A CHARACTERIZATION OF CUBIC LATTICE GRAPHS

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Institute of Statistics Mimeo Series No. 515

March 1967

Abstract

A cubic lattice graph may be defined as a graph G, whose vertices can be identified with ordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two common symbols in the same positions. If $d(x,y)$ denotes the distance between two vertices x and y and $\Delta(x,y)$ dentoes the number of vertices adjacent to both x and y, then a cubic lattice graph G has the following properties: (b_1) The number of vertices is n^3. (b_2) G is connected and regular of valence $3(n-1)$. (b_3) G is edge regular, with edge-degree n-2. (b_4) $\Delta(x,y)=2$, if $d(x,y)=2$. (b_5) If $d(x,y)=2$, there exist exactly n-1 vertices z, adjacent to x such that $d(y,z)=3$. We show that if $n > 7$, then any graph G (without loops and with utmost one edge connecting two vertices) having the properties (b_1)-(b_5) must be a cubic lattice graph.

This research was supported by the Air Force Office of Scientific Research Grant No. AF-AFOSR-760-65.

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I Introduction

1. We shall consider only finite undirected graphs, with at most one edge joining a pair of vertices and no edge joining a vertex to itself.

The valence $d(u)$ of the vertex u of a graph G is defined to be the number of vertices adjacent to u. If all vertices of G have the same valence n_1, the graph G is said to be a regular graph of valence n_1.

A chain $x_1, x_2, \ldots x_n$ is a sequence of vertices of G, not necessarily all different, such that any two consecutive vertices in the chain are adjacent. Thus the pairs $(x_1,x_2),(x_2,x_3) \ldots (x_{n-1},x_n)$ are edges of G. The number of edges n-1 is said to be the length of the chain. The chain is said to begin at x_1 and terminate at x_n, and is said to join x_1 and x_n.

The graph G is said to be connected if for every pair of distinct vertices x and y, there is a chain beginning at x and terminating at y. For a connected graph the distance $d(x,y)$ between two vertices x and y is defined to be the length of the shortest chain joining x and y.

For any two vertices u and v, $\Delta(u,v)$ denotes the number of vertices ω, adjacent to both u and v. If u and v are adjacent, i.e. $d(u,v) = 1$, $\Delta(u,v)$ is called the edge degree of the edge (u,v). A regular graph G for which all edges have the same edge-degree Δ, is said to be edge-regular, with edge-degree Δ.

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2. A graph G is said to be L_2, if the vertices of G can be identified with ordered pairs on n symbols such that, two vertices are adjacent if and only if the corresponding pairs have a symbol in common in the same position. Clearly such a graph G has the following properties:

(a_1) The number of vertices is n^2.

(a_2) G is regular of valence 2(n-1).

(a_3) G is edge-regular, with edge degree n-2.

(a_4) $\Delta(x,y)=2$, if x and y are any two non-adjacent vertices in G.

Shrikhande [2] showed, with somewhat different terminology, that if $n > 4$, the above properties (a_1)-(a_4) characterize a L_2 graph. In other words if a graph G satisfies conditions (a_1)-(a_4) and $n > 4$, then G must be a L_2 graph.

3. In this paper we consider the problem of characterization of cubic lattice graphs. A cubic lattice graph may be defined as a graph G whose vertices can be identified with ordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two common symbols in the same positions. It is readily seen that G has the following properties:

(b_1) The number of vertices in G is n^3.

(b_2) G is connected and regular of valence 3(n-1).

(b_3) G is edge-regular, with edge-degree n-2, i.e. $\Delta(x,y) = n-2$, if $d(x,y) = 1$.

(b_4) $\Delta(x,y) = 2$ if $d(x\ y) = 2$.

(b_5) If $d(x\ y) = 2$, there exist exactly n-1 vertices z, adjacent to x such that $d(y,z) = 3$.

In this paper (Section II) we prove that for $n > 7$, the properties $(b_1) - (b_5)$ characterize a cubic lattice graph, i.e. if G possesses properties $(b_1) - (b_5)$ and $n > 7$, then it is possible to establish a (1,1) correspondence between the vertices of G, and the ordered triplets on n symbols, such that two vertices of G are adjacent if and only if the corresponding triplets have two common symbols in the same positions.

The proof is based on certain theorems regarding the existence or non-existence of cliques and claws in edge-regular graphs, which were proved in [1], and for the sake of completeness we will state those theorems in the next section para 2.

II Characterization of Cubic Lattice Graphs

1. As mentioned earlier in the introduction a cubic lattice graph G is a graph whose vertices can be identified with n^3 ordered triplets on n symbols, such that any two vertices are adjacent if and only if the corresponding triplets have two common symbols in the same positions. Then G clearly possesses the properties $(b_1) - (b_5)$ given in Section I, para 3. We shall here prove that if $n > 7$, the converse also holds.

2. Consider a graph G which has the following properties :

(c_1) G is connected and regular of valence $r(k-1)$.

(c_2) G is edge-regular with edge-degree $(k-2) + \alpha$.

