A METHOD OF CONSTRUCTING PBIB DESIGNS OF T_m TYPE

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Contents

1. Introduction

An association scheme with m associate classes, called T_m type, has been introduced in [5] as a generalization of the so-called triangular type: Let n and m be any given positive integers such that $1 \leq m \leq n/2$, and let us put $v = \binom{n}{m}$. Let, for any given positive integer c, V_c denote the set of all subsets of the set of n integers, $\{1,2,\ldots,n\}$, which contain c distinct integers; the cardinality of V_c is clearly $\binom{n}{c}$.

Suppose now that we are given a set of v treatments, $\{\phi_1,\ldots,\phi_v\}$. Let us take and fix a one-to-one correspondence between these v treatments and the v elements of V_m in an arbitrary way. Under this correspondence, suppose two treatments, ϕ and ϕ', correspond to two elements of V_m, $(r_1,\ldots,r_m)$ and $(r'_1,\ldots,r'_m)$, respectively. Then, ϕ and ϕ' are said to be u-th associates if and only if $(r_1,\ldots,r_m)$ and $(r'_1,\ldots,r'_m)$ have m-u integers in common, u=1,...,m. In particular, any treatment is 0-th associate of itself. This is called <u>the T_m association scheme</u>.

For later use, it is convenient to say that two elements of V_c are <u>in s-relation</u> if they have exactly s integers in common. Then, one can say that two treatments are u-th associates if the corresponding elements of V_m are in (m-u)-relation.

Parameters of the T_m association scheme have been shown [5] to be

$$(1.1) \qquad n_u = \binom{m}{u}\binom{n-m}{u}, \quad u = 0,1,\ldots,m,$$

$$p^t_{su} = \sum_{i=0}^{m-t} \binom{m-t}{i}\binom{t}{m-s-i}\binom{t}{m-u-i}\binom{n-m-t}{s+u-m+i}, \ s,t,\ u = 0,1,\ldots,m.$$

It is evident that these parameters coincide with those of the triangular association scheme when m = 2.

An allocation of v treatments with T_m association scheme to b blocks of size k each in such a way that each treatment occurs in r blocks, each block contains k different treatments and two treatments which are u-th associates occur together in λ_u blocks, $u = 1,\ldots,m$, is called a <u>PBIB design of T_m type</u>. In particular, λ_o is defined to be r.

Parameters of PBIB designs of T_m type satisfy the relations

$$vr = bk \quad \text{and} \quad \sum_{u=o}^{m} n_u \lambda_u = rk. \tag{1.2}$$

In passing, it is noted that non-existence criteria have been given for regular and symmetrical PBIB designs of T_2 (triangular) type [3], of T_3 type [4], and in general of T_m type [5].

In this article, a general method of constructing PBIB designs of T_m type is devised (sections 2 and 3). This gives a class of trivial BIB designs when $m = 1$ (section 4), and two classes of triangular PBIB designs (section 5), most of which have been solved already except for a few designs. PBIB designs of T_3 type which can be constructed by this method are shown in section 6.

2. Preliminary results

Let d be any given integer such that $1 \leqq d \leqq n$. We say, as before, that $(r_1,\ldots,r_m)$ in V_m and $(t_1,\ldots,t_d)$ in V_d are in s-relation to one another if and only if they have s integers in common, $s = 0, 1,\ldots,m$.

Then the following two results are straightforward.

<u>Lemma 2.1</u>. For any given $(r_1,\ldots,r_m)$ in V_m, the number of those $(t_1,\ldots,t_d)$ in V_d which are in s-relation to $(r_1,\ldots,r_m)$ is given by

$$R(s) = \binom{m}{s}\binom{n-m}{d-s}, \quad (s=0,1,\ldots,m), \tag{2.1}$$

using the convention $\binom{x}{y} = 0$ if $x < y$ or $y < 0$.

Lemma 2.2. For any given $(t_1,\ldots,t_d)$ in V_d, the number of those $(r_1,\ldots,r_m)$ in V_m which are in s-relation to $(t_1,\ldots,t_d)$ is given by

$$(2.2) \qquad C(s) = \binom{d}{s}\binom{n-d}{m-s}, \quad (s = 0,1,\ldots,m),$$

with the same convention as in the above lemma.

Note that the above two values, $R(s)$ and $C(s)$, are independent of the choices of $(r_1,\ldots,r_m)$ and $(t_1,\ldots,t_d)$ respectively.

It is also easy to see the following:

Lemma 2.3. Let $(r_1,\ldots,r_m)$ and $(r'_1,\ldots,r'_m)$ be any two members of V_m which are in s-relation to one another. Then, the number of $(t_1,\ldots,t_d)$'s in V_d which are in s-relation to $(r_1,\ldots,r_m)$ and at the same time in s'-relation to $(r'_1,\ldots,r'_m)$ is independent of the choices of $(r_1,\ldots,r_m)$ and $(r'_1,\ldots,r'_m)$ and is given by

$$(2.3) \qquad I_u(s,s') = \Sigma\binom{m-u}{p}\binom{u}{q}\binom{u}{q'}\binom{n-m-u}{d-p-q-q'}, \quad (0 \leqq s,\ s' \leqq n),$$

with the same convention as in Lemma 2.1, where the summation is taken over all non-negative integers p,q and q' restricted by the conditions $p+q = s$ and $p+q' = s'$.

