REMARKS ON THE STRUCTURE OF STATIONARY POINT PROCESSES

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Summary

In this report the analytical approach of Khintchine is used in defining a basic structure for stationary point processes. The main section consists of a simple unified treatment of three important theorems - Khintchine's existence theorem for the intensity of the process, Korolyuk's Theorem, and a fundamental lemma of Dobrushin. It is also shown how the proof of the existence of certain other limits of basic importance, may be deduced by means of Khintchine's Theorem.

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1. Introduction. There are a number of somewhat different viewpoints from which one can approach the structure of a stationary point process. For example the earliest formal treatment of point processes appears to be that of Khintchine [3], in which an "analytical" approach is used. More will be said ofthis below, but briefly the properties of such a process are developed by Khintchine in terms of random variables N(s,t) representing the number of events occurring in the time interval (s,t]. There is very little reference to the basic probabilistic structure except through such random variables.

On the other hand Ryll Nardzewski [7] has given a very elegant general theory from an abstract point of view - concentrating on measure theoretic properties of the underlying basic probability space. This point of view has been further developed by Matthes [6] in providing a very polished and general theory indeed. In this Matthes uses the device of associating a "mark" with the occurrence of each event of the process, in a particularly profitable way.

Another approach to the theory is to begin by considering a sequence of random variables (representing the times between consecutive events) for which there is an obvious probability space. This approach is often implicit in theories of point processes of special structure (such as renewal processes) and has been used for general point processes, for example by Beutler and Leneman [1]. It leads to explicit results concerning the distribution functions and moments relevant to the process.

The approach of Ryll Nardzewski [7] and Matthes [6] is perhaps aesthetically the most desirable from the point of view of obtaining a very general theory. However at the same time it is less intuitive and requires more mathematical machinery than the other two approaches. Of these other two approaches we prefer the lines of development due to Khintchine (which we shall take here) since the starting point is with the probabilities of primary interest - the distributions of the numbers of events in given intervals. This is, however, a matter of personal choice.

In Section 2 we shall briefly review the construction of a basic probabilistic structure along the lines of Khintchine's development. This basic structure will be such that the quantities $N(s,t)$ defined on the basic sample space (and representing the number of events in $(s,t]$), are genuine random variables.

It is this family of such random variables $N(s,t)$ which, from our point of view, constitute the "point process." Following Khintchine this point process will be termed <u>stationary</u> if the joint distribution of the random variables

$$N(s_1 + \tau,\ t_1 + \tau) \quad \ldots . \quad N(s_k + \tau,\ t_k + \tau)$$

is independent of τ, for any fixed integer k and choice of (s_i, t_i), $i=1\ldots k$.

<u>Khintchine's existence theorem</u> asserts that for such a stationary point process there exists a non negative constant $\lambda \leq \infty$ such that

$$P\{N(0,t) \geq 1\} \sim \lambda t \quad \text{as } t \downarrow 0.$$

This result depends <u>only</u> on the assumption of stationarity stated above.

A further property which may be possessed by stationary point process is that of <u>regularity</u> (or orderliness, in the sense of Khintchine [3]). Such a process will be called <u>regular</u> if

$$P\{N(0,t) > 1\} = o(t) \text{ as } t \downarrow 0.$$

<u>Korolyuk's Theorem</u> ([3]) states that for a stationary regular process we have $\lambda = \mathcal{E}\{N(0,1)\}$.

In general multiple events may occur - that is it is possible to have more than one event at the same time point. It is easy to see that this is not possible for a stationary, regular process. (For under these conditions we have $P\{\text{a multiple event occurs somewhere in } 0 < t \leq 1\} \leq \sum_{i=0}^{n-1} P\{N(\frac{i}{n}, \frac{i+1}{n}) > 1\} = nP\{N(0, \frac{1}{n}) > 1\} \longrightarrow 0$ as $n \longrightarrow \infty$.)

