A CHARACTERIZATION OF THE GRAPH OF THE T_m ASSOCIATION SCHEME

by

Thomas A. Dowling

University of North Carolina

Institute of Statistics Mimeo Series No. 535

June, 1967

Abstract

The graph G of the T_m association scheme is a graph on $\binom{n}{m}$ vertices, where $2 \leq m \leq n/2$, such that the vertices may be identified with unordered m-plets $(a_1, a_2, \ldots a_m)$ from among the n symbols $1,2,\ldots,n$ and two vertices of G are adjacent if and only if the corresponding m-plets have exactly m-1 common symbols. If $d(x,y)$ denotes the distance between two vertices x and y and $\Delta(x,y)$ denotes the number of vertices adjacent to both x and y, then G is a connected graph possessing the following properties: (a_1) The number of vertices in G is $\binom{n}{m}$; (a_2) G is regular of valence $m(n-m)$; (a_3) $\Delta(x,y) = n-2$ if $d(x,y) = 1$; (a_4) $\Delta(x,y) = 4$ if $d(x,y) = 2$. In this paper it is shown that, conversely, if G is a connected graph and (a_1)-(a_4) hold for some $m \geq 2$ and $n > 2m(m-1) + 4$, then G is necessarily the graph of the T_m association scheme. This result generalizes previously known results for the cases m = 2, 3 corresponding to the triangular and tetrahedral schemes.

This research was supported by the National Science Foundation Grant No. GP-3792 and the Air Force Office of Scientific Research Grant No. AF-AFOSR-760-65.

DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.

A CHARACTERIZATION OF THE GRAPH OF THE T_m ASSOCIATION SCHEME

by

Thomas A. Dowling
University of North Carolina
Chapel Hill, N. C.

1. Introduction. An m-class association scheme, called the T_m scheme, was introduced in [7] as a generalization of the well-known triangular scheme with two associate classes. The T_m scheme may be defined as follows: Let m and n be positive integers satisfying $2 \leq m \leq n/2$, and suppose we are given $v = \binom{n}{m}$ treatments $\varphi_1, \varphi_2, \ldots, \varphi_v$. To each of the v unordered m-plets $(a_1, a_2, \ldots, a_m)$ of distinct symbols from among $1, 2, \ldots, n$, we associate a treatment φ in some one-to-one manner. Two treatments φ and φ' are said to be u-th associates of each other if and only if the corresponding m-plets $(a_1, a_2, \ldots, a_m)$ and $(a_1', a_2', \ldots, a_m')$ contain exactly m-u symbols in common. The triangular association scheme clearly corresponds to the case m = 2.

The parameters of the T_m scheme are given in [7] as follows: The number of treatments is

(1) $$v = \binom{n}{m}.$$

The number of u-th associates of each treatment is

(2) $$n_u = \binom{m}{u}\binom{n-m}{u}, \quad u = 0, 1, \ldots, m.$$

For any two treatments φ and φ' which are t-th associates, the number of treatments which are s-th associates of φ and at the same time u-th associates of φ' is

(3) $$p_{su}^{t} = \sum_{i=0}^{m-t} \binom{m-t}{i} \binom{t}{m-s-i} \binom{t}{m-u-i} \binom{n-m-t}{s+u-m+i}, \quad u, s, t = 0, 1, \ldots, m,$$

where as usual we take $\binom{x}{y} = 0$ if $x < y$ or $y < 0$.

Given any association scheme, the graph of the scheme is constructed by identifying the v treatments with vertices, and joining two vertices by an edge if and only if the corresponding treatments are first associates.

This research was supported by the National Science Foundation Grant No. GP-3792 and the Air Force Office of Scientific Research Grant No. AF-AFOSR-760-65.

Thus two vertices of the graph G of a T_m scheme are adjacent if and only if the corresponding m-plets have exactly m-1 symbols in common. More generally, one can easily show that two vertices of G are at distance u from each other if and only if the corresponding m-plets have exactly m-u symbols in common, i.e., if and only if the corresponding treatments are u-th associates.