The following two lemmas are easily proved:

Lemma 2.4.

$$(2.4) \qquad \binom{n}{m} R(s) = \binom{n}{d} C(s), \quad s = 0,1,\ldots,m.$$

Lemma 2.5.

$$(2.5) \qquad I_u(s,s') = I_u(s',s) \ , \ u,s,s' = 0,1,\ldots,m,$$

and

$$R(s) \ C(s') = R(s') \ C(s) \ , \ s,s' = 0,1,\ldots,m.$$

Finally, we have

Lemma 2.6. For any given s and s',

$$(2.7) \qquad \sum_{u=o}^{m} n_u \ I_u(s,s') = R(s) \ C(s').$$

Proof. Form a rectangular array, i.e., a matrix A, with $\binom{n}{m}$ rows and $\binom{n}{d}$ columns, listing the members of V_m down the row headings and the members of V_d across the column headings. If $(r_1,\ldots,r_m)$ in V_m is in s-relation to $(t_1,\ldots,t_d)$ in V_d, we enter the integer s in the position whose row corresponds to $(r_1,\ldots,r_m)$ and column to $(t_1,\ldots,t_d)$. There is no harm in assuming that the first R(s) positions on the first row of A are occupied by any given integer s.

Consider first the case s = s'. In this case, both sides of (2.7) give the number of entries "s" contained in the first R(s) columns of A.

Second, consider the case s ≠ s'. In this case, $I_o(s,s') = 0$ and both sides of (2.7) give the number of entries "s'" contained in the first R(s) columns of A.

This proves the lemma.

3. Construction of PBIB designs of T_m type.

Suppose we are given $b = \binom{n}{d}$ blocks.

We make a one-to-one correspondence between these b blocks and the members of V_d in any way.

Now, let us divide the set of integers, $\{0, 1, \ldots, m\}$, into two non-empty subsets, J_o and J_1, and form a v x b matrix N from the matrix A given in the proof of Lemma 2.6 in such a way that the (i,j) position of N receives "1" if the (i,j) element of A belongs to J_1, and "0" otherwise.

Then, we have the following

Theorem 3.1. N is the incidence matrix of a PBIB design of T_m type with parameters

$$v = \binom{n}{m}, \quad b = \binom{n}{d},$$

$$(3.1) \qquad r = \sum_{s \in J_1} R(s), \quad k = \sum_{s \in J_1} C(s),$$

$$\lambda_u = \sum_{(s,s') \in J_1 \times J_1} I_u(s,s'), \quad u = 1, \ldots, m.$$

Proof. From Lemmas 2.1 and 2.2 it follows that each row and each column of N contain exactly r and k entries "1", respectively.

By Lemma 2.3 it is seen that any pair of treatments which are u-th associates occurs in λ_u blocks.

Lemma 2.4 implies that vr = bk. Since $I_o(s,s) = R(s)$, putting $\lambda_o = r$, we can see from Lemma 2.6 that

$$\sum_{u=o}^{m} n_u \lambda_u = \sum_{(s,s') \in J_1 \times J_1} R(s)C(s') = rk.$$

This proves the theorem.

It is noted that, by Lemma 2.5, λ_u in (3.1) can be written as

$$(3.2) \qquad \lambda_u = \sum_{s \in J_1} I_u(s,s) + 2 \sum_{\substack{s < s' \\ s,s' \in J_1}} I_u(s,s'), \quad u = 1, \ldots, m.$$

4. A class of BIB designs.

In the case $m = 1$, (3.1) gives the parameters of BIB designs: Putting $J_o = \{0\}$ and $J_1 = \{1\}$, we have

$$(4.1) \qquad v = n,\ k = d,\ b = \binom{n}{d},\ r = \binom{n-1}{d-1},\ \lambda = \binom{n-2}{d-2},\ (2 \leq d \leq n-1),$$

and putting $J_o = \{1\}$ and $J_1 = \{0\}$,

$$(4.2) \qquad v = n,\ k = n-d,\ \ b = \binom{n}{d},\ r = \binom{n-1}{d},\ \lambda = \binom{n-2}{d},\ (1 \leq d \leq n-2).$$

It is clear that (4.2) is the complementary design of (4.1), and is obtained by changing d in (4.1) for n-d. Hence the class of BIB designs (4.1) is 'self-complementary' in the sense that the complementary design of any design of this class is also contained in the class.

Designs of this class are quire trivial. It would be of some interest, however, to investigate a "splitting property" of some of these designs: For example, if we take $n = 6$ and $d = 3$ in (4.1), we have a BIB design with $v = 6$, $k = 3$, $b = 20$, $r = 10$ and $\lambda = 4$, which is a duplicate of the existent BIB design with $v = 6$, $k = 3$, $b = 10$, $r = 5$ and $\lambda = 2$. In general, suppose the last three parameters of (4.1) have a common diviser $c(> 1)$, and put $b = cb'$, $r = cr'$ and $\lambda = c\lambda'$. Then, what conditions guarantee the existence of BIB design with parameters v, k, b', r' and λ'?