Conversely Dobrushin's Lemma (See Volkonski [8]) asserts that if multiple events cannot occur, and if $\mathcal{E}N(0,1) < \infty$, then the stationary point process is also regular.

These three results (Khintchine's Existence Theorem, Korolyuk's Theorem and Dobrushin's Lemma) are of fundamental importance and have been proved separately (by moderately lengthy arguments) in the references cited (cf. also Cramér and Leadbetter [2]). In Section 3 we shall give a unified treatment for these results by means of a simple basic technique.

Finally in Section 4 it will be briefly shown how certain distribution functions (concerning for example, times between consecutive events) may be defined within this framework.

2. – The basic probabilistic structure.

In this section we shall sketch one approach to the construction of a basic probability space for a point process. This particular construction is, as noted, along the lines of that of Khintchine [3]. (For further discussion see also Cramér and Leadbetter [2 , Sections 3.7, 3.8].

Specifically we wish to construct a probability space on which the quantities $N(s,t)$ will be random variables and will have finite dimensional distributions given by

$$P\{N(s_i,t_i) = r_i,\ i=1,\dots,n\} = p_{I_1 \dots I_n}(r_1,\dots,r_n)$$

where I_i denotes the interval $(s_i, t_i]$, r_i are non negative integers and the quantities $p_{I_1 \ldots I_n}(r_1 \ldots r_n)$ are probabilities on such sets of non negative integers, which have been specified in advance in a consistent way. "Consistency" here is to be taken to mean that relations such as

$$p_{I_1 \ldots I_k}(r_1 \ldots r_k) = \sum_{r_{k+1}=0}^{\infty} \ldots \sum_{r_n=0} p_{I_1 \ldots I_n}(r_1 \ldots r_n)$$

hold together with the condition that each $p_{I_1 \ldots I_n}(r_1 \ldots r_n)$ should be unchanged when $I_1 \ldots I_n$ and $r_1 \ldots r_n$ are permuted in the same way. We require also that $p_{(t,t+h]}(0) \longrightarrow 1$ as $h \downarrow 0$, for each t, (which is necessary in view of the fact that $N(t,t+h)$ will be interpreted as the number of events in the semiclosed interval $(t,t+h]$ and this tends to zero as $h \downarrow 0$).

Define now a real stochastic process $\{x(t): -\infty < t < \infty\}$ by its finite dimensional distributions as follows. Let $x(0) = 0$ and for $0 < t_1 \ldots < t_n$,

$$P\{x(t_1) = r_1 \ldots x(t_n) = r_n\} = p_{I_1 \ldots I_n}(r_1, r_2 - r_1 \ldots r_n - r_{n-1})$$

where $0 \leq r_1 \ldots \leq r_n$ are non negative integers and $I_1 = (0,t_1] \ldots I_n = (t_{n-1}, t_n]$, (the probabilities being zero for all other values of $r_1 \ldots r_n$). From these probabilities we may define a consistent family of finite dimensional distribution functions $F_{t_1 \ldots t_n}(x_1 \ldots x_n)$, $0 < t_1 \ldots < t_n$ by summing over the probabilities defined above.

The definition is extended to the case where some of the t_i may be negative in an obvious way by writing for $u_1 < u_2 \ldots < u_m < 0 < v_1 \ldots < v_n$

$$P\{x(u_1) = r_1 \ldots x(u_m) = r_m, x(v_1) = s_1 \ldots x(v_n = s_n\}$$

$$= p_{I_1 \ldots I_{m-1} I_m J_1 J_2 \ldots J_n}(r_2 - r_1, \ldots r_m - r_{m-1}, -r_m, s_1, s_2 - s_1 \ldots s_n - s_{n-1})$$

where $I = (u_1, u_2] \ldots I_m = (u_m, o]$, $J_1 = (o, v_1] \ldots J_n = (v_{n-1}, v_n]$, and r_i, s_j are integers, $r_1 \leq r_2 \ldots \leq r_m \leq 0 \leq s_1 \leq s_2 \ldots \leq s_n$.