Using (1), (2), and (3) we have in particular for a T_m scheme $v = \binom{n}{m}$, $n_1 = m(n-m)$, $p_{11}^1 = n-2$, $p_{11}^2 = 4$. If for any finite, connected graph we denote by $d(x,y)$ the distance between the vertices x and y, and by $\Delta(x,y)$ the number of vertices adjacent to both x and y, then the above parameters imply that the graph G of a T_m scheme possesses the following properties:

(a_1) The number of vertices in G is $\binom{n}{m}$;

(a_2) G is regular of valence $m(n-m)$;

(a_3) $\Delta(x,y) = n-2$ if $d(x,y) = 1$;

(a_4) $\Delta(x,y) = 4$ if $d(x,y) = 2$.

In the case m = 2, corresponding to the triangular association scheme, Connor [2] showed (using somewhat different terminology) that if $n > 8$ the triangular graph is characterized by properties (a_1)-(a_4). Shrikhande [8], Li-Chien [5,6], and Hoffman [3,4] completed Connor's work by showing that the same result holds if $n < 8$, but that if $n = 8$ then there exist graphs satisfying (a_1)-(a_4) which are not triangular. Recently Bose and Laskar [1] have investigated the graph of the T_3 scheme, which they called a tetrahedral graph, and were able to show that a connected graph satisfying (a_1)-(a_4) for m = 3 is tetrahedral if $n > 16$. In this paper we generalize these results to an arbitrary $m \geq 2$. The main theorem states that if G is a connected graph satisfying (a_1)-(a_4) for some integers $m \geq 2$ and $n > 2m(m-1) + 4$, then G is the graph of a T_m scheme. Equivalently, we can say that the T_m scheme is characterized by the four parameters v, n_1, p_{11}^1, p_{11}^2 when $n > 2m(m-1) + 4$. Little is known at present concerning the uniqueness of the T_m scheme when $m \geq 3$ and $n \leq 2m(m-1) + 4$.

2. Characterization of the T_m graph.

We shall consider only finite, undirected graphs G with no loops and no multiple edges. For completeness it is convenient to recall here several definitions concerning such graphs.

A chain $C = (x_1, x_2, \ldots, x_t)$ is a sequence of $t \geq 2$ distinct vertices of G, not necessarily all distinct, such that any two consecutive vertices in the sequence are adjacent. Thus the pairs (x_1,x_2), $(x_2,x_3), \ldots,$ (x_{t-1},x_t) are edges of G. The number of edges t-1 is the length of C, and C is said to join x_1 and x_t.

G is a connected graph if there exists a chain joining x and y for every pair of distinct vertices x,y. For a connected graph the distance $d(x,y)$ is defined to be the length of the shortest chain joining x and y. Clearly $d(x,y) = 1$ if and only if x and y are adjacent vertices. By convention we take $d(x,x) = 0$.

The valence $v(x)$ of a vertex x is the number of vertices in G adjacent to x. If there exists an integer n_1 such that $v(x) = n_1$ for every vertex x in G, then G is said to be regular of valence n_1.

For any two distinct vertices x,y of G, we denote by $\Delta(x,y)$ the number of vertices adjacent to both x and y. Obviously $\Delta(x,y) = 0$ if $d(x,y) > 2$. If x and y are adjacent vertices, then $\Delta(x,y)$ is called the edge-degree of the edge (x,y). A regular graph G for which all edges have the same edge-degree Δ is said to be edge-regular with edge-degree Δ.

A clique K is a set of mutually adjacent vertices in G. K is said to be complete if there does not exist a vertex $x \notin K$ such that $x \cup K$ is a clique.

We now quote a result due to Bose and Laskar [1] which is of fundamental importance in the sequel. Consider an edge-regular graph G with valence $r(k-1)$ and edge-degree $(k-2) + \alpha$ such that $\Delta(x,y) \leq 1 + \beta$ for every pair of non-adjacent vertices x,y, where r,k,α,β are fixed integers satisfying $r \geq 1$, $k \geq 2$, $\alpha \geq 0, \beta \geq 0$, and $r\beta - 2\alpha \geq 0$. A grand clique K in G is defined to be a complete clique satisfying $|K| \geq k - (r-1)\alpha$. Then we have the following theorem [1] for G:

Theorem. If $k > \max[p(r,\alpha,\beta), \rho(r,\alpha,\beta)]$, where

$$p(r,\alpha,\beta) = 1 + \frac{1}{2}(r+1)(r\beta-2\alpha),$$

$$\rho(r,\alpha,\beta) = 1 + \beta + (2r-1)\alpha,$$

then

(i) each vertex of G is contained in exactly r grand cliques;

(ii) each pair of adjacent vertices is contained in exactly one grand clique.

