A GEOMETRIC CHARACTERIZATION OF THE LINE GRAPH OF A PROJECTIVE PLANE

by

T.A Dowling and Renu Laskar
University of North Carolina

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Abstract

The line graph of a graph G is defined to be the graph whose vertices are the edges of G, two vertices being adjacent if and only if the corresponding edges of G have a common vertex. If π is a projective plane of order n, then the bipartite graph of π, denoted by $G(\pi)$, is the graph whose vertices are the $2(n^2+n+1)$ points and lines of π, two vertices of $G(\pi)$ being adjacent if and only if one is a point and the other is a line containing the point. The line graph of $G(\pi)$ will be denoted by $L(\pi)$. A characterization of the class $\{L(\pi)\}$ in terms of the eigenvalues of the adjacency matrix of $L(\pi)$ has been given by Hoffman. An alternative characterization of the class $\{L(\pi)\}$ is presented here. If $d(x,y)$ denotes the distance between two vertices x and y and $\Delta(x,y)$ denotes the number of vertices adjacent to both x and y, then $G = L(\pi)$ has the following properties: (b_1) The number of vertices in G is $(n+1)(n^2+n+1)$. (b_2) G is connected and regular of valence 2n. (b_3) $\Delta(x,y) = n-1$ if $d(x,y) = 1$. (b_4) $\Delta(x,y) = 1$ if $d(x,y) = 2$. (b_5) If $d(x,y) = 2$, the number of vertices z, such that $d(x,z) = 1$ and $d(y,z) = 2$, is at most n-1. In this paper we show that any graph G with no loops and no multiple edges having the properties (b_1)-(b_5) must be the line graph of a projective plane of order n.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.

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1. Introduction. We shall consider only finite, undirected graphs G with no loops and no multiple edges. The line graph of G is the graph whose vertices are the edges of G, two vertices being adjacent if and only if the corresponding edges of G have a common vertex. If π is a projective plane of order n, then the bipartite graph of π, denoted by $G(\pi)$, is the graph whose vertices are the $2(n^2+n+1)$ points and lines of π, two vertices of $G(\pi)$ being adjacent if and only if one is a point and the other is a line containing the point. The line graph of $G(\pi)$ will be denoted by $L(\pi)$. Hoffman [2] showed that if G is a regular connected graph on $(n+1)(n^2+n+1)$ vertices such that the adjacency matrix of G has the same distinct eigenvalues as those of the adjacency matrix of $L(\pi)$, then $G = L(\pi')$ where π' is a projective plane of order n. Since the eigenvalues of the adjacency matrix of $L(\pi)$ depend only on n, they are the same for every projective plane π of order n. Thus Hoffman's result states in effect that these eigenvalues characterize the class $\{L(\pi)\}$ consisting of the line graphs of all projective planes of order n. In this paper we obtain an alternative characterization of the class $\{L(\pi)\}$ based on properties of $L(\pi)$ other than the

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eigenvalues of its adjacency matrix.

2. Properties of $L(\pi)$. In this section we shall determine several properties of $L(\pi)$ which will subsequently be shown to characterize the class of graphs $\{L(\pi)\}$. It is convenient here to recall a few definitions.

Let G be a finite, undirected graph with no loops and no multiple edges. A path $C = (x_1, x_2, \ldots, x_m)$ is a sequence of $m \geq 2$ vertices of G, not necessarily all distinct, such that any two consecutive vertices in the sequence are adjacent and the pairs (x_1,x_2), $(x_2,x_3),\ldots,(x_{m-1},x_m)$ are distinct edges of G. The number of edges m-1 is the length of C, and C is said to join x_1 and x_m. If no vertex appears twice in C, then C is called an arc. If $x_1 = x_m$, but no other vertex appears twice, C is called a circuit.

G is said to be connected if there is a path joining x and y for every pair of distinct vertices x,y in G. For a connected graph the distance $d(x,y)$ is defined to be the length of the shortest path joining x and y. Clearly $d(x,y) = 1$ if and only if x and y are adjacent. In addition, we define $d(x,x) = 0$.