(c_3) $\Delta(x,y) \leq 1 + \beta$, for all pairs of non-adjacent vertices, x and y of G, where r, k, α, β are fixed positive integers, such that $r \geq 1$, $k \geq 2$, $\alpha \geq 0$, $\beta \geq 0$, and $r\beta - 2\alpha \geq 0$.

A clique K of a graph is a set of vertices adjacent to each other. A clique K is complete if we cannot find a vertex x, not contained in K such that $x \cup K$ is a clique. A clique K of G will be called a major clique if $|K| \geq 1 + k - \gamma(r,\alpha)$ where

$$\gamma(r,\alpha) = 1 + (r-1)\alpha.$$

A clique K of G will be called a grand-clique if it is both major and complete.

A claw [p,s] of G, consists of a vertex p, called the vertex of the claw, and a non-empty set S of vertices of G, not containing p, such that p is adjacent to every vertex in S, but any two vertices in S are non-adjacent. The order of the claw is defined to be the number $s = |S|$.

For a graph G, satisfying the conditions (c_1), (c_2), (c_3) given above, the following properties were proved in [1].

Theorem. If $k > \max[p(r,\alpha,\beta), \ \rho(r,\alpha,\beta)]$, where

$$p(r,\alpha,\beta) = 1 + \tfrac{1}{2}(r+1)(r\beta-2\alpha),$$

$$\rho(r,\alpha,\beta) = 1 + \beta + (2r-1)\alpha,$$

then

i) there cannot exist a claw of order r+1 in G,

ii) any two adjacent vertices p and q of G are contained in exactly one grand clique,

iii) each vertex in G is contained in exactly r grand cliques.

If we set $r=3$, $k=n$, $\alpha=0$, $\beta=1$, then the conditions $(b_2),(b_3),(b_4)$ given in Section I, para 3 are the same as the conditions $(c_1),(c_2),(c_3)$ given in this section. Also $p(r,\alpha,\beta)=7$, $\rho(r,\alpha,\beta)=2$, $\gamma(r,\alpha)=1$. Hence a clique is a major clique if $|K| \geq n$ and if it is complete it is a grand clique. Moreover, if we take $n > 7$, then the condition $K > \max[p(r,\alpha,\beta), \rho(r,\alpha,\beta)]$ is satisfied.

In the following pages G is a graph satisfying conditions (b_1)-(b_5) and such that $n > 7$, then from the theorem stated above we have:

Lemma (2.2.1) (i) There cannot exist a claw of order 4 in G.

(ii) Any two adjacent vertices p and q of G are contained in exactly one grand clique.

(iii) Each vertex in G is contained in exactly 3 grand cliques.

The null set we will denote by ϕ. The unique grand clique containing two given adjacent vertices x and y, may be denoted by K(x,y).

Lemma (2·2·2). Each grand clique of G contains exactly n vertices.

Consider a grand clique K, and any two vertices x and y in K. Since $|K| \geq n$, K has at least n-2 vertices, other than x and y. But from (b_3), $\Delta(x,y) = n-2$. Hence K has exactly n vertices.

Lemma (2·2·3). There are exactly $3n^2$ grand cliques in G.

Consider ordered pairs (x,K) where x is a vertex and K is a grand clique containing **x**. Since each vertex is contained in 3 grand cliques we get $3v$ such pairs, where $v=n^3$ is the number of vertices in G. But each grand clique accounts for n pairs. Hence the number of grand cliques is $3\ v/n = 3n^2$.

Lemma (2.2.4). If K_1, K_2, K_3 are grand cliques in G, such that $K_1 \cap K_2 = x_o$, and $K_3 \cap K_1 = x_1$, $x_1 \not\sim x_o$, then $K_3 \cap K_2 = \emptyset$.

Suppose, on the contrary, that $K_3 \cap K_2 = x_2$. By lemma (2.2.1) $x_2 \not\sim x_o$, $x_2 \not\sim x_1$. Then the (n-2) vertices other than x_o, x_1 in K_1, and x_2 are vertices adjacent to both x_o, x_1, and this number exceeds n-2 contradicting (b_3).

In the next three lemmas we will study the intersection of the grand cliques containing two distinct vertices x and y of G.

Lemma (2·2·5). Let x and y be two distinct vertices of G such that $d(x,y) > 2$. Let K_i (i=1,2,3) be the grand cliques containing x, and S_j (j=1,2,3) be the grand cliques containing y. Then

$$K_i \cap S_j = \emptyset, (i,j = 1,2,3).$$

If K_i and S_j have a common vertex z, then x,y,z, is a chain of length two joining x and y, which makes $d(x,y) \leq 2$ in contradiction to the hypothesis.

Lemma (2.2.6). Let x and y be two distinct vertices of G, such that $d(x,y)=2$. Let K_i $(i=1,2,3)$ be the grand cliques containing x, and S_j $(j=1,2,3)$ be the grand cliques containing y. Then one of the grand cliques K_i, say K_1, disjoint with each of S_j, and one of the three grand cliques S_j, say S_1, is disjoint with each of K_i. Thus,

$$K_1 \cap S_j = \emptyset, (j=1,2,3)$$

$$S_1 \cap K_i = \emptyset, (i=1,2,3).$$

Among the other grand cliques K_2, K_3, S_2, S_2, there is a (1,1) correspondence between K_i, S_j, such that two grand cliques intersect, if and only if, they correspond.

