ASYMPTOTICALLY EFFICIENT TESTS BY THE METHOD OF n RANKINGS

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1. Summary and introduction. When the block-effects are not additive or the method of ranking after alignment is not practicable, the method of n rankings [cf. Friedman (1937), Kendall (1955)] is quite useful for the analysis of two way layouts. Consider n observers, each of which ranks (independently) p objects $O_1,\ldots,O_p$; the ranks of $O_1,\ldots,O_p$ by the ith observer are denoted by $r_{i1},\ldots,r_{ip}$, respectively, for $i=1,\ldots,n$. Well-known tests for the hypothesis of no difference among the p objects are due to Friedman (1937) and Brown and Mood (1951), and are respectively based on the test statistics

$$(1.1)\qquad \chi_r^2 = [12/np(p+1)]\sum_{j=1}^{p}(R_j-n(p+1)/2)^2;\quad R_j = \sum_{i=1}^{n} r_{ij},\ j=1,\ldots,p;$$

and

$$(1.2)\qquad M_r = \frac{p(p-1)}{na(p-a)}\sum_{j=1}^{p}(M_j-na/p)^2;\quad M_j = \sum_{i=1}^{n} m_{ij},\ j=1,\ldots,p,$$

where m_{ij} is 1 or 0 according as r_{ij} is $\leq$ or $>$ a: $1 \leq a < p$; usually a is taken to be the largest integer contained in $(p+1)/2$. Generalizations of the χ_r^2-test to non-orthogonal designs are due to Durbin (1951) and Benard and Elteren (1953) and their asymptotic power-efficiencies are studied by Elteren and Noether (1959). Similar works on the other test are due to Bhapkar (1961, 1963). The present investigation is concerned with (i) the derivation of a class of asymptotically efficient test for the same problem and (ii) the characterizations of the optimality of the tests based on (1.1) and (1.2).

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2. Preliminary notions. We conceive of some stochastic variables $(X_{i1},\ldots,X_{ip})$ (may or may not be observable) underlying the ranks $(r_{i1},\ldots,r_{ip})$, for $i=1,\ldots,n$. It is then assumed that $X_{i1},\ldots,X_{ip}$ are independently distributed according to continuous cumulative distribution functions (cdf's) $F_{i1}(x),\ldots,F_{ip}(x)$, respectively, for $i=1,\ldots,n$. The null hypothesis states that

$$H_o:\ F_{i1} \equiv \cdots \equiv F_{in} \equiv F_i, \text{ for all } i=1,\ldots,n, \tag{2.1}$$

whatever be $F_1,\ldots,F_n$. We are interested in translation alternatives viz.,

$$F_{ij}(x) = F_i(x-t_j),\ j=1,\ldots,p,\ i=1,\ldots,n;\ \Sigma_{j=1}^{p} t_j = 0, \tag{2.2}$$

$\underset{\sim}{t} = (t_1,\ldots,t_p)$ being a real p-vector. For our study, we shall assume that

(i) n (the number of observers or blocks) is large;

(ii) $F_i(x)$ is absolutely continuous having a continuous density function $f_i(x)$ where

$$\int_{-\infty}^{\infty} f_i^2(x)\,dx < \infty \text{ for all } i=1,\ldots,n; \tag{2.3}$$

(2.4) (iii) $\underset{\sim}{t} = \underset{\sim}{t}_n = n^{-\frac{1}{2}}\underset{\sim}{\theta}$; $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_p)$ has real and finite elements.

Also, we shall confine ourselves to the following class of rank tests. Let $\{J(r,p),\ r=1,\ldots,p\}$ be p real valued functions, where $J(r,p)$ is a function of r and p, $r=1,\ldots,p$. We define

$$\bar{J} = (1/p)\Sigma_{r=1}^{p} J(r,p) \quad \text{and} \quad A^2(J) = \frac{1}{p-1}\sum_{r=1}^{p}\{J(r,p)-\bar{J}\}^2 . \tag{2.5}$$

Concerning $\{J(r,p)\}$ we assume that for any finite p, they are all finite and are not all equal. Under these assumptions both $\bar{J}$ and $A^2(J)$ are finite and $A^2(J)$ is strictly positive. We then define a rank statistic (vector) $\underset{\sim}{T}_n = (T_{n,1},\ldots,T_{n,p})$, where

(2.6) $$T_{n,j} = (1/n)\Sigma_{i=1}^{n} J(r_{ij},p), \text{ for } j=1,\ldots,p.$$

The tests to be considered here are based on the following type of statistics

(2.7) $$S_n = n\, A^{-2}(J)\, \Sigma_{j=1}^{p}\{T_{n,j}-\bar{J}\}^2 .$$

It is easily verified that both (1.1) and (1.2) are particular cases of (2.7). Our main contention is to select $\{J(r,p)\}$ in such a manner that the corresponding S_n leads to asymptotically efficient test.