The valence $v(x)$ of a vertex x in G is the number of vertices adjacent to x. If there exists an integer v such that $v(x) = v$ for every vertex x in G, then G is said to be regular of valence v.

For any two distinct vertices x and y, $\Delta(x,y)$ denotes the number of vertices adjacent to both x and y. If x and y are adjacent vertices, $\Delta(x,y)$ is called the edge-degree of the edge (x,y). A regular graph G for which all edges have the same edge-degree Δ is said to be edge-regular with edge-degree Δ.

Lemma (2.1). If $G = L(\pi)$, where π is a projective plane of order n, then

(b_1) G contains $(n+1)(n^2+n+1)$ vertices;

(b_2) G is connected and regular of valence 2n;

(b_3) G is edge-regular with edge-degree n-1, i.e., $\Delta(x,y) = n-1$ if $d(x,y) = 1$;

(b_4) $\Delta(x,y) = 1$ if $d(x,y) = 2$;

(b_5) if $d(x,y) = 2$, the number of vertices z, such that $d(x,z) = 1$ and $d(y,z) = 2$, is at most n-1.

Proof. (b_1): The vertices of $L(\pi)$ can be represented by pairs (P,ℓ), where P is a point in π and ℓ is a line in π containing P. Since π contains n^2+n+1 points and each point is incident with n+1 lines, the number of such pairs is $(n+1)(n^2+n+1)$.

(b_2): Two distinct vertices (P,ℓ) and (P_1,ℓ_1) of $L(\pi)$ are adjacent if and only if $P = P_1$ or $\ell = \ell_1$. Thus the vertices adjacent to (P,ℓ) fall into two categories: (i) those of the form (P,m) where $P \in m$ and $m \neq \ell$; and (ii) those of the form (Q,ℓ) where $Q \in \ell$ and $Q \neq P$. Since there are n lines through P other than ℓ and n points on ℓ other than P, the number of vertices adjacent to (P,ℓ) is clearly 2n.

Let (P,ℓ) and (P_1,ℓ_1) be two distinct vertices of $L(\pi)$. We wish to show that there is a path C joining (P,ℓ) and (P_1,ℓ_1). If they are adjacent the result is obvious, so we may assume $P \neq P_1$, $\ell \neq \ell_1$. There exists a unique line m in π containing P and P_1. If $m = \ell$, we can take $C = ((P,\ell), (P_1,\ell), (P_1,\ell_1))$. If $m = \ell_1$, we let $C = ((P_1,\ell), (P,\ell_1), (P_1,\ell_1))$. If $m \neq \ell$, $m \neq \ell_1$, then $C = ((P,\ell), (P,m), (P_1,m), (P_1,\ell_1))$ is a path joining (P,ℓ) and (P_1,ℓ_1). It follows that $L(\pi)$ is connected. In addition, we note that (P,ℓ) and (P_1,ℓ_1) are at distance two if and only if $P \neq P_1$, $\ell \neq \ell_1$, but either $P \in \ell_1$ or $P_1 \in \ell$.

(b_3): If (P,ℓ) and (P_1,ℓ_1) are adjacent, then either $P = P_1$ or $\ell = \ell_1$. Consider first the case $P = P_1$. Clearly every vertex of the form (P,m) where $P \in m$ and $m \neq \ell$, $m \neq \ell_1$, is adjacent to both (P,ℓ) and (P,ℓ_1). There are n-1 lines m and hence n-1 such vertices. Clearly no other vertex can be adjacent to both (P,ℓ) and (p,ℓ_1). Next suppose $\ell=\ell_1$. Then every vertex of the form (Q,ℓ) where $Q \in \ell$ and $Q \neq P$, $Q \neq P_1$, is adjacent to both (P,ℓ) and (P_1,ℓ). There are n-1 points Q and hence n-1 such vertices, and these are clearly the only vertices adjacent to both (P,ℓ) and (P_1,ℓ). Hence in either case we have $\Delta(x,y) = n-1$ when $x = (P,\ell)$ and $y = (P_1,\ell_1)$ are adjacent.