This is left for further investigations.

5. Construction of triangular PBIB designs

In the case $m = 2$, PBIB designs with parameters (1.1) and (3.1) are triangular PBIB designs, for which a general method of construction was proposed in Section 3.

All possible choices of J_1 from $\{0, 1, 2\}$ are

$$(5.1) \qquad J_1 = \{0\},\ J_1 = \{1\},\ J_1 = \{2\}:\ \ J_1 = \{1,2\},\ J_1 = \{0,2\},\ J_1 = \{0,1\},$$

among which the last three cases give the complementary designs of the first three respectively. Hence, in this section, we consider only the first three cases.

For m = 2, the parameters of the association, (1.1), are

$$n_1 = 2(n-2) \quad , \quad n_2 = \binom{n-2}{2},$$

$$(5.2) \qquad p^t_{su} = \sum_{i=0}^{2-t} \binom{2-t}{i}\binom{t}{2-s-i}\binom{t}{2-u-i}\binom{n-2-t}{s+u-2+i} \ , \ s,t,u=0,1,2.$$

5.1. The case $J_1 = \{0\}$

In this case, the parameters (3.1) are given by

$$(5.3) \qquad v=\binom{n}{2},\ k=\binom{n-d}{2},\ b=\binom{n}{d},\ r=\binom{n-2}{d},\ \lambda_1=\binom{n-3}{d},\ \lambda_2 = \binom{n-4}{d},$$

for $1 \leqq d \leqq n-3$.

5.2. The case $J_1 = \{1\}$.

The parameters of the design are given by

$$(5.4) \qquad v=\binom{n}{2},\ k=d(n-d),\ b=\binom{n}{d},\ r=2\binom{n-2}{d-1},\ \lambda_1=\binom{n-2}{d-1},\ \lambda_2 = 4\binom{n-4}{d-2},$$

for $1 \leq d \leq (n-1)/2$; changing d for n-d gives the same design.

5.3. The case $J_1 = \{2\}$

The parameters of the design are given by

$$(5.5) \qquad v=\binom{n}{2},\ k=\binom{d}{2},\ b=\binom{n}{d},\ r=\binom{n-2}{d-2},\ \lambda_1=\binom{n-3}{d-3},\ \lambda_2=\binom{n-4}{d-4},$$

for $3 \leq d \leq n-1$. This gives the same class of triangular PBIB designs given by Masuyama [6], and it is easily seen that the parameters (5.5) are obtained by changing d in (5.3) for n-d.

Some of the designs of the first two classes are listed below together with their complementary designs. Table 5.1 below is the same as that of [6].

Table 5.1.

Triangular PBIB designs with parameters (5.3) and their complementary designs.

n	d	v	k	b	r	λ_1	λ_2		v'	k'	b'	r'	λ'_1	λ'_2	
5	1	10	6	5	3	2	1	*	10	4	5	2	1	0	*
	2	10	3	10	3	1	0	*	10	7	10	7	5	4	*
6	1	15	10	6	4	3	2	*	15	5	6	2	1	0	*
	2	15	6	15	6	3	1	**	15	9	15	9	6	4	**
	3	15	3	20	4	1	0	*	15	12	20	16	13	12	
7	1	21	15	7	5	4	3		21	6	7	2	1	0	*
	2	21	10	21	10	6	3	**	21	11	21	11	7	4	
	3	21	6	35	10	4	1	**	21	15	35	25	19	16	
	4	21	3	35	5	1	0	**	21	18	35	30	26	25	
8	1	28	21	8	6	5	4		28	7	8	2	1	0	*
	2	28	15	28	15	10	6		28	13	28	13	8	4	**
	3	28	10	56	20	10	4	**	28	18	56	36	26	20	
	4	28	6	70	15	5	1	**	28	22	70	55	45	41	
	5	28	3	56	6	1	0	**	28	25	56	50	45	44	
9	1	36	28	9	7	6	5		36	8	9	2	1	0	*
	2	36	21	36	21	15	10		36	15	36	15	9	4	**
	3	36	15	84	35	20	10	**	36	21	84	49	34	24	
	4	36	10	126	35	15	5		36	26	126	91	71	61	
	5	36	6	126	21	6	1		36	30	126	105	90	85	
	6	36	3	84	7	1	0	**	36	33	84	77	71	70	
10	1	45	36	10	8	7	6		45	9	10	2	1	0	*
	2	45	28	45	28	21	15		45	17	17	45	10	4	**
	3	45	21	120	56	35	20		45	24	120	64	43	28	
	4	45	15	210	70	35	15		45	30	210	140	105	85	
	5	45	10	252	56	21	6		45	35	252	196	161	146	
	6	45	6	210	28	7	1		45	39	210	182	161	155	
	7	45	3	120	8	1	0	*	45	42	120	112	105	104	
11	1	55	45	11	9	8	7		55	10	11	2	1	0	*
	-														
	8	55	3	165	9	1	0	*	55	52	165	156	148	147	
12	1	66	55	12	10	9	8		66	11	12	2	1	0	
	-														
	9	66	3	220	10	1	0	*	66	63	220	210	201	200	

* Bose-Clatworthy-Shrikhande [1], Clatworthy [2], Chang Li-chien et al [8].

