A CHARACTERIZATION OF TETRAHEDRAL GRAPHS

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Abstract

A tetrahedral graph may be defined as a graph G, whose vertices may be identified with the $n(n-1)(n-2)/6$ unordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two symbols in common. If $d(x,y)$ denotes the distance between two vertices x and y and $\Delta(x,y)$ denotes the number of vertices adjacent to both x and y, then a tetrahedral graph G has the following properties: (b_1) The number of vertices is $n(n-1)(n-2)/6$. (b_2) G is connected and regular of valence $3(n-3)$. (b_3) For any two adjacent vertices x and y, $\Delta(x,y)=n-2$. (b_4) $\Delta(x,y)=4$ if $d(x,y)=2$. We show that if $n > 16$, then any graph G (without loops and with utmost one edge connecting two vertices) having the properties (b_1)-(b_4) must be a tetrahedral graph.

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I. Introduction

1. We shall consider only finite undirected graphs, with at most one edge joining a pair of vertices and no edge joining a vertex to itself.

The valence d(u) of the vertex u of a graph G, is defined to be the number of vertices adjacent to u. If all vertices of G have the same valence n_1, the graph G is said to be a regular graph of valence n_1.

A chain $x_1, x_2, \ldots, x_n$ is a sequence of vertices of G, not necessarily all different, such that any two consecutive vertices in the chain are adjacent. Thus the pairs $(x_1,x_2)(x_2,x_3), \ldots (x_{n-1}, x_n)$ are edges of G. The number of edges n-1 is said to be the length of the chain. The chain is said to begin at x_1 and terminate at x_n, and is said to join x_1 and x_n.

The graph G is said to be connected if for every pair of distinct vertices x and y, there is a chain beginning at x and terminating at y. For a connected graph the distance d(x,y) between two vertices x and y is defined to be the length of the shortest chain joining x and y.

For any two vertices u and v, Δ (u,v) denotes the number of vertices w, adjacent to both u and v. If u and v are adjacent, i.e. d(x,y)=1, Δ (u,v) is

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called the edge degree of the edge (u, v). A regular graph G for which all edges have the same edge-degree Δ, is said to be edge-regular, with edge-degree Δ.

2. A graph G is said to be triangular if the vertices of G can be identified with unordered pairs on n symbols, so that two pairs are adjacent if and only if, the correspondiag pairs have one symbol in common. A triangular graph G obviously possesses the following properties:

(a_1) The number of vertices in G is $n(n-1)/2$.

(a_2) G is regular of valence $2(n-2)$.

(a_3) G is edge regular with edge degree n-2, i.e. $\Delta(u,v) = n-2$ if u and v are adjacent.

(a_4) $\Delta(u,v) = 4$, if u and v are non-adjacent.

Connor [4] showed (with a somewhat different terminology) that for $n > 8$, the properties (a_1)-(a_4), characterize a triangular graph, i.e. if G has the properties (a_1)-(a_4), then G must be triangular. Shrikhande [9], Li-chien [7,8] and Hoffman [5,6] completed Conner's work by demonstrating that the same result holds for $n < 8$, but if n=8, then there exist other non-isomorphic graphs which are not triangular.

3. In this paper we consider the problem of characterization of tetrahedral graphs. A tetrahedral graph may be defined as a graph G whose vertices can be identified with unordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two common symbols. It is readily seen that G has the following properties:

(b_1) The number of vertices in G is $n(n-1)(n-2)/6$.

(b_2) G is connected and regular of valence $3(n-3)$.

(b_3) G is edge-regular, with edge degree n-2, i.e. $\Delta(x,y)=n-2$ if $d(x,y)=1$.

(b_4) $\Delta(x,y)=4$ if $d(x,y)=2$.

In Section III we prove that for $n > 16$, the properties (b_1)-(b_4) characterize a tetrahedral graph, i.e., if G possesses properties (b_1)-(b_4) and $n > 16$, then it is possible to establish a (1,1) correspondence between the vertices of G, and the unordered triplets on n symbols, such that two vertices of G are adjacent if and only if the corresponding triplets have a pair of symbols in common.

The proof is based on certain theorems regarding the existence or non-existence of cliques and claws in edge-regular graphs, which are proved in Section II. These theorems generalize the previous work of Bruck [2] and (one of the authors) Bose [1]. Other applications of these theorems will be given in subsequent communications.

II. Claws and Cliques in Edge-regular graphs.

1. In this section we shall consider a graph G which has the following properties:

(c_1) G is connected and regular of valence $r(k-1)$.

(c_2) G is edge-regular with edge-degree $(k-2) + \alpha$.

(c_3) $\Delta(x,y) \leq 1 + \beta$, for all pairs of non-adjacent vertices, x and y of G.

In the above r, k, α, β, are fixed positive integers, such that $r \geq 1$, $k \geq 2$, $\alpha \geq 0$, $\beta \geq 0$ and $r\beta - 2\alpha \geq 0$. All the lemmas and theorems in paragraphs 2 and 3 of this section are about the edge regular graph G, with properties (c_1), (c_2), (c_3).

We define here some functions of the parameters r, k, α, and β, which play an important role in subsequent developments.

(2.1.1) $$\gamma(r,\alpha) = 1 + (r-1)\alpha .$$

(2.1.2) $$q(r,\alpha) = 1 + (2r-1)\alpha .$$

(2.1.3) $$\rho(r,\alpha,\beta) = 1 + \beta + (2r-1)\alpha .$$

(2.1.4) $$p(r,\alpha,\beta) = 1 + \tfrac{1}{2}(r+1)(r\beta - 2\alpha) .$$

We shall denote as usual the cardinality of a set S by $|S|$.