3. Asymptotic distribution of S_n. This is presented briefly, as it will be required in the sequel. Let us define for each $i(=1,\ldots,n)$

(3.1) $$p_{i,r}^{(j)} = P\{r_{ij}=r\},\ p_{i,rs}^{(j,j')} = P\{r_{ij}=r, r_{ij'}=s\},\ r\neq s=1,\ldots,p,\ j\neq j'=1,\ldots,p.$$

Conventionally, we let

(3.2) $$p_{i,rs}^{(j,j)} = p_{i,r}^{(j)}\cdot\delta_{rs},\ p_{i,rr}^{(j,j')} = \delta_{jj'}p_{i,r}^{(s)},\ r,s,j,j'=1,\ldots,p,$$

where $\delta_{jj'}$ and δ_{rs} are the Kronecker deltas. Let then

(3.3) $$\mu_{n,j} = \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{p} J(r,p)p_{i,r}^{(j)},\ j=1,\ldots,p;\ \underset{\sim}{\mu}_n = (\mu_{n,1},\ldots,\mu_{n,p});$$

(3.4) $$\sigma_{n,jj'} = \frac{1}{n}\sum_{i=1}^{n}\sum_{r=1}^{p}\sum_{s=1}^{p} J(r,p)J(s,p)p_{i,rs}^{(j,j')} - \mu_{n,j}\mu_{n,j'},\ j,j'=1,\ldots,p;$$

(3.5) $$\underset{\sim}{\Sigma}_n = ((\sigma_{n,jj'}))_{j,j'=1,\ldots,p} .$$

THEOREM 3.1. $n^{\frac{1}{2}}(\underset{\sim}{T}_n-\underset{\sim}{\mu}_n)$ has asymptotically a normal distribution with a null mean vector and dispersion matrix $\underset{\sim}{\Sigma}_n$.

OUTLINE OF THE PROOF. It suffices to show that for any real and non-null $\underset{\sim}{a}$ = $(a_1,\ldots,a_p)$ (other than a(1,...,1)) $Z_n = n\, \underset{\sim}{a} \cdot \underset{\sim}{T}_n'$ converges in law to a normal distribution. By using (2.6), we may rewrite Z_n as

$$(3.6) \qquad Z_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} U_i, \text{ where } U_i = \Sigma_{j=1}^{p} a_j J(r_{ij},p),\ i=1,\ldots,n.$$

As J(r,p), r=1,...,p are all finite, $U_1,\ldots,U_n$ are all independent and finite valued random variables. The rest of the proof follows by an application of the classical central limit theorem [cf. Loeve (1963, p. 277)] and some routine analysis. Hence, the theorem.

Under the null hypothesis (2.1), it is easily seen that $\underset{\sim}{\mu}_n = \bar{J} \cdot \underset{\sim}{\ell}, \underset{\sim}{\ell} = (1,\ldots,1)$, and $\underset{\sim}{\Sigma}_n = A^2(J)[\underset{\sim}{I}_p - \frac{1}{p} \underset{\sim}{\ell}' \underset{\sim}{\ell}]$, where $\underset{\sim}{I}_p$ is the identity matrix of order p. In this case, using (3.6), the fact that $U_1,\ldots,U_n$ are identically distributed and the Berry-Essen theorem [cf. Loeve (1962, p. 288)], it can be shown that the convergence of $N^{\frac{1}{2}}(\underset{\sim}{T}_n - \underset{\sim}{\mu}_n)$ to a multinormal variable is uniform in $(F_1,\ldots,F_n)$. Further, considering the random variables $n^{\frac{1}{2}}(T_{n,j} - \bar{J})$, j=1,...,p-1, taking the reciprocal of their covariance matrix as a suitable discriminant of their quadratic form, symmetrizing it and using some well known results on the asymptotic distribution of quadratic forms associated with asymptotically multinormal distributions, we arrive at the following.