By Kolmogorov's Theorem we may thus define the stochastic process $x(t)$ having the above finite dimensional distributions (the basic probability space consisting of real functions on the real line).

From the definition of the finite dimensional distributions we thus have $x(s) \leq x(t)$ with probability one for each fixed $s \leq t$. The exceptional (null) set may depend on s,t. In order to remove this possibility we form an equivalent version $y(t)$ of the process defined as follows. Write $y(t) = x(t)$ for all rational t. If t is irrational, however, let $y(t) = \lim_{k \to \infty} \{x(t_k): t_k \text{ rational}, t_k \downarrow t\}$. This limit exists with probability one since $x(t)$ is, with probability one, non decreasing on the rationals by virtue of their countability and the remark above. It follows that, with probability one, $y(t)$ is non decreasing and (since $x(t) - x(s)$ is an integer with probability one for given s,t) is a step function, increasing only by jumps of integer values, of which there are only a finite number in any bounded interval. Finally, since $P\{x(t_k) \neq x(t)\} = 1 - p_{(t,t_k]}(o) \to o$ as $t_k \downarrow t$ by assumption, it follows that $x(t_k) \to x(t)$ in probability and hence $x(t) = y(t)$ with probability one. That is $y(t)$ and $x(t)$ are equivalent processes and in particular, $y(t)$ also has the same finite dimensional distributions defined above for $x(t)$.

Finally we may now write $N(s,t) = y(t) - y(s)$ which is a well defined random variable. Further, with probability one, $N(s,t)$ is non negative and integer valued for all $s \leq t$. We may thus regard the jumps of $y(t)$ as the occurrence of events of the point process and $N(s,t)$ as the number of such events in the interval $(s,t]$.

While the quantities $N(s,t)$ are the basic random variables it is clear that other random variables of interest may be defined from this structure. For example the time of occurrence of the k-th event after $t = o$ is simply the smallest value of t for which $y(t) \geq k$. However there are certain other quantities discussed in the literature as if they were random variables, but which are not well defined as such. These are usually associated with the concept of an "arbitrary event" of the process. Such concepts may have heuristic value but care must be exercised in stating the results within a precise framework (See e.g., Leadbetter [4]).

Finally in this section we repeat that the point process will be termed <u>stationary</u> if the joint distributions of the now well defined random variables $N(s_1 + \tau, t_1 + \tau) \ldots N(s_k + \tau, t_k + \tau)$ are independent of τ. The stationarity assumption will be made throughout the following sections.

3. <u>The basic theorems</u>. In this section we shall give a short unified treatment for the three theorems described in Section 1. For reference and comparison we state these results again here, using the notation developed above. Korolyuk's Theorem is stated in slightly sharpened form - to apply to a stationary point process without multiple events, instead of assuming regularity. That this is a slightly sharper form follows from the result (shown in Section 1) that if a stationary point process is also regular, then there is zero probability that a multiple event will occur anywhere. (We note again also that Dobrushin's Lemma is a partial converse to this latter fact).

Khintchine's Theorem. Any stationary point process has an intensity λ, $0 \leq \lambda \leq \infty$. That is $P\{N(0,t) \geq 1\} \sim \lambda t$ as $t \downarrow 0$.

Korolyuk's Theorem (slightly sharper form). Consider a stationary point process for which there is zero probability of the occurrence of multiple events. Then $\mathcal{E}N(0,1) = \lambda \leq \infty$.

Dobrushin's Lemma. If in addition to the above assumptions for Korolyuk's Theorem we have $\mathcal{E}N(0,1) < \infty$ (or equivalently $\lambda < \infty$), then the stationary point process is also regular.