A clique K of a graph is a set of vertices adjacent to each other. A clique K will be called complete if we cannot find a vertex x, not contained in K such that $x \cup K$ is a clique. Thus a complete clique cannot be extended to a larger clique by the adjunction of a new vertex belonging to the graph.

Now consider the graph G with the properties, (c_1), (c_2), (c_3). A clique K of G will be called a major clique if

(2.1.5) $$|K| \geq 1 + k - \gamma(r,\alpha) = k - (r-1)\alpha .$$

A clique K of G will be called a grand clique if it is both major and complete.

A claw [p, S] of G, consists of a vertex p, the vertex of the claw and a non-empty set S of vertices of G, not containing p, such that p is adjacent to every vertex in S, but any two vertices in S are non-adjacent. The order of the claw is defined to be the number $s = |S|$.

In the next two paragraphs of this section, we obtain a number of theorems about claws and cliques in a graph G having the properties (c_1), (c_2), (c_3).

These theorems are very similar to those obtained in an earlier paper [1], but are now proved under less restrictive conditions, thereby substantially increasing their range of applicability. This will be illustrated in Section III, where they will be used to obtain a geometric characterization of tetrahedral graphs (defined in Section I). Further applications will be given in subsequent communciations.

2. Theorem (2.2.1). If $k > p(r,\alpha,\beta)$, there cannot exist a claw of order $r+1$ in G.

Suppose there exists in G a claw $[p, S]$ of order s. Let T be the set of vertices of G, not belonging to $[p, S]$, and adjacent to p. Let $f(x)$ denote the number of vertices q in T, such that q is adjacent to exactly x vertices in S. Counting the number of vertices in T we have from (c_1),

$$\sum_{x=0}^{s} f(x) = r(k-1)-s = rk-r-s, \tag{2.2.1}$$

Counting the number of ordered pairs (b,q) where b and q are adjacent, b belongs to S, and q belongs to T, we have from (c_2)

$$\sum_{x=0}^{s} x\, f(x) = s(k-2+\alpha). \tag{2.2.2}$$

Again counting the triplets (b_1, b_2, q), where b_1, b_2 is an ordered pair of vertices in S, q is a vertex in T, and b_1, b_2 are both adjacent to q, we have from (c_3)

$$\sum_{x=0}^{s} x(x-1)\, f(x) \leq s(s-1)\beta. \tag{2.2.3}$$

If a claw of order $r+1$ exists, putting $s=r+1$, we have from (2.2.1), (2.2.2) and (2.2.3)

$$(2.2.4) \qquad f(0) + \frac{1}{2} \sum_{x=1}^{r+1} (x-1)(x-2)\, f(x) \leq -k + 1 + \frac{1}{2}(r+1)(r\beta-2\alpha) = -k+p(r,\alpha,\beta)$$

Since the left hand side is essentially non-negative whereas $k > p(r,\alpha,\beta)$ by hypothesis, we have a contradiction. This proves our theorem.

Theorem (2.2.2). If $k > \gamma(r,\alpha)$, then any claw of G of order $s < r$, can be extended to a claw of order r.

Suppose there exists in G a claw [p,S] of order s. From (2.2.1) and (2.2.2)

$$(2.2.5) \qquad f(0) - \sum_{x=1}^{s} (x-1)\, f(x) = (k-1)(r-s)-\alpha s.$$

If $k > \gamma(r,\alpha) = 1+(r-1)\alpha$, and $s < r$

$$f(0) > \alpha\, r(r-s-1) \geq 0$$

Hence $f(0) > 0$, which shows that there exists a vertex q in T, which is not adjacent to any vertex of S. If $S^* = S \cup q$ we can extend the claw [p,S] to the claw [p,S*] of order s+1. If $s+1 < r$ we can continue the process till we arrive at a claw of order r.

Theorem (2.2.3). Given a claw [p,S] of G of order r-1 there exist at least $k-\gamma(r,\alpha)$ distinct vertices q of G such that $[p, S \cup q]$ is a claw of order r.

Putting $s=r-1$, in (2.2.5) we have

$$\begin{aligned} f(0) &\geq k-1-\alpha(r-1) \\ &= k-\gamma(r,\alpha)\,. \end{aligned}$$

Hence there exist at least $k-\gamma(r,\alpha)$ vertices q in T such that [p,SUq] is a claw of order r.

3. Lemma (2.3.1). If $k > \gamma(r,\alpha)$ and if G has no claw of order r+1, then any pair of adjacent vertices p and q is contained in at least one major clique.

From theorem (2.2.2) we can extend the claw [p,q] to a claw [p,S] of order r. Let $b_1, b_2, \ldots b_r$ be vertices in S other than q. Let Ω be the set of vertices ω, which when adjoined to S-q give a claw [p,S*] of order r, where $S^* = (S-q)U\omega$. Of course q is contained in Ω and from theorem (2.2.3)

$$|\Omega| \geq k - \gamma(r,\alpha).$$

The vertices in Ω are all adjacent to one another. If any two were not adjacent they could be added to $b_1, b_2, \ldots b_r$ to give a claw of order r+1. Let $K = pU\Omega$. Then K is a major clique since

$$|K| \geq 1 + k - \gamma(r,\alpha).$$

Corollary (2.3.1). In Lemma (2.3.1), the hypothesis may be replaced by

$$k > \max\ [\gamma(r,\alpha),\ \ p(r,\alpha,\beta)].$$

This follows at once from theorem (2.2.1).

Corollary (2.3.2). When the conditions of Lemma (2.3.1), or corollary (2.3.1) are satisfied, then any pair of adjacent vertices p and q, is contained in at least one grand clique.