Corollary 3.1. _Under_ H_o _in (2.1),_ S_n, _defined by (2.7), converges in law to a chi-square distribution with (p-1) degrees of freedom._

We shall now consider the asymptotic distribution of S_n under the sequence of alternatives in (2.2) and (2.4). For this, we define by

$$(3.7) \qquad \beta_{s,p-2}^{(i)} = \binom{p-2}{s} \int_{-\infty}^{\infty} [F_i(x)]^s [1-F_i(x)]^{p-2-s} f_i^2(x)\,dx,\ s=0,\ldots,p-2$$

and conventionally, we let $\beta_{-1,p-2}^{(i)} = \beta_{p-1,p-2}^{(i)} = 0$, for all i=1,...,n; (2.3) ensures the finiteness of (3.7). Let then

(3.8) $$\bar{\beta}^{(n)}_{s,p-2} = (1/n)\ \Sigma_{i=1}^{n}\beta^{(i)}_{s,p-2},\ \text{for } s=-1,0,\ldots,p-1;$$

(3.9) $$\lambda_{r,n} = \bar{\beta}^{(i)}_{r-1,p-2} - \bar{\beta}^{(n)}_{r-2,p-2},\ \text{for } r=1,\ldots,p.$$

THEOREM 3.2. Under (2.2) through (2.5), S_n, defined by (2.7), has asymptotically a non-central chi-square distribution with p-1 degrees of freedom and the non-centrality parameter

(3.10) $$\Delta_n(J) = p^2A^{-2}(J)(\Sigma_{j=1}^{p}\theta_j^2)(\Sigma_{r=1}^{p}J(r,p)\lambda_{r,n})^2 .$$

PROOF. By virtue of the result of theorem 3.1, it is sufficient to show that under (2.2) through (2.5)

(3.11) $$n^{\frac{1}{2}}(\mu_{n,j} - \bar{J}) = -\ p\theta_j\ \Sigma_{r=1}^{p}J(r,p)\lambda_{r,n} + o(1),\ j=1,\ldots,p;$$

(3.12) $$[\underset{\sim}{\Sigma}_n - A^2(J)\{\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{\ell}'\underset{\sim}{\ell}\}]\ \text{converges to a null matrix as } n\to\infty .$$

Now, by definition in (3.1)

(3.13) $$p^{(j)}_{i,r} = \Sigma_{S_j}\int_{-\infty}^{\infty}F_{is_1}(x)\ldots F_{is_{r-1}}(x)[1-F_{is_{r+1}}(x)]\ldots[1-F_{is_p}(x)]dF_{ij}(x),$$

where the summation S_j extends over all possible choice of $(s_1,\ldots,s_{r-1})$ from $(1,\ldots,j-1,j+1,\ldots,p)$, $(s_{r+1},\ldots,s_p)$ being the complementary set. Now, by (2.2) and (2.4), we have

(3.14) $$F_{i\ell}(x) = F_{ij}(x-n^{-\frac{1}{2}}[\theta_\ell-\theta_j])\ \text{for all } j,\ell=1,\ldots,p;\ \Sigma_{r=1}^{p}\theta_r = 0.$$

Using (3.13), (3.14) and some routine analysis, it follows that under (2.3),

$$(3.15)\qquad p_{i,r}^{(j)} = (1/p) + n^{-\frac{1}{2}}[p\theta_j(\beta_{r-2,p-2}^{(i)} - \beta_{r-1,p-2}^{(i)})] + o(n^{-\frac{1}{2}}),$$

for all $r,j=1,\ldots,p$, $i=1,\ldots,n$. (3.3) and (3.15) ascert (3.11). Proceeding in a similar manner, it can be shown that under (2.2) through (2.4)

$$(3.16)\qquad p_{i,rs}^{(j,j')} = 1/p(p-1) + O(n^{-\frac{1}{2}}) \text{ for all } j\neq j', r\neq s=1,\ldots,p,\ i=1,\ldots,n.$$

Hence, from (3.2), (3.4), (3.5), (3.15) and (3.16), we obtain (3.12). Hence the theorem.