** Masuyama [7].

Table 5.2.

Triangular PBIB designs with parameters (5.4) and their complementary designs.

n	d	v	k	b	r	λ_1	λ_2	v'	k'	b' (complementary)	r'	λ_1'	λ_2'	
5	1	10	4	5	2	1	0(*)	10	6	5	3	2	1	(*)
	2	10	6	10	6	3	4 *	10	4	10	4	1	2	*
6	1	15	5	6	2	1	0(*)	15	10	6	4	3	2(*)	
	2	15	8	15	8	4	4	15	7	15	7	3	3	BIB
	3	15	9	20	12	6	8	15	6	20	8	2	4	**
7	1	21	6	7	2	1	0(*)	21	15	7	5	4	3	
	2	21	10	21	10	5	4(**)	21	11	21	11	6	5	
	3	21	12	35	20	10	12	21	9	35	15	5	7	
8	1	28	7	8	2	1	0(*)	28	21	8	6	5	4	
	2	28	12	28	12	6	4	28	16	28	16	10	8	
	3	28	18	56	30	15	16	28	10	56	26	11	12	
	4	28	16	70	40	20	24	28	12	70	30	10	14	
9	1	36	8	9	2	1	0(*)	36	28	9	7	6	5	
	2	36	14	36	14	7	4	36	22	36	22	15	12	
	3	36	18	84	42	21	20	36	18	84	42	21	20	
	4	36	20	126	70	35	40	36	16	126	56	21	26	
10	1	45	9	10	2	1	0(*)	45	36	10	8	5	4	
11	1	55	10	11	2	1	0(*)	55	45	11	9	8	7	
12	1	66	11	12	2	1	0	66	55	12	10	9	8	

6. Construction of PBIB designs of T_3 type.

In the case m=3, Section 3 offers a method of construction of T_3 PBIB designs.

In this case, the parameters of the association are given by

$$n_1 = 3(n-3),\ n_2 = 3\binom{n-3}{2},\ n_3 = \binom{n-3}{3},$$

(6.1)

$$p_{su}^{t} = \sum_{i=0}^{3-t}\binom{3-t}{i}\binom{t}{3-s-i}\binom{t}{3-u-i}\binom{n-3-t}{s+u-3+i},\ s,t,u=0,1,2,3,$$

with the same convention as in Lemma 2.1.

To give the parameters of the design, we first give the following:

$$R(0) = \binom{n-3}{d},\ R(1)=3\binom{n-3}{d-1},\ R(2)=3\binom{n-3}{d-2},\ R(3)=\binom{n-3}{d-3},$$

(6.2)

$$C(0) = \binom{n-d}{3},\ C(1) = d\binom{n-d}{2},\ C(2)=(n-d)\binom{d}{2},\ C(3)=\binom{d}{3},$$

and

$$I_1(0,0)=\binom{n-4}{d},\ I_1(0,1)=\binom{n-4}{d-1},\ I_1(1,1) = 2\binom{n-4}{d-1}+\binom{n-4}{d-2},$$

$$I_1(0,2)=0,\ I_1(2,2) = \binom{n-4}{d-2}+2\binom{n-4}{d-3},\ I_1(0,3) = 0,$$

$$I_1(3,3)=\binom{n-4}{d-4},$$

$$I_2(0,0)=\binom{n-5}{d},\ I_2(0,1)=2\binom{n-5}{d-1},\ I_2(1,1)= \binom{n-5}{d-1}+4\binom{n-5}{d-2},$$

(6.3) $$I_2(0,2)=\binom{n-5}{d-2},\ I_2(2,2)=4\binom{n-5}{d-3}+\binom{n-5}{d-4},\ I_2(0,3) = 0,$$

$$I_2(3,3)=\binom{n-5}{d-5},$$

$$I_3(0,0)=\binom{n-6}{d},\ I_3(0,1)=3\binom{n-6}{d-1},\ I_3(1,1)=9\binom{n-6}{d-2},$$

$$I_3(0,2)=\binom{n-6}{d-2},\ I_3(2,2)=9\binom{n-6}{d-4},\ I_3(0,3)=\binom{n-6}{d-3},$$

$$I_3(3,3)=\binom{n-6}{d-6}.$$

All possible choices of J_1 from $\{0,1,2,3\}$ are given by the following 7 pairs, each of which gives a class of T_3 PBIB designs and their complementary designs:

$$\left\{\begin{array}{l} J_1=\{0\} \\ J_1=\{1,2,3\}, \end{array}\right. \left\{\begin{array}{l} J_1=\{1\} \\ J_1=\{0,2,3\}, \end{array}\right. \left\{\begin{array}{l} J_1=\{2\} \\ J_1=\{0,1,3\}, \end{array}\right. \left\{\begin{array}{l} J_1=\{3\} \\ J_1=\{0,1,2\}, \end{array}\right. \tag{6.4}$$

$$\left\{\begin{array}{l} J_1=\{0,1\} \\ J_1=\{2,3\}, \end{array}\right. \left\{\begin{array}{l} J_1=\{0,2\} \\ J_1=\{1,3\}, \end{array}\right. \left\{\begin{array}{l} J_1=\{0,3\} \\ J_1=\{1,2\}. \end{array}\right.$$

6.1. <u>The case</u> $J_1 = \{0\}$.