We can extend the major clique K by adding new vertices till it is complete and therefore a grand clique.

Lemma (2.3.2). If K and L are cliques of G, and $K \cup L$ is not a clique, then $|K \cap L| \leq 1+\beta$.

Since $K \cup L$ is not a clique, there exists in $K \cup L$ a pair of vertices c and d which are non-adjacent, such that c belongs to K and d to L. Any vertex belonging to $K \cap L$ must be adjacent to both c and d. Hence $\Delta(c,d) \geq |K \cap L|$. The lemma follows from (c_3).

Lemma (2.3.3). If K and L are cliques of G and $K \cap L$ contains at least two vertices a and b, then $|K \cup L| \leq k + \alpha$.

Every vertex in $K \cup L$, other than a and b is adjacent to both a and b. Hence

$$\Delta(a,b) \geq |K \cup L| - 2.$$

It follows from (c_1), that $|K \cup L| \leq k + \alpha$.

Lemma (2.3.4). If K and L are cliques of G, $K \cup L$ is not a clique and $K \cap L$ contains at least two vertices, then

$$|K| + |L| \leq 1 + k + \alpha + \beta.$$

This follows at once from Lemmas (2.3.2) and (2.3.3) by noting that

$$|K| + |L| = |K \cap L| + |K \cup L|.$$

Theorem (2.3.1A). If $k > \rho(r,\alpha,\beta)$ and G has no claw of order r+1, then any pair of adjacent vertices p and q is contained in one and only one grand clique.

The existence of at least one grand clique containing p and q follows at once from corollary (2.3.2) by noting that $\rho(r, \alpha, \beta) \geq \gamma(r, \alpha)$.

Suppose there exist at least two distinct grand cliques K and L both containing the adjacent vertices p and q. Since K and L are complete, $K \cup L$ is not a clique. Hence from Lemma (2.3.4)

$$|K| + |L| \leq 1 + k + \alpha + \beta .$$

But K and L are both major cliques. Hence

$$|K| + |L| \geq 2 \{ k-(r-1)\alpha \}.$$

which shows that

$$k \leq 1 + \beta + (2r-1)\alpha = \rho(r, \alpha, \beta),$$

contrary to the hypothesis.

The previous theorem can be written in the following alternative form:

Theorem (2.3.1B). If $k > \max [\rho(r, \alpha, \beta) \; p(r, \alpha, \beta)]$, then any two adjacent vertices p and q of G are contained in exactly one grand clique.

This follows at once from Theorems (2.2.1) and (2.3.1A).

Theorem (2.3.2A). If $k > q(r,\alpha)$, there exists no claw of order $r+1$ in G, and every pair of adjacent vertices of G is contained in utmost one grand clique of G, then each vertex of G is contained in exactly r grand cliques.

If we note that $q(r,\alpha) \geq \gamma(r,\alpha)$ it follows from corollary (2.3.2), that any pair of adjacent vertices is contained in exactly one grand clique. Again from Theorem (2.2.2), p is the vertex of at least one claw of order r. Let $[p,S]$ be a claw of order r, where $S = \{b_1, b_2,\ldots,b_r\}$. As in Theorem (2.2.1), let T be the set of vertices not belonging to S, which are adjacent to p.

Let H_j be the set consisting of p, b_j and q belonging to T, such that q is adjacent to b_j but not adjacent to b_i, $i \neq j$. As in Theorem (2.2.1) let f(x) denote the number of vertices in T, which are adjacent to exactly x vertices in S. Then f(0) = 0, otherwise there would exist a claw of ord r+1. Putting s=r in (2.2.1) and (2.2.2), we have

(2.3.1) $$\sum_{x=1}^{r} f(x) = r(k-2).$$

(2.3.2) $$\sum_{x=1}^{r} x\, f(x) = r(k-2+\alpha).$$

Hence

$$\sum_{x=2}^{r} (x-1)\, f(x) = r\alpha .$$

Since

$$-f(1) + \sum_{x=1}^{r} f(x) = \sum_{x=2}^{r} f(x)$$

$$\leq \sum_{x=2}^{r} (x-1)\, f(x)$$

$$= r\,\alpha ,$$

it follows that

(2.3.3) $$f(1) \geq r(k-2-\alpha).$$

Any two vertices of H_j are adjacent to one another, otherwise there would exist a claw of order r+1. Thus H_j is a clique.

Put $H_j^* = H_j - (b_j \cup p)$. Then H_j^* consists of exactly those vertices of T which are adjacent to b_j but to no other vertex of S. Hence $H_1^*, H_2^*, \ldots H_r^*$ are disjoint sets, and the total number of vertices in these sets in f(1).

Now there is a unique grand clique K_j containing b_j and p. The number of vertices in K_j cannot be less than the number of vertices in H_j. If possible let $|K_j| < |H_j|$. Since K_j is a grand clique it follows that H_j is a major clique and contained in some grand clique K_j'. Since b_j and p are contained in K_j and K_j', they must coincide. Hence K_j contains H_j, which contradicts $|K_j| < |H_j|$.

Now consider the r grand cliques, $K_1, K_2, \ldots, K_r$. Then K_1-p, K_2-p,..., K_r-p are disjoint. For if K_i-p and K_j-p, $i \neq j$, have a common vertex q, then K_i and K_j would coincide, and would contain both b_i and b_j which is impossible since b_i is not adjacent to b_j. Remembering (2.3.3), we have

$$(2.3.4) \qquad \sum_{j=1}^{r} |K_j - p| \geq \Sigma\, |H_j - p| = r + \sum_{j=1}^{r} |H_j^*| = r + f(1) \geq r(k-1-\alpha).$$

If possible, suppose there is another grand clique K_{r+1} containing p. The vertices in K_{r+1}-p must be disjoint from the vertices in K_1-p, K_2-p, ... , K_r-p. Since K_{r+1} is a grand and therefore a major clique, $|K_{r+1} - p| \geq k-1-(r-1)\alpha$.