(b_4): If $x = (P,\ell)$, $y = (P_1,\ell_1)$, and $d(x,y) = 2$, then, as noted in the proof of (b_2), we have $P \neq P_1$, $\ell \neq \ell_1$, but either $P \in \ell_1$ or $P_1 \in \ell$. Suppose first that $P \in \ell_1$. This implies $P_1 \notin \ell$. The only pairs (Q,m) which could have one element in common with both (P,ℓ) and (P_1,ℓ_1) are (P,ℓ_1) and (P_1,ℓ). The first is a vertex of $L(\pi)$ since $P \in \ell_1$; the second, however, is not a vertex since $P_1 \notin \ell$. A similar argument shows that (P_1,ℓ) is the only vertex adjacent to both (P,ℓ) and (P_1,ℓ_1) in the case $P_1 \in \ell$. Hence in either case we have $\Delta(x,y) = 1$ if $d(x,y) = 2$.

(b_5): Again if $x = (P,\ell)$, $y = (P_1,\ell_1)$, and $d(x,y) = 2$, then $P \neq P_1$, $\ell \neq \ell_1$, but either $P \in \ell_1$ or $P_1 \in \ell$. Suppose first that $P_1 \in \ell$. Then clearly the n-1 vertices of the form $z = (P,m)$ where $P \in m$ and $m \neq \ell$, $m \neq \ell_1$, are adjacent to (P,ℓ) and at distance two from (P_1,ℓ_1). Any other vertex adjacent to (P,ℓ) is of the form (Q,ℓ) where $Q \in \ell$ and $Q \neq P$. If (Q,ℓ) is at distance two from (P_1,ℓ_1), then either $P_1 \in \ell$ or $Q \in \ell_1$. But $P \in \ell_1$ implies $P_1 \notin \ell$ and $Q \notin \ell_1$. Hence there are exactly n-1 vertices adjacent to (P,ℓ) and at distance two from (P_1,ℓ_1).

Next suppose $P_1 \in \ell$. In this case the n-1 vertices of the form $z = (Q,\ell)$ where $Q \in \ell$ and $Q \neq P$, $Q \neq P_1$, are adjacent to (P,ℓ) and at distance two from (P_1,ℓ_1). Any other vertex adjacent to (P,ℓ) is of the form (P,m) where $P \in m$ and $m \neq \ell$. If (P,m) is at distance two from (P_1,ℓ_1), then either $P \in \ell_1$ or $P_1 \in m$. But $P_1 \in \ell$ implies $P \notin \ell_1$ and $P_1 \notin m$. Hence again we have exactly n-1 vertices z adjacent to $x = (P,\ell)$ and at distance two from $y = (P_1,\ell_1)$. It follows a fortiori that (b_5) holds for $L(\pi)$.

3. Geometric characterization of $\{L(\pi)\}$. We shall prove in this section that properties (b_1)-(b_5) of Lemma (2.1) characterize the class $\{L(\pi)\}$. The proof requires a theorem due to Bose and Laskar on the grand cliques of an edge-regular graph. For completeness we shall quote the theorem here along with some definitions.

Let G be a connected, edge-regular graph with valence r(k-1) and edge-degree $k - 2 + \alpha$, and in addition let G satisfy $\Delta(x,y) \leq 1 + \beta$ for every pair of non-adjacent vertices x and y, where r, k, α, β are fixed integers such that $r \geq 1$, $k \geq 2$, $\alpha \geq 0$, $\beta \geq 0$, and $r\beta - 2\alpha \geq 0$. A grand clique K in G is a set of mutually adjacent vertices satisfying $|K| \geq k - (r-1)\alpha$, and such that for any vertex $y \notin K$, there exists at least one vertex in K not adjacent to y. Bose and Laskar [1] proved the following theorem for G:

Theorem. If $k > \max[p(r,\alpha,\beta), \rho(r,\alpha,\beta)]$, where $p(r,\alpha,\beta) = 1 + \frac{1}{2}(r+1)(r\beta-2\alpha)$, $\rho(r,\alpha,\beta) = 1 + \beta + (2r-1)\alpha$, then each vertex of G is contained in exactly r grand cliques and each pair of adjacent vertices is contained in exactly one grand clique.