Since $d(x,y)=2$, it follows that one of the grand cliques containing x, say K_2 must intersect one of the grand cliques containing y, say S_2. By lemma (2.2.1) K_2 and S_2 cannot have more than one common vertex, and by lemma (2.2.4) K_2 cannot intersect any other grand clique containing y. Hence, another grand clique containing x say K_3, must intersect another grand clique containing y say S_3 to account for $\Delta(x,y)=2$ in (b_4). Obviously, the third grand clique K_1 containing x, and the third grand clique S_1 containing y are such that,

$$K_1 \cap S_j = \emptyset, (j=1,2,3)$$

$$S_1 \cap K_i = \emptyset, (i=1,2,3).$$

Corollary. The grand cliques K_1 and S_1 in the above lemma are such that, all vertices $z \neq x$ in K_1 are at distance 3 from y, and all vertices $\bar{z} \neq y$ in S_1 are at distance 3 from x.

Clearly neither K_2 nor K_3 can contain a vertex z which is at distance 3 from y, otherwise by lemma (2.2.5) K_2 cannot intersect S_2, and K_3 cannot intersect S_3. But by (b_5), there are exactly (n-1) vertices z, adjacent to x, such that d(y,z)=3, and hence K_1 must contain all those vertices z. Similarly S_1 must contain all vertices $\bar{z} \neq y$, such that $d(x,\bar{z})=3$.

Lemma (2.2.7). Let x and y be a pair of adjacent vertices of G. Let L be the grand clique containing x and y, and K_1, K_2 be the other grand cliques containing x and S_1, S_2 be those containing y. Then

$$K_i \cap S_j = \emptyset, \ (i=1,2), \ (j=1,2).$$

This follows immediately from lemma (2.2.4).

Lemma (2.2.8). Let K_i, i=1,2,3 be the grand cliques containing a vertex x. There cannot exist a grand clique L, distinct from K_i, such that one of the grand cliques K_i, say K_1, intersects L in y and the grand cliques K_2, K_3 contain vertices x_2, x_3, distinct from x such that x_2 is adjacent to $y_2 \neq y$ in L and x_3 is adjacent to $y_3 \neq y_2$, y in L.

Suppose, on the contrary, such a grand clique L exists. Consider the vertices x and y_3. Clearly $d(x,y_3)=2$ and the vertices y and x_3 are both adjacent to x and y_3, and y is in K_1 and x_3 is in K_3, and hence by Corollary of lemma (2.2.6) the third grand clique K_2 containing x must be such that all vertices in K_2, other than x, are at distance 3 from y_3. But $x_2 \neq x$ is in K_2 and $d(x_2,y_3)=2$, which is a contradiction.

3. Let x be a vertex of G and K a set of vertices of G, the associated number of x relative to K denoted by

$$\delta(x,K)= \max d(x,y), \text{ for all y in K}.$$

Let K_1, K_2 be two sets of vertices of G, the associated number relative to each other denoted by

$$\delta(K_1, K_2) = \max d(x,y) \text{ for all } x \text{ in } K_1 \text{ and all } y \text{ in } K_2.$$

Lemma (2.3.1). Let K be a given grand clique of G, and x, a vertex of G not contained in K, such that $\delta(x,K)=2$. Then there exists exactly one vertex y in K, which is adjacent to x.

Since $\delta(x, K) = 2$, there exists a vertex z in K, such that $d(x,z) = 2$. Also all vertices x^* in K are such that $d(x,x^*) \leqslant 2$. Let S_i, $i=1,2,3$ be the grand cliques containing x. By lemma (2.2.6), one of the grand cliques S_i, say S_1, does not intersect any grand clique containing z, and each of S_2 and S_3 intersects exactly one distinct grand clique containing z. If S_2 or S_3 intersects K, then the lemma follows. Suppose, K does not intersect any grand clique containing x, and $K_2 \cap S_2 = p$ and $K_3 \cap S_3 = q$, where K_2 and K_3 are the grand cliques containing z, other than K.

Let $x^* \neq z$ be a vertex in K, and $d(x,x^*) = 2$. Then, since $K \cap S_i = \emptyset$, $i= 1,2,3$, each of the two grand cliques, say $K_1(x^*)$, $K_2(x^*)$ containing x^*, other than K, must intersect exactly one of S_i, $i = 1,2,3$, to account for $\Delta(x,x^*) = 2$ in (b_4). Also both of $K_1(x^*)$ and $K_2(x^*)$ cannot intersect the same S_i, otherwise it would contradict lemma (2.2.4). But this implies that, a grand clique containing x^*, other than K, must intersect at least one of S_2, S_3. Suppose $K_1(x^*) \cap S_3 = y$. Then consider the three grand cliques K_3, K_2, K and the grand clique S_3. Now $K_3 \cap S_3 = q$, y in S_3 is adjacent to x^* in K, and x in S_3 is adjacent to p in K_2, and all the vertices p,y,x,q,z are distinct. This contradicts lemma (2.2.8). We get similar contradiction if $K_i(x^*) \cap S_j \neq \phi$, $i=1,2$, $j=2,3$.