The parameters of the design (3.1) are given, by (6.2) and (6.3), as follows:

$$\begin{aligned} &v=\binom{n}{3},\ k=\binom{n-d}{3},\ b=\binom{n}{d},\ r=\binom{n-3}{d}, \\ &\lambda_1=\binom{n-4}{d},\ \lambda_2=\binom{n-5}{d},\ \lambda_3=\binom{n-6}{d},\quad (1 \le d \le n-4;\ n \ge 6) \end{aligned} \tag{6.5}$$

6.2. <u>The case</u> $J_1 = \{1\}$.

The parameters (3.1) are given by

$$\begin{aligned} &v=\binom{n}{3},\ k = d\binom{n-d}{2},\ b=\binom{n}{d},\ r=3\binom{n-3}{d-1}, \\ &\lambda_1 = 2\binom{n-4}{d-1}+\binom{n-4}{d-2},\ \lambda_2=\binom{n-5}{d-1}+4\binom{n-5}{d-2},\ \lambda_3=9\binom{n-6}{d-2},\ (1 \le d \le n-3). \end{aligned} \tag{6.6}$$

6.2'. <u>The case</u> $J_1 = \{2\}$.

The parameters (3.1) are given by

$$\begin{aligned} &v=\binom{n}{3},\ k = (n-d)\binom{d}{2},\ b=\binom{n}{d},\ r=3\binom{n-3}{d-2}, \\ &\lambda_1 =\binom{n-4}{d-2}+2\binom{n-4}{d-3},\ \lambda_2=4\binom{n-5}{d-3}+\binom{n-5}{d-4},\ \lambda_3=9\binom{n-6}{d-4},(2 \le d \le n-2). \end{aligned} \tag{6.6'}$$

It is easily noticed that these parameters are obtained by changing d in (6.6) for n-3, and hence (6.6') give the same class as (6.6).

6.1! The case $J_1=\{3\}$.

The parameters (3.1) are given by

$$v=\binom{n}{3},\ k=\binom{d}{3},\ b=\binom{n}{d},\ r=\binom{n-3}{d-3},$$

(6.5')

$$\lambda_1=\binom{n-4}{d-4},\ \lambda_2=\binom{n-5}{d-5},\ \lambda_3=\binom{n-6}{d-6},\quad (4 \leq d \leq n),$$

which give the same class of T_3 designs as (6.5).

6.3. The case $J_1=\{0,1\}$.

The parameters (3.1) are given by

$$v=\binom{n}{3},\ k=\binom{n-d}{3}+d\binom{n-d}{2},\ b=\binom{n}{d},\ r=\binom{n-3}{d}+3\binom{n-3}{d-1},$$

(6.7)
$$\lambda_1=\binom{n-4}{d}+4\binom{n-4}{d-1}+\binom{n-4}{d-2},\ \lambda_2=\binom{n-5}{d}+5\binom{n-5}{d-1}+4\binom{n-5}{d-2},$$

$$\lambda_3=\binom{n-6}{d}+6\binom{n-6}{d-1}+9\binom{n-6}{d-2},\quad (1 \leq d \leq n-2).$$

6.4. The case $J_1=\{0,2\}$.

The parameters (3.1) are given by

$$v=\binom{n}{3},\ k=\binom{n-d}{3}+(n-d)\binom{d}{2},\ b=\binom{n}{d},\ r=\binom{n-3}{d}+3\binom{n-3}{d-2},$$

(6.8)
$$\lambda_1=\binom{n-4}{d}+\binom{n-4}{d-2}+2\binom{n-4}{d-3},\ \lambda_2=\binom{n-5}{d}+2\binom{n-5}{d-2}+4\binom{n-5}{d-3}+\binom{n-5}{d-4},$$

$$\lambda_3=\binom{n-6}{d}+6\binom{n-6}{d-2}+9\binom{n-6}{d-4},\quad (1 \leq d \leq n-1).$$

6.5. The case $J_1=\{0,3\}$.

The parameters (3.1) are given by

$$v=\binom{n}{3},\ k=\binom{n-d}{3}+\binom{d}{3},\ b=\binom{n}{d},\ r=\binom{n-3}{d}+\binom{n-3}{d-3},$$

(6.9)

$$\lambda_1=\binom{n-4}{d}+\binom{n-4}{d-4},\ \lambda_2=\binom{n-5}{d}+\binom{n-5}{d-5},\ \lambda_3=\binom{n-6}{d}+2\binom{n-6}{d-3}+\binom{n-6}{d-6},$$

$$(1 \le d \le n/2),$$

for which changing d for n-d gives the same design.