In the remainder of this paper we shall let G denote a graph possessing properties (b_1)-(b_5) of Lemma (2.1) for some positive integer n.

Lemma (3.1). Each vertex of G is contained in exactly two grand cliques, and each pair of adjacent vertices is contained in exactly one grand clique.

Proof. It is easily verified that G satisfies the conditions of the previous theorem when $r = 2$, $k = n+1$, $\alpha = 0$, $\beta = 0$. In this case $p(r,\alpha,\beta) = \rho(r,\alpha,\beta) = 1$. Hence if $k > 1$, i.e., $n > 0$, the theorem holds for G.

Lemma (3.2). Each grand clique in G contains n+1 vertices.

Proof. If K is a grand clique in G, then by definition $|K| \geq k - (r-1)\alpha = n+1$. If $|K| \geq n+2$, then given any two vertices x,y in K, there would exist at least n vertices adjacent to both, which contradicts (b_3).

Lemma (3.3). The number of grand cliques in G is $2(n^2+n+1)$.

Proof. Let N be the number of grand cliques in G. By counting pairs (x,K) where x is a vertex in G and K is a grand clique containing x, we have from (b_1) and Lemmas (3.1) and (3.2)

$$(n+1)\ N = 2(n+1)(n^2+n+1),$$

from which the lemma follows immediately.

From Lemma (3.1) it is clear that two distinct grand cliques of G have at most one vertex in common. If K and L have a common vertex x, we say that K and L intersect and write $x = K \cap L$. Since each vertex is contained in exactly two grand cliques, the above representation of x as the intersection of two grand cliques is unique. Hence there is a one-one correspondence between vertices of G and pairs of intersecting grand cliques such that $x \longleftrightarrow (K,L)$ if and only if $x = K \cap L$.

A clique path $C = (S_1,S_2,\ldots,S_m)$ is a sequence of $m \geq 2$ grand cliques of G, not necessarily all distinct, such that any two consecutive grand cliques in the sequence intersect and the vertices $S_i \cap S_{i+1}$, $i = 1,2,\ldots,m-1$ are all distinct. The number $m-1$ of such vertices will be called the length of C and denoted by $\ell(C)$. C is said to join S_1 and S_m. If no grand clique appears twice in C, then C is a <u>clique arc</u>. If $S_1 = S_m$, but no other grand clique appears twice, then C is called a <u>clique circuit</u>.

<u>Lemma (3.4)</u>. If C is a clique circuit in G, then $\ell(C) \geq 6$.

<u>Proof</u>. Let $C = (S_1, S_2, \ldots, S_{\ell(C)}, S_1)$. The cases $\ell(C) = 1,2$ are clearly impossible, so it is sufficient to consider the cases $3 \leq \ell(C) \leq 5$.

(i) Suppose $\ell(C) = 3$. Consider the adjacent vertices $x = S_1 \cap S_2$ and $y = S_2 \cap S_3$. There are $n-1$ vertices in S_2 adjacent to both x and y. If $\ell(C) = 3$, then $z = S_3 \cap S_1$ is adjacent to both x and y and $z \notin S_2$. Hence we have $\Delta(x,y) \geq n$, contradicting (b_3).

(ii) Suppose $\ell(C) = 4$. Let $x = S_1 \cap S_2$ and $y = S_3 \cap S_4$. Clearly $d(x,y) = 2$ for otherwise there would exist a clique circuit of length 3, contradicting (i) above. But if $\ell(C) = 4$, then $z_1 = S_4 \cap S_1$ and $z_2 = S_2 \cap S_3$ are adjacent to both x and y, which contradicts (b_4).

(iii) Suppose $\ell(C) = 5$. Let $x = S_1 \cap S_2$ and $y = S_3 \cap S_4$. Since $d(x,y) = 2$, by (b_5) there can be at most $n-1$ vertices z such that $d(x,z) = 1$, $d(y,z) = 2$. But each of the $n-1$ vertices in S_2 other than x and $x^* = S_2 \cap S_3$ is of this type, and in addition so is $z' = S_5 \cap S_1$. It follows that $\ell(C) \geq 6$.

