WEAK SUFFICIENT CONDITIONS FOR FATOU'S LEMMA AND LEGESGUE'S DOMINATED CONVERGENCE THEOREM

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NOTICE

The aim of this mimeograph is in the first place didactical. A tabular survey of a number of (mostly too restrictive) versions of Fatou's lemma taken from the literature, motivates some new counterexamples, which disprove one half of a theorem that occurs in the literature and show that if any interesting necessary conditions exist at all they must be in terms of properties different from or finer than those occurring in sufficient conditions.

0. Introduction. In many expositions the Lebesgue-Stieltjes integral, $\int f(x)\ \mu(dx) = \int f\ d\mu$, or briefly $\int f$, of a measurable function f is defined as the limit of a sequence of integrals $\int s_n d\mu$, where the s_n are simple functions which in some sense tend to f as $n \to \infty$. So, when we are interested in the limit of a sequence $\int f_n d\mu$, where all f_n are measurable (rather than simple) functions, we have to deal with a double limit process. The monotone convergence theorem (MCT), Fatou's lemma, and Lebesgue's dominated convergence theorem (DCT) belong in this category. In the literature these results are discussed under a variety of mostly too restrictive conditions (cf. section 2 below), which we have found tend to obscure their true nature in the mind of many students. The aim of this note is to present Fatou's lemma as a special case of the MCT, and the DCT as a special case of Fatou's lemma, being as general as possible as to conditions of boundedness and finiteness; also to indicate a method by which to construct the dominating function in the DCT. Of these objectives the last one seems to have some novelty. However, our main concern is pedagogical.

1. Notations and terminology. All functions discussed are assumed to be defined on a totally σ-finite measure space $(X, \mathfrak{A}, \mu)$ into the extended real number system R^* (for the properties of R^* see for instance Halmos, p. 2). All functions discussed will be measurable (i.e., if B is $\{+\infty\}$, $\{-\infty\}$, or a Borel subset of the real line R, then $f^{-1}(B) \in \mathfrak{A}$). Given $\varphi : X \to R^*$, the symbols φ^+ and φ^- have the usual meaning, $\varphi^+ = \frac{1}{2}(\varphi + |\varphi|)$, $\varphi^- = \frac{1}{2}(-\varphi + |\varphi|)$, so $\varphi = \varphi^+ - \varphi^-$. Integration is always over some set A belonging to the σ-algebra $\mathfrak{A}$. For our purposes the choice of A is irrelevant (all properties stated concerning integrands are to hold on A), and we shall omit all reference to it. Whenever we write $\int\varphi$, or $\int\varphi\, d\mu$, or $\int\varphi(x)\,\mu(dx)$, we imply that the integral is defined, either as a finite number, or as $+\infty$, or as $-\infty$, and we shall call such φ integrable. In fact, we shall say that φ is integrable iff any one of the following three cases applies:

a) $\int\varphi^+$ finite, $\int\varphi^-$ finite, $\int\varphi = \int\varphi^+ - \int\varphi^-$, φ summable,

b) $\int\varphi^+$ finite, $\int\varphi^-$ infinite, $\int\varphi = -\infty$,

c) $\int\varphi^+$ infinite, $\int\varphi^-$ finite, $\int\varphi = +\infty$.

A function φ such that $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for μ-almost all x $(\in A)$ will be denoted by $\lim_n \varphi_n$.

Given a sequence $\{\alpha_k\}$, define $\beta_n = \inf_{k\geq n} \alpha_k$; $\gamma_n = \sup_{k\geq n} \alpha_k$. Then $\{\beta_n\}$ is a non-decreasing sequence, and $\{\gamma_n\}$ is a non-increasing sequence. So $\lim_{n\to\infty} \beta_n = \lim_n \beta_n$ and $\lim_{n\to\infty} \gamma_n = \lim_n \gamma_n$ exist (the first being finite or $+\infty$, the second being finite or $-\infty$). We shall write $\lim\inf_n \alpha_n$ for $\lim_n \beta_n$ and $\lim\sup_n \alpha_n$ for $\lim_n \gamma_n$. A function φ such that $\lim\inf_n \varphi_n(x) = \varphi(x)$ for μ-almost all x will be denoted by $\lim\inf_n \varphi_n$; a similar definition applies to $\lim\sup_n \varphi_n$; $\lim_n \varphi_n$ exists iff $\lim\inf_n \varphi_n = \lim\sup_n \varphi_n$ (μ-almost everywhere).