But from (c_1), the number of vertices adjacent to p is exactly r(k-1). Hence from (2.3.4)

$$r(k-1) \geq \sum_{j=1}^{r+1} |K_j - p| \geq r(k-1) + k - 1 - (2r-1)\alpha.$$

$$\therefore \ k \leq 1 + (2r-1)\alpha = q(r,\alpha)$$

which is a contradiction. Thus p is contained in exactly r grand cliques.

Theorem (2.3.2B). If $k > \rho(r, \alpha, \beta)$, and there exists no claw of order r+1 in G, then each vertex of G is contained in exactly r grand cliques.

This follows at once from the previous theorem and Theorem (2.3.1A), remembering $\rho(r, \alpha, \beta) \geq q(r,\alpha)$.

Theorem (2.3.2C). If $k > \max\,[\rho(r, \alpha,\beta), p(r, \alpha, \beta)]$, then each vertex in G is contained in exactly r grand cliques.

This follows from the previous theorem and theorem (2.2.1).

III. Characterization of Tetrahedral Graphs

1. As mentioned earlier in the introducation a tetrahedral graph G is a graph whose vertices can be identified with the $n(n-1)(n-2)/6$ unordered triplets on n symbols, such that any two vertices are adjacent if and only if the corresponding triplets have a pair of common symbols. Then G clearly possesses the properties (b_1)-(b_4) given in Section I, paragraph 3. We shall here prove that if $n > 16$, the converse also holds.

In the following lemmas G is a graph satisfying the conditions (b_1)-(b_4), and such that $n > 16$.

If we set $\gamma = 3$, $K = n-2, \alpha = 2$, $\beta = 3$, then the conditions (b_2), (b_3), (b_4) are the same as (c_1),(c_2), (c_3) of Section II. Also from (2.1.1), (2.1.3), and (2.1.4)

$$\gamma(r,\alpha) = 5, \quad \rho(r,\alpha,\beta) = 14, \quad p(r,\alpha,\beta) = 11.$$

Hence a clique K of G is a major clique if $|K| \geq n-6$, and if it is complete it is a grand clique. Since $n > 16$, the condition $K > \max\,[\rho(r,\alpha,\beta)\,,\, p(r,\alpha,\beta)]$ is satisfied. Hence from theorems (2.3.1B) and (2.3.2C) we have:

Lemma (3.1.1). Any two adjacent vertices of G are contained in exactly one grand clique. Each vertex of G is contained in exactly 3 grand cliques.

The unique grand clique containing any two given adjacent vertices x and y, may be denoted by $K(x,y)$.

The null set will be denoted by $\emptyset$.

The following six lemmas are directed towards proving that $|K| = n-2$, for any grand clique K in G.

Lemma (3.1.2). If K is a grand clique in G, then

$$n-4 \leq |K| \leq n.$$

Let x and y be any two vertices in K. There are $|K|-2$ vertices in K other than x and y, and by definition each of these is adjacent to both x and y. If $|K| > n$, then $|K|-2 > n-2$ which would contradict (b_3). Hence

$$|K| \leq n$$

Let A be the set of all vertices adjacent to both x and y but not contained in K. Then from (b_4)

$$|K|-2 + |A| = n-2.$$

i.e.

(3.1.1) $$|K| + |A| = n.$$

If S_1 and S_2 are the two other cliques containing x and T_1, T_2 be those containing y, then any vertex in A must be of the form

$$Z_{ij} = S_i \cap T_j, \quad i,j = 1,2,$$

if it exists at all. Since two distinct cliques can have at most one vertex in common, we have

$$|A| \leq 4.$$

Hence from (3.1.1)

$$|K| \geq n-4.$$

Lemma (3.1.3). If K is a grand clique in G, then

$$|K| \neq n-4.$$

Suppose $|K| = n-4$, and let x, y, A, S_i, T_j (i,j = 1,2) be as in lemma (3.1.2). Since there are only n-6 vertices in K adjacent to both x and y, we must have $|A| = 4$. Hence, $S_i \cap T_j \neq \emptyset$ for i,j = 1,2. It follows from (b_2) that

(3.1.2) $$|K-x| + |S_1-x| + |S_2-x| = 3n-9.$$

Since $$|K-x| = n-5, \quad \text{we have}$$

(3.1.3) $$|S_1-x| + |S_2-x| = 2n-4.$$

Thus, at least one S_i-x, say S_1-x, has at least $n-2$ vertices, and therefore

$$(3.1.4) \qquad |S_1| \geq n-1.$$

Consider the two vertices x and z_{11} in S_1. They are both adjacent to $|S_1|-2$ other vertices in S_1 as well as to y and z_{21} not in S_1. From (3.1.4) it follows that $\Delta(x, z_{11}) \geq n-1$, which contradicts (b_3).

Lemma (3.1.4). If K is a grand clique in G, then $|K| \neq n-3$.

Suppose $|K| = n-3$. Then from (3.1.1), $|A| = 3$, and one of the grand cliques, say S_1, containing x must intersect both of the other two grand cliques containing y and the other grand clique S_2, containing x, must intersect exactly one of T_1, T_2, say T_2.