In the remainder of this paper we shall denote by G a connected graph possessing properties (a_1)-(a_4) of Section 1 for some integers $m \geq 3$ and $n > 2m(m-1) + 4$.

<u>Lemma 1</u>. Each vertex of G is contained in exactly m grand cliques, and each pair of adjacent vertices is contained in exactly one grand clique. If K is a grand clique, then $|K| \geq n-m(m-1)$.

<u>Proof</u>. If we set $r = m$, $k = n-m+1$, $\alpha = m-1$, $\beta = 3$, then G satisfies the conditions of the previous theorem. In this case we have

$$p(r\ \alpha,\beta) = 1 + \frac{1}{2}(m+1)(m+2),$$
$$\rho(r,\alpha,\beta) = 4 + (2m-1)(m-1).$$

Thus

$$\rho(r,\alpha,\beta) - p(r,\alpha,\beta) = \frac{3}{2}(m-1)(m-2) \geq 0.$$

Hence the theorem holds for G if

$$n-m+1 > 4 + (2m-1)(m-1),$$

i.e., if

$$n > 2m(m-1) + 4,$$

a condition which we have assumed to be satisfied.

By definition we have for any grand clique K in G,

$$|K| \geq k - (r-1)\alpha$$
$$= n - m(m-1).$$

It is clear from Lemma 1 that any two grand cliques in G can intersect in at most one vertex. Also if x is any vertex and y is a vertex adjacent to x, then y is contained in one and only one of the m grand cliques containing x. Since there are m(n-m) vertices adjacent to x by (a_2), and each of these belongs to a unique grand clique through x, the average number of vertices other than x in a grand clique containing x is n-m. Thus the average clique size among the m grand cliques through x is n-m+1. The next two lemmas are needed to show that actually we have $|K| = n-m+1$ for every grand clique K in G.

Let K_0 and K_1 be two intersecting grand cliques in G, and let $x = K_0 \cap K_1$. Denote by $E(K_0,K_1)$ the total number of edges (x_0,x_1) such that $x_0 \in K_0-x$, $x_1 \in K_1-x$.

Lemma 2. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0,K_1) \leq 1 + \max\{|K_0|, |K_1|\}.$$

Proof. Let $x = K_0 \cap K_1$. For each vertex $x_0 \in K_0-x$ denote by $\varphi(x_0)$ the number of vertices in K_1-x which are adjacent to x_0. Also define

$$\varphi = \max_{x_0 \in K_0-x} \{ \varphi(x_0) \}.$$

We shall establish an upper bound for $E(K_0,K_1)$ for three cases corresponding to different values of φ.

Case 1. $\varphi \geq 3$. Let $x_0 \in K_0-x$ be such that $\varphi(x_0) = \varphi$. Denote the φ vertices in K_1-x adjacent to x_0 by $y_1, y_2, \ldots, y_\varphi$. Since $x_0 \notin K_1$ and K_1 is a grand clique and is therefore complete, there exists a vertex $y \in K_1$ not adjacent to x_0. Thus $d(x_0,y) = 2$, and by (a_4) we have $\Delta(x_0,y) = 4$. But $x, y_1, y_2, \ldots, y_\varphi$ are $\varphi+1$ vertices adjacent to both x_0 and y, and so we cannot have $\varphi \geq 4$. Thus $\varphi \geq 3$ implies $\varphi = 3$. Again if $\varphi=3$ and y is adjacent to a vertex $z \in K_0-x$, then x_0, y_1, y_2, y_3, z are five vertices adjacent to both x_0 and y, contradicting (a_4). Hence the only vertices in K_1-x which can be adjacent to vertices in K_0-x are y_1, y_2, y_3. Each of these can be adjacent to at most three vertices in K_0-x (including x_0) by the argument above. Hence if $\varphi \geq 3$ we have

(4) $$E(K_0,K_1) \leq 9.$$

Case 2. $\varphi=2$. Again let $x_0 \in K_0-x$ be such that $\varphi(x_0)=2$, and let y_1, y_2 be the two vertices in K_1-x adjacent to x_0. If y is one of the $|K_1|-3$ vertices in K_1 not adjacent to x_0, then y can be adjacent to at most one vertex in K_0-x, for otherwise we again contradict (a_4). If either y_1 or y_2 is adjacent to three vertices in K_0-x including x_0, then $E(K_0,K_1) \leq 9$ by the argument in Case 1. Otherwise each of y_1 and y_2 is adjacent to at most two vertices in K_0-x including x_0. Hence in any case, if $\varphi=2$ then