Hence K contains a vertex, adjacent to x, and this must be the only one, otherwise it would contradict lemma (2.2.4).

Lemma (2.3.2). Given two distinct non-intersecting grand cliques K_1, K_2 such that $\delta(K_1, K_2) = 2$, there exists a (1,1) correspondence between the vertices of K_1 and K_2 such that the corresponding vertices are contained in a grand clique.

Let x be a vertex of K_1. As $\delta(x, K_2) = 2$ by lemma (2.3.1), there exists exactly one vertex y in K_2 such that x and y are adjacent. Thus for every vertex x in K_1, there is exactly one vertex y in K_2, such that x and y belong to a unique grand clique. Also from lemma (2.2.4) no two vertices of K_1 can be adjacent to one vertex of K_2. In the same way we can start from a vertex y in K_2, and show that there is exactly one vertex in K_1 adjacent to y. Thus K_1 and K_2 are intersected by n different non-intersecting grand cliques at different vertices.

Lemma (2.3.3). Let K_1 and K_2 be two distinct non-intersecting grand cliques. $\delta(K_1, K_2) = 2$, if and only if, there exist at least 2 distinct non-intersecting grand cliques, S_1, S_2 such that both intersect K_1 and K_2.

If $\delta(K_1, K_2) = 2$, it follows from lemma (2.3.2) that there exist n non-intersecting grand cliques intersecting both K_1, K_2.

Suppose two non-intersecting grand cliques S_1, S_2 intersect both K_1, K_2 and $\delta(K_1, K_2) \neq 2$. Clearly then $\delta(K_1, K_2) = 3$, and hence there exists a pair of vertices x and y, x is in K_1 and y is in K_2, such that $d(x,y) = 3$. By lemma (2.2.5) no grand clique containing y, and in particular, K_2 can intersect any grand clique containing x. Let $K_2 \cap S_2 = z$. Since $d(x,z) = 2$, there exists exactly a pair of grand cliques containing x and z respectively, other than K_1 and S_2, which intersect. This grand clique containing z cannot be K_2 and hence a grand clique containing x other than K_1 must intersect the third grand clique containing z. Consider the grand clique K_1 and the three grand cliques containing z. One of the grand cliques containing z, namely S_2 intersects K_1 and two other grand cliques have vertices adjacent to distinct vertices of K_1, which contradicts lemma (2.2.8).

Lemma (2.3.4). Let K_1 and K_2 be two non-intersecting grand cliques such that $\delta(K_1, K_2) = 2$. Let distinct non-intersecting grand cliques $S_1, S_2, \ldots S_n$ intersect both K_1, K_2. Then there exist exactly n distinct grand cliques, including K_1 and K_2, which intersect all the grand cliques S_i, $i=1,2,\ldots n$.

It follows from lemma (2.3.3) that $\delta(S_i, S_j) = 2$, $i \neq j$, $i,j = 1,2,\ldots n$, and hence there exists a (1,1) correspondence between the vertices of S_i and S_j such that the corresponding vertices are adjacent. The lemma follows immediately if we show that, if x_i of S_i is adjacent to x_j of S_j and x_j of S_j is adjacent to x_k of S_k, then x_i of S_i is adjacent to x_k of S_k.

Suppose, on the contrary, x_i is not adjacent to x_k. Consider the grand clique K_1 and the three grand cliques containing x_j, namely $K(x_i,x_j)$, S_j and $K(x_j,x_k)$. One of the grand cliques containing x_j, namely S_j, intersects K_1 and other two grand cliques contain vertices which are adjacent to distinct vertices of K_1, which contradicts lemma (2.2.8).

A set **C** of grand cliques is said to form a planar system, if and only if, for any two grand cliques K_1,K_2 in **C**,

(i) $\delta(K_1,K_2) = 2$

(ii) the system is complete in the sense that we cannot add a new grand clique to the system so that the property(i) is maintained.

Lemma (2.3.5). Any two grand cliques S,K with $\delta(S,K) = 2$ determine a unique planar system $\mathcal{C}$ which contains exactly 2 n grand cliques.

Let S,K be grand cliques with $\delta(S,K) = 2$. If S and K are intersecting, let $S \cap K = x_0$. Let $x \neq x_0$ be a vertex in K and $y \neq x_0$ be a vertex in S. Clearly $d(x,y) = 2$. Hence by lemma (2.2.6) there exist a grand clique other than S, containing y and a grand clique other than K, containing x, such that these two grand cliques intersect in a vertex z. The grand cliques K and K(y,z) are non-intersecting, but intersected by two distinct non-intersecting grand cliques, namely, S and K(x,z) and hence by lemma (2.3.3), $\delta(K, K(y,z)) = 2$. Similarly, $\delta(S, K(x,z)) = 2$. Thus by the correspondence described in lemma (2.3.2), and by lemma (2.3.4) we get 2n grand cliques, any two of these grand cliques, say, L,M are either intersecting or non-intersecting, but $\delta(L,M) = 2$.

Now, suppose S and K are non-intersecting, but $\delta(S,K) = 2$. Then by lemmas (2.3.3) and (2.3.4) we get 2n grand cliques, any two of these grand cliques say L,M, are either intersecting or non-intersecting but $\delta(L,M) = 2$.