Since $|K-x| = n-4$, from (3.1.2) we have

$$(3.1.5) \qquad |S_1-x| + |S_2-x| = 2n-5 .$$

It follows then, that one of S_1-x and S_2-x has at least n-2 vertices. If $|S_1-x| \geq n-2$, then $|S_1| \geq n-1$ and considering the vertices x and z_{12} we have at most one vertex not in S_1 adjacent to both. This is contradicted since z_{22} and y are adjacent to both. If $|S_2| \geq n-1$, the same argument can be applied to x and z_{22}.

Lemma (3.1.5). If K is a grand clique in G, then $|K| \neq n-1$.

Suppose $|K| = n-1$, then from (3.1.1), $|A| = 1$. Hence, exactly one of S_1, S_2 intersects exactly one of T_1, T_2.

Suppose, $z_{11} = S_1 \cap T_1$ and $S_2 \cap T_2 = \emptyset$.

Since $|K-x| = n-2$, it follows from (3.1.2) that

$$|S_1-x| + |S_2-x| = 2n-7.$$

Hence for one i, $|S_i-x| \leq n-4$ and then $|S_i| \leq n-3$. But by lemmas (3.1.2), (3.1.3), (3.1.4), we have $|K| \geq n-2$ for every K in G.

Lemma (3.1.6). If K is a grand clique in G, then

$$|K| \neq n .$$

Suppose $|K| = n$, then $|A| = 0$ and hence $S_i \cap T_j = \emptyset$, $(i = 1, 2,)$ $(j = 1, 2,)$. Since $|K-x| = n-1$, from (3.1.2) we have

$$|S_1-x| + |S_2-x| = 2n-8.$$

Hence at least one of S_1-x and S_2-x, say S_1-x, has at most $n-4$ vertices. Thus $|S_1| \leq n-3$ which contradicts at least one of the lemmas (3.1.2), (3.1.3), (3.1.4).

Lemma (3.1.7). If K is a grand clique in G, then $|K| = n-2$.

This follows immediately from lemmas (3.1.2)—(3.1.6).

Lemma (3.1.8). Let x be a vertex in G and let L be a grand clique not containing x. Then the three grand cliques K_1, K_2, K_3 containing x cannot all intersect L.

Suppose K_1, K_2, K_3 all meet L, and Let $y_i = K_i \cap L$, $i = 1, 2, 3$. From Lemma (3.1.1) the vertices y_i, $i = 1, 2, 3$ are all distinct. Let S_i, $i = 1, 2, 3$ be the third grand clique containing y_i in addition to K_i and L.

Suppose $S_i \cap K_j \neq \emptyset$, for some pair i,j, $i \neq j$, and let $z = S_i \cap K_j$. Then, the vertices z, y_j and y_k are such that, each of these is adjacent to both x and y_i but none contained in the grand clique K_i containing x and y_i (i, j, K are different). From lemma (3.1.7) $|K_i| = n-2$, and hence the $n-4$ vertices in K_i, other than x and y_i, together with the three vertices z, y_j and y_k constitute a set of $n-1$ vertices which are adjacent to both

x and y_i . This contradicts (b_3). Hence

(3.1.6) $$S_i \cap K_j = \emptyset \text{ for } i \neq j, \quad i, j=1, 2, 3$$

By lemma (3.1.7) there are $n-5$ vertices in L other than y_1, y_2 and y_3. Each of these vertices must be non-adjacent to x , for otherwise we would have 4 grand cliques containing x contradicting lemma (3.1.1) . Let z be one of these $n-5$ vertices in $L-y_1-y_2-y_3$. Then, $d(z,x) = 2$ and by (b_4) there are exactly 4 vertices adjacent to both z and x. Clearly three of these are y_1, y_2, y_3 and the fourth vertex must be on some K_i, $(1 \leq i \leq 3)$ and be distinct from x and y_i. Thus for each of the $n-5$ vertices in $L-y_1-y_2-y_3$, there is exactly one vertex in the set

$$T = \bigcup_{i=1}^{3} \{ K_i-x-y_i \}$$

which is adjacent to it. Let us define three sets A_1, A_2, A_3 where A_i consists of all those vertices in K_i-x-y_i which are adjacent ot at least one vertex of L other than y_i . Since from (3.1.6) no vertex $z' \in K_i-x-y_i$ can be adjacent to y_j , $j \neq i$, it is clear that A_i consists of all those vertices in K_i-x-y_i which are adjacent to a vertex of $L-y_1-y_2-y_3$.

Now, since each vertex in $L-y_1-y_2-y_3$ is adjacent to exactly one vertex T, and since

$$|L-y_1-y_2-y_3| = n-5,$$

we have

(3.1.7) $$\sum_{i=1}^{3} |A_i| \leq n-5.$$

Again, since $|K_i-x-y_i| = n-4$, and the K_i-x-y_i , $i = 1, 2, 3$ are disjoint, we have

$$(3.1.8) \qquad |T| = \sum_{i=1}^{3} |K_i-x-y_i| = 3n-12.$$

If we define B_i to be the set of all those vertices in K_i-x-y_i which are adjacent to no vertex of L other than y_i, then clearly A_i is disjoint from B_i and

$$A_i \cup B_i = K_i-x-y_i \quad .$$

Hence, from (3.1.8),

$$(3.1.9) \qquad \sum_{i=1}^{3} |A_i| + \sum_{i=1}^{3} |B_i| = (3n-12)$$

Combining (3.1.7) and (3.1.9), we have

$$(3.1.10) \qquad \sum_{i=1}^{3} |B_i| \geq 2n-7 .$$

It follows from (3.1.10) that for at least one $i = i_0$, we have

$$(3.1.11) \qquad |B_{i_0}| \geq \frac{2n-7}{3} .$$

Now, let $b \in B_{i_0}$ and consider the vertices b and y_{i_0}. Since $|K_{i_0}| = n-2$ from lemma (3.1.7), it follows from (b_3) that there are exactly two vertices adjacent to both b and y_{i_0} and not in K_{i_0}. From the definition of B_{i_0}, neither of these two vertices can be in L. Hence they must both lie in $S_{i_0}-y_{i_0}$.