Let $x = K \cap L$ be an arbitrary vertex of G. Denote the n grand cliques intersecting K other than L by $L_1, L_2, \ldots, L_n$, and the n grand cliques intersecting L other than K by $K_1, K_2, \ldots, K_n$. For $i = 1,2,\ldots,n$, let $K_{i1}, K_{i2}, \ldots, K_{in}$ be the n grand cliques intersecting L_i other than K, and let $L_{i1}, L_{i2}, \ldots, L_{in}$ be the n grand cliques intersecting K_i other than L.

Lemma (3.5). The $2(n^2+n+1)$ grand cliques $K,L,K_i,L_i,K_{ij},L_{ij}$, $i,j = 1,2,\ldots,n$, defined above are all distinct, and hence constitute all the grand cliques of G.

Proof. This lemma follows directly from Lemma (3.4), for if any two of these grand cliques are identical then the common clique will be contained in a clique circuit of length 5 or less. For example, if $K_{ij} = L_{i'j'}$, then the sequence $\mathcal{C} = (K,L_i,K_{ij},K_{i'},L,K)$ is a clique circuit of length 5. A similar result holds for any pair of grand cliques in the above collection as can be easily verified.

Lemma (3.6). The $2(n^2+n+1)$ grand cliques of G can be divided into two sets $\mathcal{K}$ and $\mathcal{L}$ each containing n^2+n+1 grand cliques, such that no two grand cliques in the same set intersect.

Proof. Let the grand cliques of G be denoted by $K,L,K_i,L_i,K_{ij},L_{ij}$, $i,j = 1,2,\ldots,n$, as defined above, and let

$$\mathcal{K} = \{K,\ K_i,K_{ij},\ i,j = 1,2,\ldots,n\},$$

$$\mathcal{L} = \{L,\ L_i,\ L_{ij},\ i,\ j = 1,2,\ldots,n\}.$$

These two sets satisfy the conditions of the lemma, for it is easily verified that if two cliques in the same set intersect, then these cliques are contained in a clique circuit of length five or less, contradicting Lemma (3.4).

Lemma (3.7). For any two distinct grand cliques S and T in G, there exists a clique path $\mathcal{C}$ joining S and T such that $\ell(\mathcal{C}) \leq 3$.

Proof. Without loss of generality we can take $S = K$ where K, L, K_i, L_i, K_{ij}, L_{ij}, i, $j = 1,2,\ldots,n$ are as defined above. Then for each possible choice of T, the sequence $\mathcal{C}$ defined below is seen to be a clique path satisfying the conditions of the lemma. If $T = L$, $C = (K,L)$; if $T = L_i$, $C = (K,L_i)$; if $T = K_i$, $C = (K,L,K_i)$; if $T = K_{ij}$, $C = (K,L_i, K_{ij})$; and if $T = L_{ij}$, $C = (K,L,K_i, L_{ij})$.

Consider now the graph H whose vertices are the grand cliques of G, where two vertices S* and T* of H are adjacent if and only if the corresponding grand cliques S and T, respectively, of G intersect.

Lemma (3.8). If H is the graph defined above, then

(i) H is regular of valence n+1;

(ii) H is bipartite and each vertex set of H contains n^2+n+1 vertices;

(iii) two distinct vertices of H are at distance two if and only if they are in the same vertex set, and in this case there is exactly one vertex adjacent to both.

Proof. (i) The fact that H is regular of valence n+1 follows immediately from Lemmas (3.1) and (3.2).

(ii) A graph is bipartite if the vertices can be assigned to disjoint sets U,V such that every edge (u,v) connects a vertex $u \in U$ to a vertex $v \in V$ [3]. Consider the two sets $\mathcal{K}^*$, $\mathcal{L}^*$ of vertices in H corresponding to the sets $\mathcal{K}$, $\mathcal{L}$ of grand cliques in G described in Lemma (3.6). If (S*,T*) is an edge in H, then S and T are intersecting grand cliques in G, and thus by Lemma (3.6) one of S and T must be in $\mathcal{K}$ and the other in $\mathcal{L}$. It follows that one of S*, T* is in $\mathcal{K}^*$ and the other is in $\mathcal{L}^*$. Hence H is bipartite and each vertex set of H contains n^2+n+1 vertices.