(5) $$E(K_0,K_1) \leq \max\{ 9, |K_1| + 1 \}$$

Case 3. $\varphi \leq 1$. In this case it is obvious that

(6) $$E(K_0,K_1) \leq |K_0| - 1.$$

Now by Lemma 1 we have $|K_1| \geq n-m(m-1)$. If $|K_1| + 1 < 9$, then $n < m(m-1) + 8$, which contradicts the assumptions $m \geq 3$ and $n > 2m(m-1) + 4$. Hence combining (5) and (6) we have

$$E(K_0,K_1) \leq \max\{ |K_1| + 1, |K_0| - 1\}$$
$$\leq 1 + \max\{ |K_0|, |K_1| \}.$$

Lemma 3. If K is a grand clique in G, then $|K| = n-m+1$.

Proof. Let x be an arbitrary vertex in G and let $K_0, K_1, \ldots, K_{m-1}$ be the m grand cliques containing x. Let $N_i = |K_i|$ and assume without loss of generality that $N_0 \leq N_1 \leq \ldots \leq N_{m-1}$. Then by (a_2) and Lemma 1 we have

$$\sum_{i=0}^{m-1} (N_i - 1) = m(n-m)$$

or

(7) $$\sum_{i=0}^{m-1} N_i = m(n-m+1).$$

Thus since the minimum clique size can be no greater than the average, we have in particular that

(8) $$N_0 \leq n-m+1.$$

Let us now count the number of edges joining vertices of K_0-x to vertices of $\bigcup_{i=1}^{m-1} (K_i-x)$. Corresponding to each vertex $x_0 \in K_0-x$ there are N_0-2 vertices adjacent to both x and x_0 in K_0. By (a_3) we have $\Delta(x,x_0)=n-2$. Hence there remain $n-N_0$ vertices adjacent to both x and x_0 which lie in $\bigcup_{i=1}^{m-1} (K_i-x)$. Since there are N_0-1 vertices in K_0-x, the required number of edges is $(n-N_0)(N_0-1)$. But this number is also given by $\sum_{i=1}^{m-1} E(K_0,K_i)$. Hence using Lemma 2 and (7) we have

$$\begin{aligned}(n-N_0)(N_0-1) &= \sum_{i=1}^{m-1} E(K_0,K_i) \\ &\leq \sum_{i=1}^{m-1} (N_i + 1) \\ &= [m(n-m+1) - N_0] + (m-1).\end{aligned}$$

On simplification, this inequality becomes

$$N_0^2 - (n+2) N_0 + [(m+1)(n-m+1) + 2(m-1)] \geq 0.$$

It follows that

(9) $$| N_0 - \tfrac{1}{2}(n+2)| \geq \tfrac{1}{2} D^{\frac{1}{2}},$$

where

(10) $$\begin{aligned} D &= (n+2)^2 - 4[(m+1)(n-m+1) + 2(m-1)] \\ &= n^2 - 4mn + 4m^2 - 8m + 8 \\ &= (n-2m-2)^2 + 4(n-4m+1).\end{aligned}$$

If $n - 4m + 1 \leq 0$, then since $n > 2m(m-1) +4$, we have

$$2m(m-1) + 4 < 4m - 1,$$

i.e., $$2(m-1)(m-2) + 1 < 0,$$

which of course is a contradiction. Hence $n-4m+1 > 0$ and thus by (10) we have

(11) $$D^{\frac{1}{2}} > n - 2m - 2.$$

Then from (9) and (11) it follows that

(12) $$| N_0 - \frac{1}{2} (n+2) | > \frac{1}{2} (n-2m-2).$$

Suppose first that $N_0 < \frac{1}{2} (n+2) - \frac{1}{2} (n-2m-2) = m + 2$. By Lemma 1, $N_0 \geq n-m(m-1)$. Thus $N_0 < m+2$ implies $n < m^2 + 2$ and therefore

$$2m(m-1) + 4 < m^2 + 2,$$

i.e. $$(m-1)^2 + 1 < 0,$$

which is again a contradiction. Hence by (12) we must have

(13) $$N_o > \frac{1}{2} (n+2) + \frac{1}{2} (n-2m-2)$$
$$= n-m.$$

From (8) and (13) and the fact that N_0 must be an integer, it follows that $N_o = n-m+1$. But $N_0 \leq N_1 \leq \dots \leq N_{m-1}$ and $\sum_{i=0}^{m-1} N_i = mN_0$ implies that $N_0 = N_1 = \dots = N_m = n-m+1$. Hence each of the m grand cliques containing x contains exactly n-m+1 vertices. Since x is an arbitrary vertex and every grand clique is non-empty, it follows that $|K| = n-m+1$ for every grand clique K in G.