We cannot add any other grand clique to the planar system $\mathcal{C}$, determined by S,K, for if there is one extra grand clique K*, it must contain a vertex of intersection x of grand cliques say K_1 and K_2 in $\mathcal{C}$. Let z be any vertex in $\mathcal{C}$, such that $d(x,z) = 2$. Since $\delta(x,K_2) = 2$ by lemma (2.3.1) there exists a vertex $z_2 \neq x$ in K_2, such that z and z_2 are adjacent. Similarly there exists a vertex $z_1 \neq x$ in K_1, such that z, z_1 are adjacent. Clearly $K(z_1,z)$ and $K(z,z_2)$ belong to the system $\mathcal{C}$. Consider the grand cliques K_1, K_2 and K* containing x, and the grand clique $K(z_1, z)$. K_1 intersects $K(z_1,z)$ and K_2 has a vertex z_2 adjacent to z in $K(z_1, z)$, hence K* cannot contain a vertex $p \neq x$, adjacent to a vertex distinct from z_1 and z_2 of $K(z_1,z)$, otherwise it would contradict lemma (2.2.8). But from lemma (2.3.3) it follows that $\delta(K^*, K(z_1,z)) \neq 2$

and hence K^* cannot belong to the system $\mathfrak{C}$.

Corollary 1. If x is any vertex in a planar system $\mathfrak{C}$, then $\mathfrak{C}$ contains exactly 2 grand cliques containing x.

Corollary 2. Let S and K be two intersecting grand cliques in a planar system $\mathfrak{C}$, and let L be any grand clique in L, then L intersects either S or K.

Corollary 3. Two planar systems cannot have more than one grand clique in common.

This follows immediately from lemma (2.3.5) noting that any two grand cliques K_1, K_2 with $\delta(K_1,K_2) = 2$ determine a unique planar system.

The unique planar system containing two given grand cliques S,K with $\delta(S,K) = 2$, may be denoted by $\mathfrak{C}(S, K)$.

Lemma (2.3.6). Each grand clique is contained in exactly 2 planar systems.

Let K_1 be a grand clique. Let x_o be a vertex in K_1, and K_2, K_2^* be the other two grand cliques containing x_o. By lemma (2.3.5), K_1,K_2 determine a unique planar system $\mathfrak{C}$ and K_1, K_2^* determine a unique planar system $\mathfrak{C}^*$. Thus K_1 is contained in $\mathfrak{C}$ and $\mathfrak{C}^*$. If possible, suppose K_1 is contained in a third planar system $\mathfrak{C}'$, distinct from $\mathfrak{C}$ and $\mathfrak{C}^*$. Let L be a grand clique in $\mathfrak{C}'$, such that L intersects K_1 in a vertex y_o. Clearly $\mathfrak{C}' = \mathfrak{C}(K_1,L)$. Let z_o be any vertex in K_2, then $d(y_o, z_o) = 2$. Hence by lemma (2.2.6) there exists another grand clique containing y_o, other than K_1, which intersects a grand clique containing z_o, other than K_2. This grand clique containing y_o cannot be L, otherwise K_1, K_2, and L will belong to the same planar system, and consequently, $\mathfrak{C} = \mathfrak{C}(K_1, K_2) = \mathfrak{C}(K_1, L) = \mathfrak{C}'$. Let $\bar{K}$ be the grand clique containing y_o, which intersects a grand clique containing z_o.

Let z be a vertex in K_2^*. By a similar argument a grand clique containing z cannot intersect L, otherwise $\mathcal{C}^* = \mathcal{C}'$. But a grand clique containing z, other than K_2^* must intersect another grand clique containing y_o, and this must be $\bar{K}$. Thus, the three grand cliques K_1, K_2 and K_2^* containing x and the grand clique $\bar{K}$ are such that K_1 intersects $\bar{K}$ and K_2 and K_2^* have vertices adjacent to distinct vertices of $\bar{K}$, which contradicts lemma (2.2.8).

Lemma (2.3.7). There exist exactly 3n planar systems in G.

Consider pairs $(K,\mathcal{C})$, where K is a grand clique and $\mathcal{C}$ is a planar system containing K. To a given K, there corresponds 2 planar systems and there are $3n^2$ grand cliques. If x is the number of planar systems, since each planar system has exactly 2n grand cliques, we must have

$$2n.x = 3n^2.2,$$

$$\text{i.e.} \quad x = 3n.$$

Lemma (2.3.8). Let x_1, x_2 be two adjacent vertices contained in the grand clique K. Let S_i, i=1,2, be the other two grand cliques containing x_1 and K_i, i=1,2, be the other two grand cliques containing x_2. Then

i) $\mathcal{C}(S_1,S_2)$ and $\mathcal{C}(K_1,K_2)$ do not have a grand clique in common.

ii) The grand cliques S_i may be put in (1,1) correspondence with K_j, so that only corresponding cliques belong to the same planar system.