(iii) If S*, T* are vertices in H such that d(S*, T*) = 2, then clearly they are both in the same vertex set. Conversely, if S* and T* are in the same vertex set, then d(S*,T*) = 2t is even. There exists a path C joining S* and T* of length 2t, but no shorter path joins S* and T*.

This implies that in G there exists a corresponding clique path C of length $2t$ joining S and T, but no shorter path joins S and T. Hence by Lemma (3.7) we have $\ell(C) = 2t \leq 3$, which implies $t=1$ and therefore $d(S^*,T^*) = 2$.

Suppose next that $d(S^*,T^*) = 2$ and that there exist two vertices U^*, V^* adjacent to both S^* and T^*. Then $C^* = (S^*, U^*, T^*, V^*, S^*)$ is a circuit of length four in H, and thus $C = (S,U,T,V,S)$ is a clique circuit of length four in G, which contradicts Lemma (3.4). Hence $\Delta(S^*,T^*) = 1$ if $d(S^*,T^*) = 2$.

Lemma (3.9). $H = G(\pi')$ for some projective plane π' of order n.

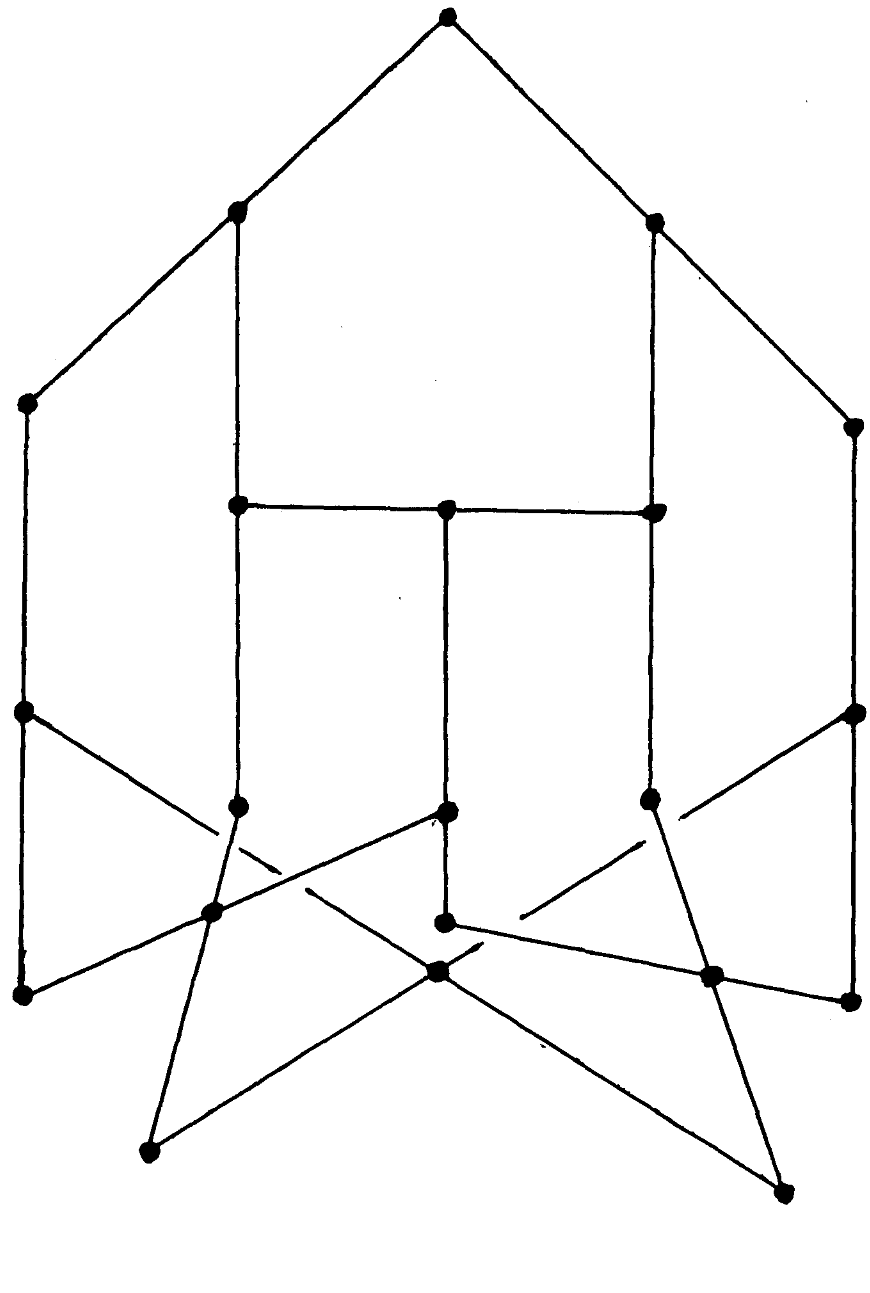
Proof. Let the vertices in one vertex set be called lines and the vertices in the other vertex set be called points. We define a point and a line to be incident if and only if the corresponding vertices are adjacent in H. Then it follows from Lemma (3.8) that there are n^2+n+1 points and n^2+n+1 lines, that each point is incident with $n+1$ lines and each line is incident with $n+1$ points, and that any two points are incident with exactly one line and any two lines are incident with exactly one point. Thus the set of points and lines together with the incidence relation defined above constitute a projective plane π' of order n. The bipartite graph of π' is clearly H.