All equalities and inequalities between functions are supposed to hold μ-almost everywhere.

When citing other authors we shall translate their statements in terms of the above terminology and notation.

In all our examples and counterexamples μ is understood to be Lebesgue measure.

TABLE 1.

Kolmogorov-Fomin, § 44, Th. 2 Šilov-Guervič, p. 37 Riesz-Sz.Nagy, § 20	$f_n \geq 0$	$\int f_n \leq K$	$\lim_n f_n$ exists	——	⇒ $\lim_n f_n$ sumable and $\int \lim_n f_n \leq K$
Jeffery, Theorem 4.17	$f_n \geq 0$	$\int f_n < +\infty$	$\lim_n f_n$ exists: sumable not sumable	——	⇒ ⇒ $\lim_n \int f_n = +\infty$
Berberian, Sect. 32 Halmos, Sect. 27, Th. F Munroe, Sect. 34.1	$f_n \geq 0$	$\int f_n < +\infty$	——	$\liminf_n \int f_n < +\infty$	⇒ $\liminf_n f_n$ summable
McShane-Botts, Ch. 5, § 4, Ex. 3	$f_n \geq g$ and g summable	f_n summable	——	$\liminf_n \int f_n < +\infty$	⇒ $\liminf_n f_n$ summable
Loève, Sect. 7.2, Th. B	$h \geq f_n \geq g$ and g, h summable	(hence f_n summable)	——	——	⇒
Goffman-Pedrick, Sect. 3.7, Th. 1	$f_n \geq 0$	$\int f_n < +\infty$	——	——	⇒
Royden, Ch. 11, Sect. 3, Th. 12	$f_n \geq 0$	——	$\lim_n f_n$ exists	——	⇒
Dunford-Schwartz, Sect. III. 6.19 Hewitt-Stromberg, Ch. 3, (12-23) Rudin, Sect. 10.31 Saks, Ch. 1, Th. 12.10	$f_n \geq 0$	——	——	——	⇒

2. Comparative survey of versions of Fatou's lemma. Table 1 lists conditions and conclusions of a number of versions of Fatou's lemma. Measurability of f_n is common to all assumptions and the inequality

$$\int \liminf_n f_n \leq \liminf_n \int f_n \tag{1}$$

is common to all conclusions: these have been omitted from our list.

Analyzing this survey we make the following remarks:

i) The conclusion $\int \lim_n f_n \leq K$ is weaker than the usual one, but the proofs given actually prove inequality (1), as is pointed out by Riesz and Sz.Nagy, l.c.

ii) Jeffery's version may suggest the question if summability of $\lim_n f_n$ (or of $\liminf_n f_n$, as the case may be) could imply that $\liminf_n \int f_n < +\infty$. Actually Munroe, l.c., claims that this is, indeed, true (see the "only if" part of his Sect. 34.1). A counter-example is provided by the sequence $\{f_n\}$:

$$f_n(x) = \begin{cases} n^2(n+1) & \text{for } (n+1)^{-1} < x < n^{-1} \\ 0 & \text{otherwise} \end{cases} .$$

Here $\liminf_n f_n(x) = \lim_n f_n(x) = 0$ for all x , so $\liminf_n f_n$ is summable. Yet $\int_0^\infty f_n = n$, hence $\liminf_n \int f_n = \lim_n \int f_n = +\infty$.

iii) Comparison of the Goffman-Pedrick version with those preceding it suggests the question whether it is possible that $f_n \geq 0$ and f_n is summable for all n , yet $\liminf_n f_n$ is not summable. The following example shows the answer is affirmative:

$$f_n(x) = \begin{cases} x^{-2} & \text{for } 2^{-n} < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here $\int_0^1 f_n < +\infty$ for all n , yet $\liminf_n f_n(x) = x^{-2}$ for $0 < x < 1$, hence not summable.

iv) Whenever the existence of $\lim_n f_n$ is assumed one may, as a matter of course, replace $\liminf_n f_n$ in inequality (1) by $\lim_n f_n$.

v) It is clear that the version as given by Dunford and Schwartz and others (not requiring the f_n to be summable) can easily generate all versions providing for finiteness or even boundedness of certain integrals.