(i) If possible, suppose K* is a grand clique common in both $\mathcal{C}(S_1,S_2)$ and $\mathcal{C}(K_1,K_2)$. Then by Corollary 2 of lemma (2.3.5), K* intersects either S_1 or S_2, and either K_1 or K_2. Suppose K* intersects S_1 and K_1. Since $\mathcal{C}(S_1,S_2)$ contains K*

$$\mathcal{C}(S_1,S_2) = \mathcal{C}(S_1,K^*).$$

Also, K and K* are two non-intersecting grand cliques, which are intersected by two grand cliques S_1, K_1 and hence K, K*, S_1, K_1 all belong to a unique planar system, which is $\mathcal{C}(S_1, K^*)$. But $\mathcal{C}(S_1, K^*)$ contains S_2, and this implies that the three grand cliques K, S_1, S_2 containing x are contained in one planar system which contradicts Corollary 1 of lemma (2.3.5). By similar arguments, we arrive at a contradiction by supposing that K* intersects S_i, K_j.

(ii) By lemma (2.2.7), $S_i \cap K_j = \emptyset, i,j = 1,2$. Let $y_1 \neq x_1$ be a vertex in S_1. Since $d(y_1, x_2) = 2$, by lemma (2.2.6), there exists a grand clique containing y_1, other than S_1, and a grand clique containing x_2, other than K, such that they intersect in a vertex say y_2. This grand clique containing x_2 is either K_1 or K_2, but not both. If it is K_1, then S_1 and K_1 are two non-intersecting grand cliques, intersected by two non-intersecting grand cliques namely K and $K(y_1,y_2)$, and hence by lemma (2.3.3), $\delta(S_1,K_1) = 2$, and these belong to a unique planar system.

Let $z_1 \neq x_1$, y_1 be a vertex in S_1. Then K_2 cannot contain a vertex z_2, adjacent to z_1. For otherwise, the three grand cliques K, K_1, K_2 containing x_2, and the grand clique S_1 will be such that, K will intersect S_1 and K_1, K_2 will have vertices adjacent to distinct vertices of S_1, contradicting lemma (2.2.8). But this implies that $\delta(S_1,K_2) \neq 2$.

By similar arguments, it can be shown that $\delta(S_2,K_2) = 2$, but $\delta(S_2,K_1) \neq 2$.

Corollary. $\mathcal{C}(S_1,K_1)$ and $\mathcal{C}(S_2,K_2)$ have the grand clique K in common.

This follows immediately by noting that as $\delta(S_1,K_1) = 2$ and K intersects both S_1 and K_1, and hence K belong to $\mathcal{C}(S_1, K_1)$. Similarly, K belongs to $\mathcal{C}(S_2, K_2)$.

Lemma (2.3.9). Let $x_1, x_2, \ldots x_n$ be vertices of a grand clique K. Let S_i, K_i be the other two grand cliques containing x_i, $i=1,2,\ldots n$. Then the planar systems C_i determined by S_i, K_i are such that no two of these have a grand clique in common.

This follows immediately from lemma (2.3.9).

Lemma (2.3.10). Let x_1, x_2, x_3 be vertices in a grand clique K. Let S_i, K_i be other two grand cliques containing x_i, $i=1,2,3$. If $\delta(S_1,S_2) = 2$, $\delta(S_2,S_3) = 2$, then $\delta(S_1,S_3) = 2$ and S_1, S_2, S_3 belong to the same planar system.

Let $x \neq x_1$ be any vertex in S_1. As $\delta(S_1,S_2) = 2$, by lemma (2.3.2) there exists a unique vertex y in S_2, such that x and y are adjacent. Again $\delta(S_2,S_3) = 2$, hence there exists a unique vertex z in S_3, such that y and z are adjacent. If x and z are not adjacent, then the three grand cliques S_2, K(x,y) and K(y,z) containing y, and the grand clique K are such that S_2 intersects K, and K(y,x) and K(y,z) contain vertices adjacent to distinct vertices of K. This contradicts lemma (2.2.8). Hence x and z are adjacent, and consequently $\delta(S_1,S_3) = 2$. Clearly S_1, S_2, S_3 belong to the same planar system.

Two planar systems C_1 and C_2 are said to be parallel if they do not have a grand clique in common.

Lemma (2.3.11). Given a planar system C_1, there are 2n planar systems, each of which has exactly one distinct grand clique common with C_1. The remaining (n-1) planar systems are parallel to C_1, and they are parallel to one another.

By lemma (2.3.7) there are exactly 3n planar systems, say, $C_1, C_2, \ldots C_{3n}$. By lemma (2.3.5) each planar system contains exactly 2n grand cliques Let C_1 contain $K_1, K_2 \ldots, K_{2n}$. Each K_i is contained in exactly one planar system, other than C_1. Suppose K_1 is contained in C_2, then K_i, $i \neq 1$ cannot be contained in C_2, otherwise C_1 and C_2 will have 2 grand cliques in common contradicting Corollary 3 of lemma (2.3.5). Suppose K_2 is contained in C_3 and so on. Thus the systems $C_3, \ldots, C_{2n+1}$ contain $K_2, K_3, \ldots, K_{2n}$ respectively. All the grand cliques in C_1 are now exhausted and hence the remaining (n-1) planar systems, say $C_{2n+2}, \ldots C_{3n}$ cannot have any grand clique in common with C_1.