Theorem. If G is an undirected graph with no loops and no multiple edges possessing the properties (b_1)-(b_5) of Lemma (2.1), then $G = L(\pi')$ where π' is a projective plane of order n.

Proof. Since $H = G(\pi')$ it will suffice to show that G is the line graph of H. We must show that there is a one-one correspondence between edges of H and vertices of G such that two edges of H have a common vertex if and only if the corresponding vertices of G are adjacent. If we define a correspondence by saying that the vertex z in G corresponds to the edge (S^*,T^*) in H if and only if $z = S \cap T$, then clearly this correspondence is one-one. If (S_1^*, T^*) and (S_2^*, T^*) are two edges in H with a common vertex T^*, then the corresponding vertices $z_1 = S_1 \cap T$ and $z_2 = S_2 \cap T$ in G are both contained in the grand clique T and are therefore adjacent. Conversely, if z_1 and z_2 are adjacent vertices in G, let T be the grand clique containing both z_1 and z_2 and let S_i, i=1,2, be the grand clique containing z_i other than T. Then the vertices $z_1 = S_1 \cap T$ and $z_2 = S_2 \cap T$ in G correspond to the edges (S_1^*,T^*), (S_2^*,T^*), respectively, in H. Hence the edges in H corresponding to the adjacent vertices z_1, z_2 in G have a common vertex T^*. This completes the proof.

It follows from the above theorem that properties (b_1)-(b_5) of Lemma (2.1) characterize the class $\{L(\pi)\}$ consisting of the line graphs of all projective planes of order n. In the early stages of this research attempts were made to show that property (b_5) is redundant (inasmuch as Bose and Laskar's theorem and hence Lemma (3.1) hold without assuming (b_5)) and that, consequently, the class $\{L(\pi)\}$ could be characterized by properties (b_1)-(b_4). The following conterexample shows that property (b_5) is, indeed, necessary. The graph G below satisfies (b_1)-(b_4) for n=2, but it contains many clique circuits of length 5. Hence H contains circuits of length 5 and cannot be bipartite. The lines in the diagram represent grand cliques.

Figure 1. Counterexample showing that condition (b_5) is not redundant (n=2).



References

[1] R. C. Bose and Renu Laskar, A characterization of the tetrahedral graph, Institute of Statistics Mimeo Series No. 509, University of North Carolina, March, 1967.

[2] A. J. Hoffman, On the line graph of a projective plane, Proceedings of the American Mathematical Society, Vol. 16, 1965.

[3] O. Ore, Theory of Graphs, American Mathematical Society Colloquium Publications, Vol. XXXVIII, 1962.