Consider then a stationary point process. It is clear that we may modify this process to form a new stationary point process by considering multiple events as single events in the modified process. Write $N^*(s,t)$ for the number of such (modified) events in (s,t); that is the number of original events in (s,t) counted without regard for their possible multiplicities. It follows at once that $N^*(s,t) \geq 1$ if and only if $N(s,t) \geq 1$.

We first prove the following result, which contains both the theorem of Khintchine and Korolyuk's Theorem.

Theorem. Consider a stationary point process and with the above notation, write $\mu^* = \mathcal{E}\, N^*(0,1) \leq \infty$. Then

$$(1) \qquad P\{N(0,t) \geq 1\} \sim \mu^* t \text{ as } t \to 0.$$

Proof: For a given integer n write $X_{in} = 1$ if $N(i/n, (i+1)/n) \geq 1$ and $X_{in} = 0$ otherwise, $i = 0,1 \ldots n-1$. Let $N_n = \sum_{i=0}^{n-1} X_{in}$. Then it is easy to see that $N_n \to N^*(0,1)$ with probability one, as $n \to \infty$. (Indeed for each "sample point" outside a null set, $N_n = N^*(0,1)$ when n is sufficiently large).

If $\mu^* < \infty$ it follows at once by dominated convergence ($N_n \le N^*(0,1)$ with probability one) that $\mathcal{E}\, N_n \to \mathcal{E}\, N^*(0,1)$ and hence from stationarity that $n\, \mathcal{E}\, X_{on} = n\, P\{N(0,n^{-1}) \ge 1\} \to \mu^*$ as $n \to \infty$. The required result then follows from the obvious inequalities

$$\frac{t^{-1}}{([t^{-1}]+1)} \quad \frac{P\{N(0,([t^{-1}]+1)^{-1}) \ge 1\}}{([t^{-1}]+1)^{-1}} \le \frac{P\{N(0,t) \ge 1\}}{t}$$

$$\le \frac{P\{N(0,[t^{-1}]^{-1}) \ge 1\}}{[t^{-1}]^{-1}} \quad \frac{t^{-1}}{[t^{-1}]}$$

(where [x] denotes the integer part of x), since the outside terms each tend to μ^* as $t \downarrow 0$.

If $\mu^* = \infty$ an application of Fatou's lemma in place of dominated convergence easily yields (1).

This result shows at once that the point process has an intensity $\lambda = \mu^*$, i.e. the theorem of Khintchine follows.

Further, if multiple events have probability zero then $\mu^* = \mathcal{E}N(0,1)$ and hence $\lambda = \mathcal{E}N(0,1)$. That is, we have the form of Korolyuk's Theorem as stated.

Finally, we prove Dobrushin's Lemma by a slight modification of the above technique.

<u>Theorem</u>. Suppose the point process considered is stationary, that the probability of the occurrence of multiple events is zero, and that $\mathcal{E}N(0,1) < \infty$. Then the process is also regular.

<u>Proof</u>. Write $X'_{in} = 1$ if $N(i/n, (i+1)/n) > 1$, $X'_{in} = 0$ otherwise, $i = 0,1,\ldots,n-1$. Let $N'_n = \sum_{i=0}^{n-1} X'_{in}$. Then N'_n is the number of intervals $(i/n, (i+1)/n)$ containing at least two events and since multiple events

have probability zero, $N'_n \to 0$ with probability one. But $N'_n \leq N(0,1)$ and hence by dominated convergence, $\mathcal{E}N'_n \to 0$ as $n \to \infty$. By stationarity we thus have $n\, P\{N(0,n^{-1}) > 1\} \to 0$ as $n \to \infty$. Hence

$$\frac{P\{N(0,t) > 1\}}{t} \leq \frac{P\{N(0,[t^{-1}]^{-1} > 1\}}{[t^{-1}]^{-1}} \frac{t^{-1}}{[t^{-1}]} \longrightarrow 0 \text{ as } t \longrightarrow 0.$$