It may be noted that if F_i possesses a finite variance σ_i^2 for all $i=1,\ldots,n$, then $(p-1)F_{(p-1),(n-1)(p-1)}$ (where $F_{(p-1),(n-1)(p-1)}$ is the classical analysis of variance test statistic,) has asymptotically (under (2.2) and (2.4)) a non-central chi-square distribution with (p-1) degrees of freedom and the non-centrality parameter

$$(3.17)\qquad \Delta_n^* = (\Sigma_{j=1}^{p}\theta_j^2)/\bar{\sigma}_n^2;\quad \bar{\sigma}_n^2 = (1/n)\ \Sigma_{i=1}^{n}\sigma_i^2 .$$

(3.10) and (3.17) will be used in the derivation of our main results.

4. Intrinsic asymptotic efficiency and the efficient S_n-test. From theorem 3.2 and (3.17) the asymptotic relative efficiency (A.R.E.) of the test based on S_n with respect to the classical analysis of variance test is deduced to be equal to

$$(4.1)\qquad e_n(J) = \bar{\sigma}_n^2 p^2(p-1)(\Sigma_{r=1}^{p} J(r,p)\lambda_{r,n})^2/\{\Sigma_{r=1}^{p}[J(r,p) - \bar{J}]^2\} .$$

As from (3.7), (3.8) and (3.9), we have $\Sigma_{r=1}^{p}\lambda_{r,n} = 0$, (4.1) may also be written as

$$(4.2)\qquad [\bar{\sigma}_n^2 p^2(p-1)\sum_{r=1}^{p}\lambda_{r,n}^2]\{(\sum_{r=1}^{p}[J(r,p) - \bar{J}]\lambda_{r,n})^2/(\sum_{r=1}^{p}\lambda_{r,n}^2)(\sum_{r=1}^{p}[J(r,p) - \bar{J}]^2)\} ,$$

where the second factor of (4.2) is uniformly bounded by 1, and the first factor is independent of $\{J(r,p),\ r=1,\ldots,p\}$. We now define

(4.3) $$e_n^* = \bar{\sigma}_n^2 p^2(p-1) \Sigma_{r=1}^{p} \lambda_{r,n}^2$$

as the intrinsic A.R.E. of the method of n rankings as it is the maximum possible value of $e_n(J)$. Further, a particular S_n-test is said to be asymptotically efficient for $(F_1,\ldots,F_n)$ if $e_n(J) = e_n^*$ for that $(F_1,\ldots,F_n)$. It follows from (4.1), (4.2) and (4.3) that $e_n(J) = e_n^*$, if and only if,

(4.4) $$J(r,p) - \bar{J} = k\ \lambda_{r,n} \text{ for all } r=1,\ldots,p, \text{ k being a constant.}$$

For p=2, we have $\lambda_{1,n} = -\lambda_{2,n}$ and also whatever be $\{J(r,p),\ r=1,2\}$, $J(1,2) - \bar{J} = [J(2,2) - \bar{J}]$. Hence, (4.4) holds. So any S_n-test will be asymptotically efficient and will attain the intrinsic A.R.E. If p=3 and $F_1,\ldots,F_n$ are all symmetric about zero (so that $\beta_{0,1}^{(i)} = \beta_{1,1}^{(i)}$ for all i=1,...,n), we have from (3.9), $\lambda_{1,n} = -\lambda_{3,n}$ and $\lambda_{2,n} = 0$. As such, the optimum S_n reduces to χ_r^2, defined by (1.1). However, if $F_1,\ldots,F_n$ are not all symmetric, this property of χ_r^2 will not hold, in general. (It may be noted that S_n corresponding to $\{J(r,p),\ r=1,\ldots,p\}$ in (4.4) also corresponds to the likelihood ratio criterion (for the model (2.2) and (2.4) with any given $\underset{\sim}{\theta}$), and the intrinsic A.R.E. e_n^* is the A.R.E. of the likelihood ratio test for the same problem. For intended brevity the proof of this statement is omitted.)

For $p \geq 4$, the choice of the optimum S_n (in accordance with (4.4)) in general, depends on $(F_1,\ldots,F_n)$ through $\lambda_{r,n}$, r=1,...,n. If, however, $F_1 \equiv \ldots \equiv F_n \equiv F$, then in most of the cases, $\lambda_{r,n}$, r=1,...,p, are quite simple and the corresponding S_n- statistics are not difficult to compute. We shall consider the following illustrations.