Let b and b' be two distinct vertices in B_{i_0} and let s_1, s_2 be the two vertices of $S_{i_0}-y_{i_0}$ adjacent to b and s_1', s_2' be the two vertices of $S_{i_0} - y_{i_0}$ adjacent to b'. Suppose

$$s_i' = s_j, \quad \text{for some pair } i,j, \ 1 \le i, j \le 2 .$$

Without loss of generality, suppose $s_1' = s_1$. Let M be the grand clique containing the adjacent vertices b and s_1. Then, there are n-4 vertices in M adjacent to both b and s_1 as well as 3 others not in M, namely, s_2, b' and y_{i_0} It follows that, there are at least n-1 vertices adjacent to both b and s_1 which contradicts (b_3). Hence the 4 vertices s_1, s_2, s_1', s_2' must all be distinct. Hence the number of distinct vertices in $s_{i_0} - y_{i_0}$ adjacent to vertices in B_{i_0} is exactly $2|B_{i_0}|$. Consequently

$$(3.1.12) \qquad |S_{i_0} - y_{i_0}| \ \ge \ 2\,|B_{i_0}|$$

or from (3.1.11) and lemma (3.1.7) ,

$$n-3 \ \ge \ 2\left(\frac{2n-7}{3}\right),$$

which contradicts the assumption $n > 16$. This completes the proof.

Lemma (3.1.9). Let x and y be two adjacent vertices in G and let K be the grand clique containing both x and y. Let S_1, S_2 be the other two grand cliques containing x and T_1, T_2 be the other two grand cliques containing y. Then the grand cliques S_i may be put in (1,1) correspondence with grand cliques T_j so that only corresponding cliques intersect,

$$S_i \cap T_i \neq \emptyset$$

$$S_i \cap T_j = \emptyset, \quad i \neq j.$$

Since $|K| = n-2$, it is clear that exactly two of the 4 intersections

$$S_i \cap T_j, \quad i, j = 1, 2$$

are non-empty. If S_1 and S_2 both intersect one of the grand cliques containing y, other than K, say T_1, then all three grand cliques containing x intersect T_1 and $x \notin T_1$. This contradicts lemma (3.1.8). Hence each of the grand cliques S_i intersects one and only one of the grand cliques and vice versa.

Lemma (3.1.10). Let x and y be two vertices in G such that, $d(x,y) = 2$. Then there is one grand clique S_3 containing x which does not intersect any grand clique containing y, and one grand clique T_3 containing y which does not intersect any grand clique containing x. The other two grand cliques S_1, S_2 containing x and the other two grand cliques T_1, T_2 containing y mutually intersect.

From (b_4), there are exactly 4 vertices adjacent to both x and y. Let z_{11} be one of these and let S_1 be the grand clique containing x and z_{11} and T_1 be the grand clique containing y and z_{11}. Clearly $S_1 \neq T_1$, since $d(x,y) = 2$.

Since $d(x,z_{11}) = 1$ by lemma (3.1.9) there exists exactly one grand clique S_2 containing x, other than S_1, which intersects T_1. Let $z_{21} = S_2 \cap T_1$. Clearly $z_{11} \neq z_{21}$, $y \neq z_{21}$. Similarly since $d(z_{11},y) = 1$, there exists exactly one grand clique T_2 containing y, which intersects S_1 in z_{12} say. Clearly $x \neq z_{12}$, $z_{11} \neq z_{12}$. Suppose $T_2 \cap S_2 = \emptyset$. Then since $d(z_{12}, x) = 1$, from the previous lemma the third grand clique S_3 containing x must intersect T_2 in z_{32}, say. Similarly, the third grand clique T_3 containing y must intersect S_2 in z_{23} say. Since $z_{23} \in S_2$ and $z_{32} \in T_2$ and $S_2 \cap T_2 = \emptyset$, we must have $z_{23} \neq z_{32}$. But then we have 5 vertices adjacent to both x and y, namely, $z_{11}, z_{12}, z_{21}, z_{23}, z_{32}$. This contradicts (b_4) and hence $S_2 \cap T_2 \neq \emptyset$. Let $z_{22} = S_2 \cap T_2$, we then have 4 vertices adjacent to both x and y, namely, z_{ij}, $i,j = 1,2$. It follows from (b_4) that

$$S_3 \cap T_j = \emptyset , \quad j = 1,2,3$$

$$S_i \cap T_3 = \emptyset , \quad i = 1,2,3.$$

Lemma (3.1.11). Given two distinct grand cliques K_1 and K_2 of G, with a common vertex x, there is a (1,1) correspondence between the vertices of K_1 and K_2 such that the corresponding vertices are contained in a grand clique.

Let y_1 be a vertex of K_1, $y_1 \neq x$. If $z \neq x$ is a vertex of K_2, then $d(y_1,z) \leq 2$. It follows as in the previous lemma that every vertex of K_2 is not adjacent to y_1. Hence there exists a vertex z in K_2 such that $d(y_1,z) = 2$. From lemma (3.1.10) there is another clique K^*, besides K_1, which contains y_1 and intersects K_2. Let $y_2 = K^* \cap K_2$. Then y_1 is adjacent to y_2. Also the third clique containing y_1 does not intersect K_2. Thus there is exactly one vertex y_2 in K_2 which is adjacent to y_1. In the same way we can start from a vertex y_2 in K_2 and show that there is exactly one vertex in K_1 adjacent to y_1.