vi) The inequality

$$(2) \qquad \limsup_n \int f_n \leq \int \limsup_n f_n$$

constitutes a complement to inequality (1). The substitution $f_n = -f_n^*$ shows that (2) holds true under the condition $f_n \leq 0$ (also see Colollary to Theorem 1 in Sect. 3.7 of Goffman and Pedrick). Thus inequalities (1) and (2) would obtain for two mutually exclusive classes of functions. This is not true, fortunately: the condition $f_n \geq 0$ for inequality (1) is unduly restrictive, as can be seen from the conditions cited by McShane and Botts, and by Loève. In fact, as pointed out by Loève, l.c., the condition, "$h \geq f_n \geq g$, and g and h summable," cited in Table 1, is sufficient for both (1) and (2). Note that $h \geq f_n$ is irrelevant for the validity of (1), and

$f_n \geq g$ is irrelevant for the validity of (2). Applications of these alternative conditions are, however, hampered by the fact that there is no construction to derive g and h from the sequence $\{f_n\}$. Our discussion will point up a natural construction for precisely this purpose.

3. Monotone convergence theorem. Let $\{\varphi_n\}$ be a non-decreasing sequence of extended real valued non-negative measurable functions defined on the measure space $(X, \mathfrak{A}, \mu)$. Then

(3) $$\int \lim_n \varphi_n = \lim_n \int \varphi_n . \qquad \text{(MCT)}$$

Proof. The proof depends on the way the integral is defined. In the present context the simplest definition is the one used by Loève (Sect. 7.1) and by Goffman and Pedrick (Sect. 3.6). Their proofs of MCT (see Loève, Sect. 7.2, Th. A, and Goffman and Pedrick, Sect. 3.7, Proposition 1) are closely related and exhibit the double limit process mentioned in our section 0 in an explicit way.

Our first concern is to weaken the condition $\varphi_n \geq 0$. We cannot just omit this condition, as is clear from the following example of an increasing sequence of functions φ_n:

(4) $$\varphi_n(x) = -1/nx \quad \text{for} \quad 0 < x < 1 .$$

Here $\int_0^1 \varphi_n = -\infty$ for all n, so $\lim_n \int_0^1 \varphi_n = -\infty$, whereas $\lim_n \varphi_n(x) = 0$ for $0 < x < 1$, so $\int_0^1 \lim_n \varphi_n = 0$. The following generalization of MCT is easily proved:

MCT^g. Let $\{\varphi_n\}$ be a non-decreasing sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which either $\int \varphi_k = +\infty$, or φ_k is summable, it follows that

(5a) $$\int \lim_n \varphi_n = \lim_n \int \varphi_n .$$

If neither of these conditions is satisfied we can only conclude that

(5b) $$\int \lim_n \varphi_n \geq \lim_n \int \varphi_n .$$

Proof. If a natural number k exists for which $\int \varphi_k = +\infty$, then both members of (5a) are seen to be equal to $+\infty$ (since $\lim_n \varphi_n \geq \varphi_k$ and since for $n \geq k$ we have $\int \varphi_n \geq \int \varphi_k$). If a natural number k exists for which φ_k is summable, the sequence $\{\varphi_n - \varphi_k\}_{n \geq k}$ satisfies the conditions for the MCT, hence

$$\lim_n \int (\varphi_n - \varphi_k) = \int (\lim_n \varphi - \varphi_k) .$$

Addition of the finite number $\int\varphi_k$ to both sides of this equality completes this part of the proof. Finally, if neither condition obtains, then $\int\varphi_n = -\infty$ for all n, so (5b) is trivially true. The example defined by equation (4) shows that in this case the inequality sign may obtain in (5b), whereas examples of strict equality are trivial.

The reader will supply a result symmetric to MCT^g by replacing non-decreasing by non-increasing, $+\infty$ by $-\infty$, $\geq$ by $\leq$. We will refer to the result discussed in the second half of this section as to $\uparrow MCT^g$, and to its symmetric counterpart as to $\downarrow MCT^g$.

The reader will observe that MCT follows from $\uparrow MCT^g$ by adjoining a member φ_0 to the sequence, where $\varphi_0(x) = 0$ for all x, and putting the number k equal to 0. Of course, this is possible only if $\varphi_n \geq 0$ for all $n = 1,2,3,\ldots$

4. Fatou's lemma and the dominating convergence theorem. The following results refer to arbitrary sequences $\{f_n\}$. They follow immediately by applying $\uparrow$MCT^g to the non-decreasing sequence $\{g_n\} = \{\inf_{\nu \geq n} f_\nu\}$ and $\downarrow$MCT^g to the non-increasing sequence $\{h_n\} = \{\sup_{\nu \geq n} f_\nu\}$.