(I) Uniform distribution: $f(x) = F'(x) = dx$: $0 \leq x \leq 1$. Here, we have

(4.5) $$\lambda_{r,n} = J(r,p) = \begin{cases} 1, & r=1 \\ 0, & 2 \leq r \leq p-1 \\ -1, & r=p\ . \end{cases}$$

Thus, for $p \geq 4$, the optimum S_n is different from χ_r^2 in (1.1).

(II) Exponential distribution: $f(x) = e^{-x}$, $0 \leq x < \infty$. Here

$$\lambda_{r,n} = \begin{cases} 1/p, & r=1 \\ -1/p(p-1, & r>1 \end{cases} \Rightarrow J(r,p) = \begin{cases} 1, & \text{if } r=1 \\ 0, & \text{if } r=2,\dots,p. \end{cases} \tag{4.6}$$

The corresponding S_n is a particular case of (1.2) (with a=1).

(III) Double exponential distribution: $f(x) = e^{-|x|}$, $-\infty < x < \infty$. By (3.7), we have

$$\beta_{s-1,p-2} = \frac{1}{p(p-1)} \left\{ p - X \sum_{r=p-s}^{p} \binom{p}{r} 2^{-p} - (p-s) \sum_{r=s}^{p} \binom{p}{r} 2^{-p} \right\}, \quad s=1,\dots,p-1. \tag{4.7}$$

Hence, it can be shown that

$$J(r,p) = p(p-1)\lambda_{r,n} = 1 - 2 \sum_{s=0}^{r-1} \binom{p}{s} 2^{-p}, \quad 1 \leq r \leq p. \tag{4.8}$$

(IV) Logistic distribution: $F(x) = (1+e^{-x})^{-1}$, $-\infty < x < \infty$. Here, we have $f(x) = F(x)[1-F(x)]$, and hence, it can be shown that

$$\lambda_{r,n} - 2\left(r - \frac{p+1}{2}\right)/p(p^2-1) \Rightarrow J(r,p) = r \text{ for } 1 \leq r \leq p. \tag{4.9}$$

Thus the optimum S_n reduces to Friedman's χ_r^2.

(V) Normal distribution: $f(x) = (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}x^2\}$, $-\infty < x < \infty$. We denote by $X_{s,p-1}$ the sth smallest observation in a sample of size p-1 from the standard normal distribution, s=1,...,p-1. Then from (3.7), we have

$$(p-1)\beta_{s-1,p-2} = E\{f(X_{s,p-1})\}, \quad s=1,\dots,p-1. \tag{4.10}$$

It may be difficult to evaluate (4.10) for various values of p ans s. Let $\mu_{s,p-1}$ and $\sigma^2_{s,p-1}$ be respectively the mean and variance of $X_{s,p-1}$. Then, for $p \geq 4$, (4.10) may be approximated (crudely) as

(4.11) $$(p-1)\beta_{s-1,p-2} = f(\mu_{s,p-1})\{1+\tfrac{1}{2}(\mu^2_{s,p-1}-1)\sigma^2_{s,p-1}\},\ s=1,\ldots,p-1.$$

As tables for the values of $\mu_{s,p-1}$ and $\sigma^2_{s,p-1}$ are available (cf [6]), (4.11) can be computed for various values of s and p.

It may be noted that for logistic cdf, the rank-sum test is asymptotically optimum (in one way layouts) and the same property holds for the method of n rankings. However, for double exponential cdf, in one way layout, the median test (cf. [4]) is asymptotically optimum, but the statistic (1.2) (with $a = [p/2]$ or $[p/2+1]$) is not so for the method of n rankings. There is also difference in the scores for the normal distribution for the two cases.

We have so far considered the case of complete layouts where each observer ranks all the p objects. Certain balanced incomplete layouts are considered by Durbin (1951) and Bhapkar (1961); the test statistics are straight forward generalizations of (1.1) and (1.2). If $F_1 \equiv \ldots \equiv F_n \equiv F$ then all what has been discussed earlier this section also holds for such (balanced) incomplete layouts. For brevity, the details are omitted. If, however, $F_1,\ldots,F_n$ are not all identical, it is quite difficult to simplify the expression for the power-efficiency of such a scheme and to consider results analogous to (4.1) to (4.11).

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