Now, let K_1 and K_2 be two grand cliques in C_1, such that K_1 and K_2 intersect in a vertex x_1. There exists a third grand clique L containing x_1, and L is not contained in C_1. Let $x_2, x_3, \ldots, x_n$ be vertices of L, and S_i, T_i be two other grand cliques containing x_i, $i=2,3,\ldots n$. Then by lemma (2.3.9) the planar systems determined by S_i, T_i denoted by $C(S_i, T_i)$ $i=2,3,\ldots n$ do not have any grand clique common with $C_1 = C(K_1, K_2)$, and also by the same lemma, no two of these have a grand clique in common. As there are exactly (n-1) planar systems parallel to C_1, $C(S_i, T_i)$ $i=2,3,\ldots n$ must be $C_{2n+2}, \ldots C_{3n}$ and they are also parallel to one another.

Lemma(2.3.12). i) In G there are 3 classes of parallel planar systems. Each parallel class contains exactly n planar systems.

ii) Two planar systems belonging to two different parallel classes have exactly one grand clique in common.

i) Let x_o be a vertex in G. Let $K_1(x_o)$, $K_2(x_o)$, $K_3(x_o)$ be the three grand cliques containing x_o. Let $x_1, x_2, \ldots x_{n-1}$ be other vertices in $K_1(x_o)$; $y_1, y_2, \ldots y_{n-1}$ be other vertices in $K_2(x_o)$ and $z_1, z_2, \ldots z_{n-1}$ be those in $K_3(x_o)$. Let $K_2(x_i)$ and $K_3(x_i)$ be other grand cliques containing x_i, $i= 1,2,\ldots n-1$. By lemma (2.3.8) we can name the grand cliques in such a way that, $\delta(K_2(x_i), K_2(x_j))= 2$, and $\delta(K_3(x_i), K_3(x_j))= 2$, $i\neq j$. By lemma (2.3.10) all grand cliques $K_2(x_i)$, $i=0,1,2,\ldots n-1$ are contained in a unique planar system. Also, by lemma (2.3.9) the planar systems C_x^i determined by $K_2(x_i)$ and $K_3(x_i)$, $i=0,1,\ldots n-1$ are all parallel to one another and hence can be said to belong to a parallel class of planar systems.

Similarly, if $K_1(y_i)$, $K_3(y_i)$, $i=1,2,\ldots n-1$ be the other two grand cliques containing y_i and they are so named that, $\delta(K_1(y_i), K_1(y_j)) = 2$ and $\delta(K_3(y_i), K_3(y_j))=2$, $i\neq j$, then the planar systems C_y^i determined by $K_1(y_i)$ and $K_3(y_i)$, $i=1,2,\ldots n-1$ together with C_y^o determined by $K_1(x_o)$ and $K_3(x_o)$ are such that these are parallel to one another and hence form another class of parallel system.

Again, if $K_1(z_i)$ and $K_2(z_i)$, $i=1,2,\ldots n-1$ are the grand cliques containing z_i, other than $K_3(x_o)$, such that $\delta(K_1(z_i), K_1(z_j)) = 2$ and $\delta(K_2(z_i), K_2(z_j)) = 2$, $i\neq j$, then the planar systems C_z^i determined by $K_1(z_i)$, $K_2(z_i)$, $i=1,2, \ldots n-1$ together with C_z^o determined by $K_1(x_o)$ and $K_2(x_o)$ form a third class of parallel planar systems. Clearly each parallel class contains exactly n grand cliques.

(ii) For the sake of definiteness, consider planar systems C_x^i of the first parallel class and C_y^j of the second parallel class. Then according to our notation C_x^i can be taken to be determined by $K_2(x_i)$, $K_3(x_i)$, and C_y^j by $K_1(y_j)$, $K_3(y_j)$.

Since $\delta(K_2(x_o), K_2(x_i)) = 2$, by the correspondence described in lemma

(2.3.2), a grand clique containing y_j in $K_2(x_o)$ must intersect $K_2(x_i)$ in a unique vertex, say x^*. Similarly, since $\delta(K_1(x_o), K_1(y_j)) = 2$, x_i of $K_1(x_o)$ must correspond to a unique vertex, say, y^* in $K_1(y_j)$. If $x^* \neq y^*$, then they must be adjacent, otherwise, we would have 4 grand cliques containing y_j, namely, $K_3(y_j)$, $K_2(x_o)$, $K_1(y_j)$ and $K(y_j, x^*)$, contradicting lemma (2.2.1). Hence y_j, y^*, x^* are adjacent to each other and belong to a unique grand clique. By similar argument, it can be shown that x_i, y^*, x^* are contained in a unique grand clique. But by lemma (2.2.1), x^*, y^*, being two adjacent vertices belong to a unique grand clique. Thus x_i, y_j, x^*, y^* belong to a grand clique, which contradicts $d(x_i, y_j) = 2$. Hence $x^* = y^*$ and $K_1(y_j) \cap K_2(x_i) = x^*$, and clearly x^* is in both the planar systems C_x^i and C_y^j. By Corollary 1 of lemma (2.3.5), exactly 2 grand cliques containing x^* must be contained in both the planar systems. C_x^i contains $K_2(x_i)$ containing x^* and C_y^j contains $K_1(y_j)$ containing x^*. Clearly none of the planar systems can contain both the grand cliques $K_2(x_i)$ and $K_1(y_j)$, otherwise they will be identical. Hence the third grand clique containing x^* is common to both the planar systems.