Fatou I. Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which either $\int g_k = \int \inf_{\nu \geq k} f_\nu = +\infty$ or $g_k = \inf_{\nu \geq k} f_\nu$ is summable, then

$$(6) \qquad \int \liminf_n f_n \leq \liminf_n \int f_n .$$

Proof. Applying $\uparrow$MCT^g to $\{g_n\}$ we find

$$(7) \qquad \int \liminf_n f_n = \lim_n \int g_n .$$

Now $g_n(x) = \inf_{\nu \geq n} f_\nu(x) \leq f_m(x)$ for all $m \geq n$.

So $$\int g_n \leq \int f_m \quad \text{for all } m \geq n ;$$

hence $$\int g_n \leq \inf_{m \geq n} \int f_m .$$

Consequently

$$(8) \qquad \lim_n \int g_n \leq \liminf_n \int f_n .$$

Combination of (7) and (8) yields inequality (6) of Fatou I.

If the conditions of Fatou I do not obtain, $\uparrow$MCT^g shows that (7) should be replaced by

$$(7^*) \qquad \int \liminf_n f_n \geq \lim_n \int g_n .$$

Inequality (8) remains valid. Combination of (7*) and (8) now seems to imply that a sequence $\{f_n\}$ with

$$\int g_k = \int \inf_{\nu \geq k} f_\nu = -\infty \quad \text{for all natural } k$$

may satisfy inequality (6) (with $\leq$) as well as its opposite (with $\geq$). In fact, in section 5 we will give an example of both types of sequences. Hence a necessary condition for the validity of Fatou's inequality (6) cannot be given in terms of integrability and/or summability of the members of the sequence $\{g_n\} = \{\inf_{\nu \geq n} f_\nu\}$ alone. However, the existence of a number k with the property cited in our Fatou I constitutes the sharpest sufficient condition to date.

The reader will have no difficulty in proving

Fatou II. Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which either $\int h_k = \int \sup_{\nu \geq k} f_\nu = -\infty$ or $h_k = \sup_{\nu \geq k} f_\nu$ is summable, then

(9) $$\limsup_n \int f_n \leq \int \limsup_n f_n \quad .$$

Indeed apply $\downarrow$MCT^g to the sequence $\{h_n\}$.

Evidently, the best sufficient condition for the validity of both (6) and (9) is given by

Fatou III. Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If at least one natural number k exists for which both $g_k = \inf_{\nu \geq k} f_\nu$ and $h_k = \sup_{\nu \geq k} f_\nu$ are summable, then

(10) $$\int \liminf_n f_n \leq \liminf_n \int f_n \leq \limsup_n \int f_n \leq \int \limsup_n f_n \quad .$$

Indeed, in order that Fatou II may apply, $\int h_k = \int \sup_{\nu \geq k} f_\nu$ has to come down from $+\infty$ for some k , and in order that Fatou I may apply, $\int g_k = \int \inf_{\nu \geq k} f_\nu$ has to come up from $-\infty$ for some k . Hence $\int h_k$ cannot ever get down to $-\infty$, and $\int g_k$ cannot ever get up to $+\infty$.

If to the conditions of Fatou III is added the condition that $\lim_n f_n$ exist, the string of inequalities in (10) becomes a string of equalities. This leads to our version of the dominated convergence theorem:

DCT. Let $\{f_n\}$ be a sequence of extended real valued measurable functions defined on $(X, \mathfrak{A}, \mu)$. If $\lim_n f_n$ exists (μ-almost everywhere) and if in addition at least one natural number k exists for which both $g_k = \inf_{\nu \geq k} f_\nu$ and $h_k = \sup_{\nu \geq k} f_\nu$ are summable, then

(11) $\lim_n \int f_n$ <u>exists, and</u>

(12) $\lim_n \int f_n = \int \lim_n f_n$ <u>is finite</u>.

By hindsight the functions g_k and h_k are as closely dominating functions as one can hope to obtain, so in critical problems this version offers more hope for success than the classical requirement that one find some summable function s such that $|f_n| \leq |s|$. In fact, our version supplies a construction of a sequence from which one can choose a dominating function s.