Lemma (3.1.12). There are exactly $n(n-1)/2$ grand cliques in G.

Consider ordered pairs (x,K) where x is a vertex and K is a grand clique of G containing x. Since each vertex is contained in 3 grand cliques we get $3v$ such pairs, where $v = n(n-1)(n-2)/6$ is the number of vertices in G. But each grand clique accounts for n-2 pairs. Hence the number of grand cliques is $3v/(n-2) = n(n-1)/2$.

Lemma (3.1.13). Each grand clique in G is intersected by exactly $2(n-2)$ other grand cliques.

This follows at once by noting that each vertex of a grand clique K is contained in exactly two other grand cliques.

Lemma (3.1.14). If K_1 and K_2 are two intersecting grand cliques in G, there exist exactly n-2 grand cliques which intersect both K_1 and K_2.

Let $x = K_1 \cap K_2$. Then through x there passes another grand clique K_3 (intersecting K_1 and K_2 in x). Again if y_1 is any one of the other n-3 points on K_1, then from Lemma (3.1.11) there exists a corresponding point y_2 on K_2 such that y_1 and y_2 are adjacent, $K(y_1, y_2)$ being a grand clique intersecting both K_1 and K_2. Each pair of corresponding points gives one such clique and vice versa.

The following three lemmas are directed towards proving that there exist exactly 4 grand cliques, which intersect each of two given non-intersecting grand cliques K_1 and K_2.

Lemma (3.1.15). If K_1 and K_2 are two non-intersecting grand cliques in G, x_1 and x_2 are two adjacent vertices belonging to K_1 and K_2 respectively, then there exist vertices y_1, y_2 belonging to K_1 and K_2 respectively, such that x_1, x_2, y_1, y_2 are mutually adjacent ($y_1 \neq x_1$, $y_2 \neq x_2$).

Since K_1 and $K(x_1,x_2)$ are intersecting cliques, by lemma (3.1.11), there exists a vertex y_1 in K_1 which corresponds to x_2 and is therefore adjacent to it. Similarly there exists a vertex y_2 in K_2, which is adjacent to x_1.

Clearly $d(y_1,y_2) \leq 2$. We want to show that y_1 and y_2 are adjacent. If not, then $d(y_1,y_2) = 2$. Then by lemma (3.1.10) there exists a clique K_1^* containing y_1, and a clique K_2^* containing y_2 such that K_1^* does not intersect any clique containing y_2, K_2^* does not intersect any clique containing y_1. Hence K_1^* must be distinct from K_1 and $K(y_1,x_2)$. Also K_2^* must be distinct from K_2 and $K(y_2,x_1)$. The cliques K_1 and $k(y_1,x_2)$ must therefore intersect the cliques K_2 and $K(y_2,x_1)$. Since by hypothesis K_1 does not intersect K_2, this is a contradiction. Hence $d(y_1,y_2) = 1$.

Lemma (3.1.16). If K_1 and K_2 are two non-intersecting grand cliques in G, such that there exist vertices x_1 and x_2 belonging to K_1 and K_2 respectively with $d(x_1,x_2) = 2$, then there exists at least one pair of adjacent vertices one of which belongs to K_1 and the other to K_2.

From lemma (3.1.10), there exist two cliques (different from K_1) containing x_1, which intersect each of the two cliques (other than K_2) containing x_2. Let z be the vertex of intersection of one of the cliques containing x_1 with one of the cliques containing x_2. Then z is different from x_1 and x_2. Since K_1 and $K(x_1,z)$ are intersecting cliques, from lemma (3.1.11), there exists a vertex y_1 in K_1 ($y_1 \neq x_1$), such that y_1 is adjacent to z. Similarly there exists a vertex y_2 in K_2 adjacent to z. x_1 and x_2 are non-adjacent by hypothesis. If the lemma is false then the pairs (x_1,y_2), (y_1,x_2) and (y_1,y_2) must also be non-adjacent. Hence $K(z,x_1)$, $K(z,x_2)$, $K(z,y_1)$ and $K(z,y_2)$ must all be different grand cliques, contradicting the lemma (3.1.1).

Lemma (3.1.7). If K_1 and K_2 are two non-intersecting grand cliques in G, there must exist a pair of adjacent vertices one of which belongs to K_1 and the other to K_2.

Let x_1 be a vertex in K_1 and x_2 a vertex in K_2. If $d(x_1,x_2) = 1$, the lemma is true. If $d(x_1,x_2) = 2$, the required result follows from the previous lemma. Suppose $d(x_1,x_2) = r > 2$. Then, there exists a chain x_1, z_1, z_2, ... z_{r-1}, x_2 joining x_1 and x_2. Clearly $d(z_{r-2},x_2) = 2$. Consider the grand cliques $K(z_{r-3},z_{r-2})$ and K_2. These must be non-intersecting, otherwise $d(z_{r-3}, x_2) = 2$, and x and y would be at distance $r-1$. Hence from the previous lemma we can find a vertex y_2 in $K(z_{r-3},z_{r-2})$ and z^*_{r-2} in K_2 such that y_2 and z^*_{r-2} are adjacent. Then $d(x_1,y_2) = r-1$. By successive repetition of the same process we can find a vertex w_2 in K_2 such that $d(x_1,w_2)=2$. the required result then follows from the previous lemma.

Lemma (3.1.18). If K_1 and K_2 are two non-intersecting grand cliques in G, then there exist exactly 4 grand cliques, which intersect each of K_1 and K_2.