Some of the designs of these classes are listed below together with their complementary designs.

Table 6.1

T_3 designs with parameters (6.5) and their complementary designs.

n	d	v	k	b	r	λ_1	λ_2	λ_3	v'	k'	b'	r'	λ'_1	λ'_2	λ'_3
6	1	20	10	6	3	2	1	0	(self-compl.)						
	2	20	4	15	3	1	0	0	20	16	15	12	10	9	9
7	1	35	20	7	4	3	2	1	35	15	7	3	2	1	0
	2	35	10	21	6	3	1	0	35	25	21	15	12	10	9
	3	35	4	35	4	1	0	0	35	31	35	31	28	27	27
8	1	56	35	8	5	4	3	2	56	21	8	3	2	1	0
	2	56	20	28	10	6	3	1	56	36	28	18	14	11	9
	3	56	10	56	10	4	1	0	56	46	56	46	40	37	36
	4	56	4	70	5	1	0	0	56	52	70	65	61	60	60
9	1	84	56	9	6	5	4	3	84	28	9	3	2	1	0
	2	84	35	36	15	10	6	3	84	49	36	21	16	12	9
	3	84	20	84	20	10	4	1	84	64	84	64	54	48	45
	4	84	10	126	15	5	1	0	84	74	126	111	101	97	96
	5	84	4	126	6	1	0	0	84	80	126	120	115	114	114

Table 6.2

T_3 designs with parameters (6.6) and their complementary designs.

n	d	v	k	b	r	λ_1	λ_2	λ_3	v'	k'	b'	r'	λ'_1	λ'_2	λ'_3
6	1	20	10	6	3	2	1	0	(self-compl.)						
	2	20	12	15	9	5	5	9	20	8	15	6	2	2	6
	3	20	9	20	9	4	4	0	20	11	20	11	6	6	2
7	1	35	15	7	3	2	1	0	35	20	7	4	3	2	1
	2	35	20	21	12	7	6	9	35	15	21	9	4	3	6
	3	35	18	35	18	9	9	9	35	17	35	17	8	8	8(BIB)
	4	35	12	35	12	5	4	0	35	23	35	23	16	15	11
8	1	56	21	8	3	2	1	0	56	35	8	5	4	3	2
	2	56	30	28	15	9	7	9	56	26	28	13	7	5	7
	3	56	30	56	30	16	15	18	56	26	56	26	12	11	14
	4	56	24	70	30	14	13	9	56	32	70	40	24	23	19
	5	56	15	56	15	6	4	0	56	41	56	41	32	30	26
9	1	84	28	9	3	2	1	0	84	56	9	6	5	4	3
	2	84	77	36	33	31	30	30	84	7	36	3	1	0	0
	3	84	65	84	65	55	50	46	84	19	84	19	9	4	0
	4	84	50	126	75	55	45	33	84	34	126	51	31	21	9
	5	84	34	126	51	31	21	9	84	50	126	75	55	45	33
	6	84	19	84	19	9	4	0	84	65	84	65	55	50	46
	7	84	7	36	3	1	0	0	84	77	36	33	31	30	30

Table 6.3

T_3 designs with parameters (6.7) and their complementary designs. (This class is self-complementary.)

n	d	v	k	b	r	λ_1	λ_2	λ_3	
6	1	20	20	6	6	6	6	6	(Rand. Block)
	2	20	16	15	12	10	9	9	
	3	20	10	20	10	6	4	0	
	4	20	4	15	3	1	0	0	
7	1	35	35	7	7	7	7	7	(Rand. Block)
	2	35	30	21	18	16	15	15	
	3	35	22	35	22	16	13	9	
	4	35	13	35	13	7	4	9	
	5	35	5	21	3	1	0	0	
8	1	56	56	8	8	8	8	8	(Rand. Block)
	2	56	50	28	25	23	22	22	
	3	56	40	56	40	32	28	24	
	4	56	28	70	35	23	17	9	
	5	56	16	56	16	8	4	0	
	6	56	6	28	3	1	0	0	
9	1	84	84	9	9	9	9	9	(Rand. Block)
	2	84	77	36	33	31	30	30	
	3	84	65	84	65	55	50	46	
	4	84	50	126	75	55	45	33	
	5	84	34	126	51	31	21	9	
	6	84	19	84	19	9	4	0	
	7	84	7	36	3	1	0	0	

Table 6.4

T_3 designs with parameters (6.8) and their complementary designs. (This class is self-complementary).

n	d	v	k	b	r	λ_1	λ_2	λ_3
6	1	20	10	6	3	2	1	0
	2	20	8	15	6	2	2	6
	3	20	10	20	10	4	6	0
	4	20	12	15	9	5	5	9
	5	20	10	6	3	2	1	0
7	1	35	20	7	4	3	2	1
	2	35	15	21	9	4	3	6
	3	35	16	35	16	6	8	6
	4	35	19	35	19	9	11	9
	5	35	20	21	12	7	6	9
	6	35	15	7	3	2	1	0
8	1	56	35	8	5	4	3	2
	2	56	26	28	13	7	5	7

Table 6.4. (continued)

	3	56	25	56	25	10	11	12
	4	56	28	70	35	15	19	15
	5	56	31	56	31	16	17	18
	6	56	30	28	15	9	7	9
	7	56	21	8	3	2	1	0
9	1	84	56	9	6	5	4	3
	2	84	42	36	18	11	8	9
	3	84	38	84	38	17	16	19
	4	84	40	126	60	25	30	27
	5	84	44	126	66	31	36	35
	6	84	46	84	46	25	24	27
	7	84	42	36	18	11	8	9
	8	84	28	9	3	2	1	0

Table 6.5.

