^{Cn\xi} a_r G_r(n\xi) = o(n^\gamma L(n)) .$$

Thus

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\sum_{n_1}^{Cn\xi} 1}{n^\gamma L(n)} &\leq \limsup_{n \rightarrow \infty} \frac{\sum_{n_1}^{n_1} a_r}{n^\gamma L(n)} \\ &= \frac{\alpha}{\Gamma(1+\gamma)} \left\{ \frac{\xi^\gamma}{\omega^\gamma} - 1 \right\} . \end{aligned}$$

But $\Sigma_1(n)$ does not depend on ω , so we may allow $\omega \uparrow \mu$ and obtain

$$\limsup_{n \rightarrow \infty} \frac{\sum_{n_1}^{Cn\xi} 1}{n^\gamma L(n)} \leq \frac{\alpha}{\Gamma(1+\gamma)} \left\{ \frac{\xi^\gamma}{\mu^\gamma} - 1 \right\} .$$

From (4.1) we then deduce that

$$\liminf_{n \rightarrow \infty} \frac{\Gamma(1+\gamma)}{n^{\gamma L(n)}} \sum_{r=1}^n a_r G_r(n\xi) \geq \alpha(1 - \eta) .$$

However, η is arbitrarily small, so we must have

$$\liminf_{n \rightarrow \infty} \frac{1}{n^{\gamma L(n)}} \sum_{r=1}^n a_r G_r(n\xi) \geq \frac{\alpha}{\Gamma(1+\gamma)} .$$

But since $G_r(n\xi) \leq 1$ necessarily, this inequality implies

$$\frac{1}{n^{\gamma L(n)}} \sum_{r=1}^n a_r G_r(n\xi) \rightarrow \frac{\alpha}{\Gamma(1+\gamma)}$$

as $n \rightarrow \infty$; this limit holding for any fixed $\xi > \mu$.

Take a constant c , $1 < c < \xi/\mu$. Then, as $n \rightarrow \infty$,

$$\frac{1}{(cn)^{\gamma L(cn)}} \sum_{r=1}^{cn} a_r G_r\left((cn)\left(\frac{\xi}{c}\right)\right) \rightarrow \frac{\alpha}{\Gamma(1+\gamma)} .$$

Thus

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left\{ \frac{1}{(cn)^{\gamma L(cn)}} \sum_{r=1}^n a_r + \frac{G_n(n\xi)}{(cn)^{\gamma L(cn)}} \sum_{r=n}^{cn} a_r \right\} \\ \geq \frac{\alpha}{\Gamma(1+\gamma)} , \end{aligned}$$

which implies

$$\frac{1}{c^{\gamma}} + \left(1 - \frac{1}{c^{\gamma}}\right) \liminf_{n \rightarrow \infty} G_n(n\xi) \geq 1$$

and thus that

$$G_n(n\xi) \rightarrow 1 \text{ as } n \rightarrow \infty ,$$

whenever $\xi > \mu$. But (4.2) states that $G_n(n\omega) \rightarrow 0$ as $n \rightarrow \infty$ whenever $\omega < \mu$. Thus we have established that S_n/n tends to μ in probability as $n \rightarrow \infty$, i.e. the W.L.L.N. holds, and the theorem is proved.

5. A counter-example

For simplicity we shall deal with the ERT and the renewal function

$$H(x) = \sum_{n=1}^{\infty} G_n(x),$$

rather than with the more complicated GERT.

Let X_n have a d.f. $F_n(x)$ such that, for some $\gamma > 1$,

$$F_n^*(s) = e^{-\gamma} \int_{n-1}^n \left(\frac{1-e^{-us}}{u} \right) du.$$

The right-hand side of this equation is recognizably the Laplace-Stieltjes transform of a non-negative infinitely divisible random variable.

Then

$$\begin{aligned} H^*(s) &= \sum_{n=1}^{\infty} e^{-\gamma} \int_0^n \left(\frac{1-e^{-us}}{u} \right) du, \\ &= \sum_{n=1}^{\infty} e^{-\gamma} \int_0^{ns} \left(\frac{1-e^{-v}}{v} \right) dv. \end{aligned}$$

Thus, as $s \downarrow 0+$,

$$(5.1) \quad sH^*(s) \rightarrow \int_0^{\infty} e^{-\gamma} \int_0^x \left(\frac{1-e^{-v}}{v} \right) dv \quad dx.$$

Let us write J for the integral on the right. Then J will be finite since $\gamma > 1$. Plainly by varying γ we can give J any real positive value. However, by a well-known Tauberion theorem we have from (5.1) the ERT result

$$H(x) \sim Jx, \quad \text{as } x \rightarrow \infty.$$

Now

$$G_n^*(s) = e^{-\gamma} \int_0^{ns} \left(\frac{1-e^{-v}}{v} \right) dv$$

and so

$$\begin{aligned} G_n^*\left(\frac{s}{n}\right) &= e^{-\gamma} \int_0^s \left(\frac{1-e^{-v}}{v} \right) dv \\ &= F_1^*(s). \end{aligned}$$

Thus $(X_1 + X_2 + \dots + X_n)/n$ has the same d.f. as has X_1 . This completes our demonstration of a sequence $\{X_n\}$ which satisfies the ERT by not the ULLN.

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