From the previous lemma we can find a vertex x_1 in K_1 and a vertex x_2 in K_2 such that x_1 is adjacent to x_2. Now from lemma (3.1.15) we can find in K_1 a vertex $y_1 \neq x_1$, and in K_2 a vertex $y_2 \neq x_2$, such that x_1, x_2, y_1, y_2 are mutually adjacent. Hence $K(x_1,x_2)$, $K(x_1,y_2)$, $K(y_1,x_2)$ and $K(y_1,y_2)$ are 4 grand cliques each of which intersects K_1 and K_2.

If possible let there be another grand clique which intersects K_1 in z_1 and K_2 in z_2. Then z_1 is distinct from x_1 and y_1 and z_2 is distinct from x_2 and y_2. Otherwise there would be 4 grand cliques containing some vertex. Again z_1 is non-adjacent to x_2. Otherwise there would be 4 grand cliques containing x_2. Hence $d(z_1, x_2) = 2$. The grand cliques K_1 and K_2 containing z_1 and x_2 respectively are non-intersecting. Since the grand cliques $K(x_2,x_1)$ and $K(x_2,y_1)$ intersect the grand clique K_1 containing z_1, it follows from lemma (3.1.10), that K_2 the third grand clique containing x_2, cannot intersect any grand clique containing z_1. This is a contradiction since $K(z_1,z_2)$ intersects K_2 in z_2.

2. <u>Theorem</u> (3.2.1). For a tetrahedral graph G, the conditions (b_1)-(b_4) hold. Conversely if $n > 16$ and conditions (b_1)-(b_4) are satisfied then G is tetrahedral.

Given n symbols, $1,2,\ldots n$, a tetrahedral graph G is the graph whose vertices are the unordered triplets on these symbols, two triplets being adjacent when the corresponding triplets have two symbols in common. The conditions (b_1)-(b_4) are easily checked.

(i) The number of unordered triplets on n symbols is clearly $n(n-1)(n-2)/6$ which is the number of vertices in G.

(ii) Let x be the vertex corresponding to the symbol (i,j,k), for a vertex adjacent to x, the corresponding triplet must contain 2 of the symbols i,j,k and one of $n-3$ symbols other than i,j,k. Hence the number of vertices adjacent to x is $3(n-3)$. Hence G is regular of valence $3(n-3)$.

Given any two triplets we can readily construct a chain of triplets beginning with the first and ending with the second so that two consecutive triplets have two symbols in common. Hence G is connected.

(iii) Let x and y be two adjacent vertices of G. Let x and y correspond to (i,j,k_1) and (i,j,k_2). Then the triplets which have two symbols in common with each of the two triplets are the triplets (i,k_1,k_2), (j,k_1,k_2) and the $n-4$ triplets (i,j,s) where s is any of the n symbols other than i,j,k_1,k_2. Hence the number of vertices adjacent to both x and y is $n-2$, i.e. $\Delta(x,y) = n-2$.

(iv) Let x and y be two vertices such that $d(x,y) = 2$. Then we may take x to correspond to (i,j_1,k_1) and y to correspond to (i,j_2,k_2). Then the only triplets which have two symbols inccommon with both these triplets are (i,j_1,j_2), (i,j_1,k_2), (i,j_2,k_1) and (i,k_1,k_2). Thus $\Delta(x,y) = 4$ if $d(x,y) = 2$.

Conversely, suppose $n > 16$ and the conditions (b_1)-(b_4) are satisfied for G. Then lemmas (3.1.1) - (3.1.18) hold. Let H be a graph whose vertices are the grand cliques of G, two vertices of H being adjacent if and only if the corresponding grand cliques of G have a non-empty intersection.

Then H satisfies the following conditions:

(a_1) The number of vertices in H is $n(n-1)/2$.

(a_2) H is regular and of valence $2(n-2)$.

(a_3) H is edge regular with edge degree $n-2$, i.e. $\Delta(u,v) = n-2$ if u and v are adjacent.

(a_4) $\Delta(u,v) = 4$ if u and v are non-adjacent.

Conditions (a_1),(a_2),(a_3),(a_4) are satisfied in virtue of lemmas (3.1.12),(3.1.13),(3.1.14) and (3.1.18). It now follows from Connor's theorem (see section I), that for $n > 8$, H is triangular. Hence to each vertex of H we can associate an unordered pair (i,j) of two distinct symbols chosen out of n symbols $1,2,\ldots,n$ so that two vertices of H are adjacent if and only if the corresponding pair have a common symbol. From the correspondence between H and G it follows that to each grand clique in G we can associate an unordered pair of symbols chosen out of n symbols, such that the pairs corresponding to two grand cliques have a common symbol if and only if the two grand cliques intersect.

Let K_1 and K_2 be any two intersecting grand cliques of G, having the vertex x in common and let (i,j) and (i,k) be the corresponding pairs. To the vertex x we associate the unordered triplet (i,j,k). The third grand clique K_3 containing x, intersects both K_1 and K_2 and hence must correspond to the pair (j,k). Note that the triplet assigned to x is unambiguously determined by any two of the three cliques intersecting in x. Thus to each of the $n(n-1)(n-2)/6$ vertices of G we can associate a unique triplet.

If two vertices x and y of G are adjacent then there is a unique grand clique K containing x and y. If (i,j) is the pair corresponding to K, then the triplets corresponding to x and y must contain the symbols i and j. Conversely if the triplets corresponding to the vertices x and y contain the symbols i and j, then x and y are contained in the grand clique K corresponding to the pair (i,j), and must therefore be adjacent. Thus two vertices of G are adjacent if and only if the corresponding triplets have a common pair of symbols. Hence G must be tetrahedral.

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