T_3 designs with parameters (6.9) and their complementary designs.

n	d	v	k	b	r	λ_1	λ_2	λ_3	v'	k'	b'	r'	λ_1'	λ_2'	λ_3'
6	1	20	10	6	3	2	1	0	(self]compl.)						
	2	20	4	15	3	1	0	0	20	16	15	12	10	9	9
	3	20	2	20	2	0	0	0							
7	1	35	20	7	4	3	2	1	35	15	7	3	2	1	0
	2	35	10	21	6	3	1	0	35	25	21	15	12	10	9
	3	35	5	35	5	1	0	2	35	30	35	30	26	25	27
8	1	56	35	8	5	4	3	2	56	21	8	3	2	1	0
	2	56	20	28	10	6	3	1	56	36	28	18	14	11	9
	3	56	11	56	11	4	1	2	56	45	56	45	38	35	36
	4	56	8	70	10	5	0	0	56	48	70	60	55	50	50
9	1	84	56	9	6	5	4	3	84	28	9	3	2	1	0
	2	84	35	36	15	10	6	3	84	49	36	21	16	12	9
	3	84	21	84	21	10	6	3	84	63	84	63	52	48	45
	4	84	14	126	21	6	1	6	84	70	126	105	90	85	90

Finally, we shall show an actual procedure of constructing a T_3 PBIB design.

Example 6.1 Construction of T_3 design with parameters

$$v=20,\ k=8,\ b=15,\quad r=6,\ \lambda_1=2,\ \lambda_2=2,\ \lambda_3=6,$$

$$n_1=9,\ n_2=9,\ n_3=1.$$

(1°) Form the matrix A:

	12	13	14	15	16	23	24	25	26	34	35	36	45	46	56
123	2	2	1	1	1	2	1	1	1	1	1	1	0	0	0
124	2	1	2	1	1	1	2	1	1	1	0	0	1	1	0
125	2	1	1	2	1	1	1	2	1	0	1	0	1	0	1
126	2	1	1	1	2	1	1	1	2	0	0	1	0	1	1
134	1	2	2	1	1	1	1	0	0	2	1	1	1	1	0
135	1	2	1	2	1	1	0	1	0	1	2	1	1	0	1
136	1	2	1	1	2	1	0	0	1	1	1	2	0	1	1
145	1	1	2	2	1	0	1	1	0	1	1	0	2	1	1
146	1	1	2	1	2	0	1	0	1	1	0	1	1	2	1
156	1	1	1	2	2	0	0	1	1	0	1	1	1	1	2
234	1	1	1	0	0	2	2	1	1	2	1	1	1	1	0
235	1	1	0	1	0	2	1	2	1	1	2	1	1	0	1
236	1	1	0	0	1	2	1	1	2	1	1	2	0	1	1
245	1	0	1	1	0	1	2	2	1	1	1	0	2	1	1
246	1	0	1	0	1	1	2	1	2	1	0	1	1	2	1
256	1	0	0	1	1	1	1	2	2	0	1	1	1	1	2
345	0	1	1	1	0	1	1	1	0	2	2	1	2	1	1
346	0	1	1	0	1	1	1	0	1	2	1	2	1	2	1
356	0	1	0	1	1	1	0	1	1	1	2	2	1	1	2
456	0	0	1	1	1	0	1	1	1	1	1	1	2	2	2

(2°) Form the incidence matrix N by changing 0 and 2 for 1 and 1 for 0, and by taking any correspondences (1:1) between treatments and rows, and between blocks and columns of A.

Treatments \ Blocks	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	0	0	0	1	0	0	0	0	0	0	1	1	1
2	1	0	1	0	0	0	1	0	0	0	1	1	0	0	1
3	1	0	0	1	0	0	0	1	0	1	0	1	0	1	0
4	1	0	0	0	1	0	0	0	1	1	1	0	1	0	0
5	0	1	1	0	0	0	0	1	1	1	0	0	0	0	1
6	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0
7	0	1	0	0	1	0	1	1	0	0	0	1	1	0	0
8	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0
9	0	0	1	0	1	1	0	1	0	0	1	0	0	1	0
10	0	0	0	1	1	1	1	0	0	1	0	0	0	0	1
11	0	0	0	1	1	1	1	0	0	1	0	0	0	0	1
12	0	0	1	0	1	1	0	1	0	0	1	0	0	1	0
13	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0
14	0	1	0	0	1	0	1	1	0	0	0	1	1	0	0
15	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0
16	0	1	1	0	0	0	0	1	1	1	0	0	0	0	1
17	1	0	0	0	1	0	0	0	1	1	1	0	1	0	0
18	1	0	0	1	0	0	0	1	0	1	0	1	0	1	0
19	1	0	1	0	0	0	1	0	0	0	1	1	0	0	1
20	1	1	0	0	0	1	0	0	0	0	0	0	1	1	1