<u>Corollary</u>. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0, K_1) \leq n-m+2.$$

<u>Proof</u>. This result follows immediately from Lemmas 2 and 3.

<u>Lemma 4</u>. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0, K_1) \geq n-m^2+2.$$

<u>Proof</u>. Let $x = K_0 \cap K_1$ and denote the remaining m-2 grand cliques containing x by $K_2, K_3, \dots, K_{m-1}$. Let us count the total number of edges connecting vertices of K_i-x to vertices of K_j-x for all $i,j = 0, 1, \dots, m-1$, $i \neq j$. If y is a vertex in K_i-x, then there are n-m-1 vertices in K_i adjacent to both x and y. Since $\Delta (x,y) = n-2$ by (a_3), there are $(n-2) - (n-m-1) = m-1$ vertices adjacent to both x and y and not in K_i. These vertices must therefore lie in $\bigcup_{\substack{j=0 \\ j \neq i}}^{m-1} (K_j-x)$. By taking each of the m(n-m) possible choices for y, we get a total of m(m-1)(n-m) edges. But each edge is included twice in this count, so that the total number of distinct edges is $\frac{1}{2} m(m-1)(n-m)$. Counting the number of such edges in another obvious way, we obtain the relation

$$\sum_{0 \leq i < j \leq m-1} E (K_i, K_j) = \frac{1}{2} m(m-1)(n-m).$$

Thus

(14) $$E(K_0,K_1) = \frac{1}{2}\, m(m-1)(n-m) - \sum_{i=2}^{m-1} E(K_0,K_i) - \sum_{i=2}^{m-1} E(K_1,K_i) - \sum_{2\le i<j\le m-1} E(K_i,K_j).$$

If we replace each of the $E(\cdot\,,\cdot)$'s on the right-hand side of (14) by its upper bound $n-m+2$ as given by the Corollary, we obtain

$$E(K_0K_1) \ge \frac{1}{2}m(m-1)(n-m) - 2(m-2)(n-m+2) - \frac{1}{2}(m-2)(m-3)(n-m+2)$$
$$= n-m^2+2.$$

Lemma 5. If K is a grand clique and x is a vertex not in K, then there are at most two vertices in K which are adjacent to x.

Proof. Let t be the number of vertices in K adjacent to x and denote these vertices by $y_1,y_2,\dots y_t$. The argument used in Case 1 of the proof of Lemma 2 shows that $t \le 4$. Suppose first that $t=4$ and denote by K_1 the grand clique containing y_1 and x. Then since x is a vertex in K_1-y_1 adjacent to three vertices y_2,y_3,y_4 in $K-y_1$, by Case 1 in the proof of Lemma 2 we have $E(K_1,K) \le 9$. Hence by Lemma 4 it follows that $n-m^2+2 \le 9$. Since $n \geq 2m(m-1)+4$, this implies

$$2m(m-1)+4 < m^2 + 7$$

or

$$(m-1)^2 < 4,$$

which contradicts the assumption that $m \ge 3$. Hence we must have $t \le 3$.

Next suppose $t = 3$. Let K_1,K_2,K_3 be the grand cliques containing x which intersect K in y_1,y_2,y_3, respectively. If y is one of the $n-m-2$ vertices in K not adjacent to x, then $d(x,y) = 2$ and hence by (a_4), $\Delta(x,y) = 4$. Thus there is exactly one vertex other than y_1,y_2,y_3 which is adjacent to both x and y. Hence the number of edges joining vertices of $K-y_1-y_2-y_3$ to vertices of $\bigcup_{i=1}^{3} (K_i-x-y_i)$ is at most $n-m-2$. It follows that for at least one K_i, say K_1, the number of edges joining vertices of $K-y_1-y_2-y_3$ to vertices of K_1-x-y_1 is at most $\frac{1}{3}(n-m-2)$. In addition each of y_2,y_3 can be adjacent to at most two vertices in K_1-y_1 including x. Thus we have