One might wonder, since g_k and h_k have to be summable for <u>some</u> k (no matter how large), if not summability of $\liminf_n f_n$ and of $\limsup f_n$ would be sufficient. Counterexample (Hartman and Mikusiński, Ch. iv, sect. 3):

$$f_n(x) = \begin{cases} -2xn^2 + 2n & \text{for } 0 < x \leq n^{-1} \\ 0 & \text{for } n^{-1} < x \leq 1 \end{cases} .$$

Here $\lim_n f_n(x) = 0$ for all $x \in]0, 1]$; $\int_0^1 \lim_n f_n(x)\,dx = 0$; $\int_0^1 f_n(x)\,dx = 1$ for all n ; $\lim_n \int_0^1 f_n(x)\,dx = 1$. So $\limsup_n f_n = \liminf_n f_n$ is summable, yet (12) does not apply. What is wrong? The function $\sup_{\nu \geq k} f_\nu$ is not summable for any k as can be shown by elementary computation.

5. Two sequences of functions violating the conditions of Fatou I. The increasing sequence of functions defined in equation (4) has the following properties:

$$g_k(x) = \inf_{\nu \geq k} \varphi_\nu(x) = -(kx)^{-1}$$

$$\int_0^1 g_k = -\infty \quad \text{for all natural } k .$$

$$\int_0^1 \liminf_n \varphi_n = \int_0^1 \lim_n \varphi_n = 0$$

$$\liminf_n \int_0^1 \varphi_n = \lim_n \int_0^1 \varphi_n = -\infty$$

Hence $\int \liminf_n \varphi_n > \liminf_n \int \varphi_n$, which constitutes an example of a sequence of functions violating the condition of Fatou I, for which the Fatou inequality (6) is invalid.

Next we will construct a sequence $\{f_n\}$ violating the condition of Fatou I, for which, however, the Fatou inequality (6) is still valid. Let $\{r_n\}$ be the sequence of all rational numbers between 0 and 1, constructed in the following manner: write all proper fractions, reduced to lowest terms, first choosing those with denominator 2 (so $r_1 = \frac{1}{2}$), then those with denominator 3 (so $r_2 = \frac{1}{3}$, $r_3 = \frac{2}{3}$) , then those with denominator 4 (hence $r_4 = \frac{1}{4}$, $r_5 = \frac{3}{4}$) , etc. Let $\rho \in]1, 2[$ be a fixed real number. Now define

$$f_n(x) = \begin{cases} -1/x & \text{if } r_n \cdot \rho^{-1} < x < r_n \cdot \rho , \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

This sequence of functions has the following properties:

(14) $\inf_{\nu \geq k} f_\nu(x) = -1/x$ for all natural k , all irrational $x \in]0, \rho[$;

hence $\int_0^2 \inf_{\nu \geq k} f_\nu(x) = -\infty$.

Also $\liminf_n f_n(x) = -\frac{1}{x}$ for $0 < x < \rho$; so $\int_0^2 \liminf_n f_n = -\infty$.

Finally $\int_0^2 f_n = -2 \log \rho$; so $\liminf_n \int_0^2 f_n = -2 \log \rho$.

Clearly $\int \liminf_n f_n < \liminf_n \int f_n$, which we wanted to show.

Proof of equality (14). For any irrational number x three exists an <u>infinite</u> number of fractions p/q such that $x \in \left]\frac{p}{q} - \frac{1}{q^2}, \frac{p}{q} + \frac{1}{q^2}\right[$ (see Perron, Chap. 5, § 36). Since for $q > \rho/(\rho-1)$ we have that $\frac{p}{q}\frac{1}{\rho} < \frac{p}{q} - \frac{1}{q^2}$ and that $\frac{p}{q}\rho > \frac{p}{q} + \frac{1}{q^2}$, it follows that each irrational number $x \in]0, 1[$ is covered by an infinite number of intervals of the form $\left]\frac{p}{q}\frac{1}{\rho}, \frac{p}{q}\rho\right[$, with $0 < p < q$, p and q relative prime, and fixed $\rho \in]1, 2[$.

Since the members of the above-constructed sequence $\{r_n\}$ are irreducible fractions whose denominators form a non-decreasing sequence of natural numbers it follows that each irrational $x \in]0, 1[$ is covered by an infinite number of intervals of form $]r_n/\rho, r_n\rho[$. Note that the irrational $x \in]1, \rho[$ are trivially so covered: just take $r_n = (m-1)/m$ with m sufficiently large. Hence definition (13) shows that for any irrational $x \in]0, \rho[$, no matter how large N may be, there is always a number $n > N$ with $f_n(x) = -\frac{1}{x}$. This proves equality (14).

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