(3°) Set up the association scheme:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	1	1	1	1	1	1	2	2	2	1	1	1	2	2	2	2	2	2	3
2	1	0	1	1	1	2	2	1	1	2	1	2	2	1	1	2	2	2	3	2
3	1	1	0	1	2	1	2	1	2	1	2	1	2	1	2	1	2	3	2	2
4	1	1	1	0	2	2	1	2	1	1	2	2	1	2	1	1	3	2	2	2
5	1	1	2	2	0	1	1	1	1	2	1	2	2	2	2	3	1	1	2	2
6	1	2	1	2	1	0	1	1	2	1	2	1	2	2	3	2	1	2	1	2
7	1	2	2	1	1	1	0	2	1	1	2	2	1	3	2	2	2	1	1	2
8	2	1	1	2	1	1	2	0	1	1	2	2	3	1	2	2	1	2	2	1
9	2	1	2	1	1	2	1	1	0	1	2	3	2	2	1	2	2	1	2	1
10	2	2	1	1	2	1	1	1	1	0	3	2	2	2	2	1	2	2	1	1
11	1	1	2	2	1	2	2	2	2	3	0	1	1	1	1	2	1	1	2	2
12	1	2	1	2	2	1	2	2	3	2	1	0	1	1	2	1	1	2	1	2
13	1	2	2	1	2	2	1	3	2	2	1	1	0	2	1	1	2	1	1	2
14	2	1	1	2	2	2	3	1	2	2	1	1	2	0	1	1	1	2	2	1
15	2	1	2	1	2	3	2	2	1	2	1	2	1	1	0	1	2	1	2	1
16	2	2	1	1	3	2	2	2	2	1	2	1	1	1	1	0	2	2	1	1
17	2	2	2	3	1	1	2	1	2	2	1	1	2	1	2	2	0	1	1	1
18	2	2	3	2	1	2	1	2	1	2	1	2	1	2	1	2	1	0	1	1
19	2	3	2	2	2	1	1	2	2	1	2	1	1	2	2	1	1	1	0	1
20	3	2	2	2	2	2	2	1	1	1	2	2	2	1	1	1	1	1	1	0

From (2^{o}), we thus have an allocation of 20 treatments with T_3 association scheme (3^{o}) into 15 blocks of size 8 each:

Blocks	Treatments							
1	1	2	3	4	17	18	19	20
2	1	5	6	7	14	15	16	20
3	2	5	8	9	12	13	16	19
4	3	6	8	10	11	13	15	18
5	4	7	9	10	11	12	14	17
6	1	8	9	10	11	12	13	20
7	2	6	7	10	11	14	15	19
8	3	5	7	9	12	14	16	18
9	4	5	6	8	13	15	16	17
10	3	4	5	10	11	16	17	18
11	2	4	6	9	12	15	17	19
12	2	3	7	8	13	14	18	19
13	1	4	7	8	13	14	17	20
14	1	3	6	9	12	15	18	20
15	1	2	5	10	11	16	19	20

Note that the upper half of the incidence matrix N in (2^{o}) is the incidence matrix of a BIB design with parameters

$$v=10,\ k=4,\ b=15,\ r=6,\ \lambda=2.$$

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References

[1] R.C. Bose, W.H. Clatworthy and S.S. Shrikhande (1954), Tables of Partially Balanced Designs with Two Associate Classes, Inst. Stat., UNC, Reprint Ser., No. 50.

[2] W.H. Clatworthy (1956), Contributions on Partially Balanced Incomplete Block Designs with Two Associate Classes, NBS, App. Math. Ser., No. 47.

[3] J. Ogawa (1959), A necessary condition for existence of regular and symmetrical experimental designs of triangular type, Ann. Math. Statist., 30, 1063-1071.

[4] K. Kusumoto (1965), A necessary condition for existence of regular and symmetrical PBIB designs of T_3 type, Ann. Inst. Stat. Math., Tokyo, 17, 149-161.

[5] M. Ogasawara (1965), A necessary condition for the existence of regular and symmetrical PBIB designs of Tm type, Inst. Stat., UNC, Mimeo Ser. No. 418.

[6] M. Masuyama (1965), A class of triangular designs, Rep. Stat. App. Res., JUSE, 12, 32-34.

[7] M. Masuyama (1965), Cyclic generation of triangular PBIB designs, Rep. Stat. App. Res., JUSE, 12, 73-81.

[8] Chang Li-chien, Liu Chang-wen and Liu Wan-ru (1965), Incomplete block designs with triangular parameters for which $k \leq 10$ and $r \leq 10$. Scientia Sinica, 14, 329-338.