(15) $$E(K,K_1) \le \frac{1}{3}\,(n-m-2) + 4 = \frac{1}{3}\,(n-m+10)\,.$$

By Lemma 4 (15) implies that $\frac{1}{3}\,(n-m+10) \ge n-m^2+2$, or $2n \le 3m^2-m+4$. Since $n > 2m(m-1)+4$, $2n > 4m(m-1)+8$ and hence we must have

$$4m(m-1) + 8 < 3m^2 - m + 4,$$

i.e.,

$$(m-1)(m-2) + 2 < 0\ ,$$

which again is a contradiction. This proves the lemma.

Corollary. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0, K_1) \leq n-m.$$

Proof. Let $x = K_0 \cap K_1$. By Lemma 5 no vertex in $K_0 - x$ can be adjacent to more than one vertex in $K_1 - x$. Thus the maximum number of edges is at most n-m, the number of vertices in $K_0 - x$.

Lemma 6. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0, K_1) = n-m,$$

and the vertices of $K_0 - x$ (where $x = K_0 \cap K_1$) may be put in one-one correspondence with the vertices of $K_1 - x$ so that corresponding vertices are adjacent.

Proof. To prove the first part we proceed exactly as in the proof of Lemma 4 except that we replace each of the $E(\cdot\ ,\ \cdot)$'s on the right-hand side of (14) by its upper bound n-m as given by the above Corollary. We then obtain

$$E(K_0, K_1) \geq \frac{1}{2}\, m(m-1)(n-m) - 2(m-2)(n-m) - \frac{1}{2}(m-2)(m-3)(n-m)$$
$$= n-m,$$

which by the Corollary shows that $E(K_0, K_1) = n-m$.

If x_0 is an arbitrary vertex in $K_0 - x$, then by Lemma 5 there is at most one vertex in $K_1 - x$ adjacent to x_0. If there exists a vertex in $K_0 - x$ which is adjacent to no vertex in $K_1 - x$, then we would have $E(K_0, K_1) < n-m$. Hence every vertex in $K_0 - x$ is adjacent to exactly one vertex in $K_1 - x$. By the same argument, every vertex in $K_1 - x$ is adjacent to exactly one vertex in $K_0 - x$, and the lemma is proved.

Lemma 7. The number of grand cliques in G is $\binom{n}{m-1}$.

Proof. Consider pairs (x,K) where x is a vertex in G and K is a grand clique containing x. Since each vertex is contained in m grand cliques, and by (a_1) there are $\binom{n}{m}$ vertices in G, the number of such pairs is $m\binom{n}{m}$. But each grand clique accounts for n-m+1 such pairs, so the total number of grand cliques in G is

$$m\binom{n}{m}/(n-m+1) = \binom{n}{m-1}.$$

Lemma 8. Each grand clique in G is intersected by exactly (m-1)(n-m+1) other grand cliques.

Proof. This follows at once by noting that each of the n-m+1 vertices in a grand clique is contained in exactly m-1 other grand cliques.

Lemma 9. If K_0 and K_1 are two intersecting grand cliques in G, then there are exactly n-2 grand cliques intersecting both K_0 and K_1.

Proof. Let $x = K_0 \cap K_1$. By Lemma 1 there is a one-one correspondence between edges joining vertices of K_0-x to vertices of K_1-x and grand cliques intersecting both K_0-x and K_1-x. Hence by Lemma 6 the number of such grand cliques is n-m. In addition there are m-2 grand cliques other than K_0 and K_1 which contain x. Thus the total number of grand cliques intersecting both K_0 and K_1 is (n-m)+(m-2) = n-2.

Lemma 10. If K_0 and K_1 are two non-intersecting grand cliques which are both intersected by a third grand clique L_1, then there are exactly four grand cliques which intersect both K_0 and K_1, including L_1.