4. <u>Other basic limits.</u>. The problem of Khintchine's existence theorem was to show that $P\{N(0,t) \geq 1\} / t$ converged to a limit as $t \downarrow 0$. There are other - apparently more involved - cases when one would like to show the existence of limits of this nature, for a stationary point process. For example consider the conditional probability

$$G_k(x,\tau) = P\{N(0,x) \geq k \mid N(-\tau,0) \geq 1\}$$

where k is a non negative integer, and $x > 0$, $\tau > 0$. If $G_k(\tau,x)$ can be shown to tend to a limit as $\tau \downarrow 0$, $F_k(x)$ say, then $F_k(x)$ may be interpreted heuristically as the probability of at least k events in $(0,x]$, "given that an event occurred at time zero." Equivalently we may regard $F_k(x)$ as the distribution function of the "time from an 'arbitrary' event to the k-th subsequent event." The reason for using this limiting process instead of a direct definition in terms of a random variable is that (as noted near the end of Section 2) the appropriate "random variable" is not properly defined within this framework. This point is discussed more fully in [4].

The existence of limits such as this may be reduced to Khintchine's existence theorem by a simple device which is in fact a special case of the use of the "marks" of Matthes [7].

Specifically suppose that the stationary point process is regular and that $0 < \lambda < \infty$. To obtain the limit referred to above we form a modified process as follows. We say that an event occurs at t_o in the modified process if an event of the original process occurred at t_o and if $N(t_o, t_o + x) \geq k$. Denote the number of such events in $(s,t]$ by $N_x(s,t)$. The modified process thus consists of those original events which are followed by k or more events within x further time units.

The modified process is clearly stationary. It is also regular since the original process is regular and $P\{N_x(0,t) > 1\} \leq P\{N(0,t) > 1\}$. Thus the modified process has the finite intensity $\lambda_x = \mathcal{E}N_x(0,1)$. Further, $N_{x+h}(0,1) - N_x(0,1) \downarrow 0$ as $h \downarrow 0$ and it follows that λ_x is continuous to the right.

Writing $H_k(x,\tau) = P\{N(0,x) \geq k, N(-\tau,0) \geq 1\}$ it is easy to see that, for $0 < \tau < \epsilon$,

$$P\{N_x(-\tau,0) \geq 1\} \leq H_k(x,\tau) + o(\tau) \leq P\{N_{x+\epsilon}(-\tau,0) \geq 1\} + o(\tau)$$

and hence that

$$\lambda_x \tau + o(\tau) \leq H_k(x,\tau) + o(\tau) \leq \lambda_{x+\epsilon} \tau + o(\tau)$$

or

$$\lambda_x + o(1) \leq H_k(x,\tau)/\tau + o(1) \leq \lambda_{x+\epsilon} + o(1)$$

Letting $\tau \downarrow 0$ and then $\epsilon \downarrow 0$, it follows from the right continuity of λ_x that $H_k(x,\tau)/\tau \longrightarrow \lambda_x$ as $\tau \downarrow 0$. But since

$$G_k(x,\tau) = H_k(x,\tau)/P\{N(-\tau,0) \geq 1\} \sim H_k(x,\tau)/\lambda\tau,$$

it follows that $G_k(x,\tau) \longrightarrow \lambda_x/\lambda$ as $\tau \downarrow 0$. Thus the assertion is proved and the required limit exists with the value λ_x/λ.

It is clear that this technique is capable of considerable generalization. For example one may define "distribution functions" for local maxima and minima of stationary stochastic processes (See Leadbetter [5]). As noted this situation is intimately related to the work of Matthes [6] who considers problems of this type in a very general and abstract setting.

Finally we note that we have here only indicated the method of proof of the existence of certain limits. For properties of the distribution functions such as $F_k(x)$ so obtained, we refer to the relevant literature ([1], [2], [3], [4], [5]).

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