4. Theorem (2.4.1). For a cubic lattice graph G, the conditions (b_1)-(b_5) hold for all $n > 1$. Conversely, if $n > 7$, and conditions (b_1)-(b_5) are satisfied, then G is cubic lattice.

Given n symbols 1, 2,...n, a cubic lattice graph G is the graph whose vertices are the ordered triplets on these symbols, two triplets being adjacent when the corresponding triplets have two common symbols in the same positions. The conditions (b_1)-(b_5) are easily checked.

i) The number of ordered triplets on n symbols is clearly n^3 which is the number of vertices in G.

ii) Let x be the vertex corresponding to the symbol (i,j,k). For a vertex adjacent to x, the corresponding triplet must contain 2 of the symbols, i,j,k in the same position, and for any two symbols in the specified positions there are n-1 ways by which the third position can be filled up, and there are 3 ways of arranging two symbols out of 3 symbols. Hence the number of vertices adjacent to x is 3(n-1). Thus G is regular of valence 3(n-1).

Given any two triplets we can readily construct a chain of triplets beginning with the first and ending with the second, so that, two consecutive triplets have two symbols common in the same positions. Hence G is connected.

iii) Let x and y be two adjacent vertices of G. Let x and y correspond to (i,j,k_1) and (i,j,k_2). Then the triplets which have two common symbols in the same positions with each of the triplets are the ones (i,j,s) where s is any one of the n symbols other than k_1, k_2. Hence, the number of vertices adjacent to both x and y is n-2, i.e. $\Delta(x,y) = n-2$.

iv) Let x and y be two vertices such that $d(x,y) = 2$. Then we may take x to correspond to (i,j_1,k_1) and y to correspond to (i,j_2,k_2). Then the only triplets which have two common symbols in the same positions with both of these are (i,j_2,k_1) and (i,j_1,k_2). Thus $\Delta(x,y) = 2$, if $d(x,y) = 2$.

v) Let x and y be two vertices of G, such that $d(x,y) = 2$. We may take x and y to correspond to (i,j_1,k_1) and (i,j_2,k_2) respectively. Let z correspond to (s, j_1,k_1) where s is different from i. Hence there are n-1 vertices z such that z is adjacent to x and $d(y,z) = 3$.

Conversely suppose $n > 7$, and (b_1)-(b_5) are satisfied for G. Then all the preceeding lemmas hold.

By lemma (2.3.7) there exist exactly 3n planar systems in G, and by lemma (2.3.12), these 3n planar systems can be divided into 3 parallel classes of planar systems , each class containing n planar systems. Let us assign the symbols x_i, i=1,2,...n for the planar systems of the first parallel class, y_j, j=1,2,...n for the second parallel class and z_k, k=1,2,...n for the third parallel class. By lemma (2.3.12) two planar systems belonging to the same parallel class have no grand clique in common, but two planar systems belonging to different parallel classes have a unique grand clique in common. Thus the planar systems say, x_i and y_j have a unique grand clique in common and it can be denoted by $(x_i, y_j, \cdot)$.

Consider a vertex x, by lemma (2.2.1) there are exactly 3 grand cliques containing x, which determine two by two 3 planar systems. Since any two of these planar systems have one grand clique in common, the planar systems can be taken as x_i, y_j, z_k. Also the 3 grand cliques have the symbols $(x_i, y_j, \cdot)$, $(x_i, \cdot, z_k)$, $(\cdot, y_j, z_k)$. Thus we can assign an ordered triplet for x, namely, (x_i, y_j, z_k). Since there are exactly n^3 vertices, all the vertices are accounted for.

Also it is clear that any vertex contained in a grand clique, must be given by ordered triplet two of whose symbols are the same in exact positions as that of the grand clique. Thus the n vertices contained in the grand clique say, $(x_i, y_j, \cdot)$ is (x_i, y_j, k) where k can take values from 1, 2,...,n. Thus any two vertices which have two common symbols in the same positions say x_i, y_j for example (x_i, y_j, u) and (x_i, y_j, υ), $u \neq \upsilon$ must be contained in the same grand clique and hence are adjacent. Conversely any two vertices which are adjacent, i.e., have 2 common symbols in the same positions must be contained in exactly one grand clique. Thus two vertices of G are adjacent if and only if the corresponding triplets have two common symbols in the same positions. Hence G must be cubic lattice.

Note:-The condition (b_5) is used only to get the Corollary of lemma (2.2.6) which led to lemma (2.2.8). Thus conditions (b_1)-(b_4) together with lemma (2.2.8) and $n > 7$, will also characterize a cubic lattice graph.

Acknowledgment. The author wishes to express her grateful thanks to Professor R.C. Bose for suggestion of the problem and many encouraging discussions. Thanks are due to Dr. T.A. Dowling for stimulating discussions.

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