Proof. Let $x_0 = K_0 \cap L_1$, $x_1 = K_1 \cap L_1$. Since K_1 and L_1 are intersecting grand cliques, it follows from Lemma 6 that there is exactly one vertex y_1 in K_1 distinct from x_1 which is adjacent to x_0. Similarly there is exactly one other vertex y_0 in K_0 adjacent to x_1. Let L_2, L_3 be the grand cliques containing the pairs (x_0,y_1), (x_1,y_0), respectively. Suppose y_0 and y_1 are not adjacent. Then $d(y_0,y_1) = 2$. Since L_3 and K_1 are intersecting grand cliques, there exists exactly one vertex z_1 in K_1 distinct from x_1 and y_1 which is adjacent to y_0. Similarly there exists a vertex z_0 in K_0 distinct from x_0 and y_0 which is adjacent to y_1. But for the same reason there must also exist a vertex z_2 in L_2, distinct from x_0 and y_1, which is adjacent to y_1. Thus x_0, x_1, z_0, z_1, z_2 are five distinct vertices adjacent to both y_0 and y_1, which contradicts the fact that $\Delta\,(y_0,y_1) = 4$ when $d(y_0,y_1) = 2$. Hence it follows that y_0 and y_1 must be adjacent. Let L_4 be the grand clique containing y_0 and y_1.

Suppose there is a fifth grand clique L_5 which intersects both K_0 and K_1. Clearly L_5 cannot contain any of the four vertices x_0,x_1,y_0,y_1 or Lemma 5 would be contradicted. Let $z_0 = K_0 \cap L_5$ and $z_1 = K_1 \cap L_5$. There is a vertex z_2 in K_1 distinct from z_1,x_1,y_1 which is adjacent to z_0.

There is also a vertex z_3 in L_5 distinct from z_0 and z_1 which is adjacent to x_1. But then again we have five vertices x_0, y_0, z_1, z_2, z_3 adjacent to both z_0 and x_1. Thus there cannot exist a fifth clique intersecting both K_0 and K_1 and the lemma is proved.

Theorem. If G is the graph of a T_m association scheme, then G is connected and the following conditions are satisfied:

(a_1) The number of vertices in G is $\binom{n}{m}$;

(a_2) G is regular of valence $m(n-m)$;

(a_3) $\Delta(x,y) = n-2$ if $d(x,y) = 1$;

(a_4) $\Delta(x,y) = 4$ if $d(x,y) = 2$.

Conversely, if G is a connected graph satisfying (a_1)-(a_4) for some integers $m \geq 2$ and $n > 2m(m-1) + 4$, then G is the graph of a T_m association scheme.

Proof. The necessity of the conditions (a_1)-(a_4) follows from (1), (2), and (3) and can easily be verified directly.

The proof of sufficiency of these conditions when $n > 2m(m-1) + 4$ will be established by induction on m. Connor [2] has shown that the theorem holds in the triangular case, i.e.,when $m = 2$. Thus let m be an arbitrary integer greater than two and assume that the theorem holds for $2,3,\ldots,m-1$. Then since $m \geq 3$ and $n > 2m(m-1) + 4$, all the results established above hold for G. Let H be a graph whose vertices are the grand cliques of G, two vertices of H being adjacent if and only if the corresponding grand cliques of G intersect. Since G is connected, it is clear that H is also connected. By Lemmas 7,8,9 and 10 it follows that H satisfies the following conditions:

(a_1') The number of vertices in H is $\binom{n}{m-1}$;

(a_2') H is regular of valence $(m-1)[n-(m-1)]$;

(a_3') $\Delta(x,y) = n-2$ if $d(x,y) = 1$.

(a_4') $\Delta(x,y) = 4$ if $d(x,y) = 2$.

But these are just the conditions of the theorem with m-1 replacing m. Also $n > 2m(m-1) + 4$ implies that $n > 2(m-1)[(m-1) - 1] + 4$. Hence by the inductive hypothesis the theorem holds for H, i.e., H is the graph of a T_{m-1} scheme. We can therefore associate the vertices of H with (m-1)-plets from among the symbols 1,2,...,n in such a way that two vertices of H are adjacent if and only if their corresponding (m-1)-plets contain exactly m-2 common symbols. From the correspondence between H and G it follows that we can associate the grand cliques of G with (m-1)-plets from among 1,2,...,n so that two grand cliques intersect if and only if their corresponding (m-1)-plets have exactly m-2 symbols in common.

Now let K_0 and K_1 be two intersecting grand cliques in G, and let $(a_1,a_2,\ldots,a_{m-2},\alpha)$ and $(a_1,a_2,\ldots,a_{m-2},\beta)$ be the corresponding (m-1)-plets. To the vertex $x = K_0 \cap K_1$ we assign the m-plet $(a_1,a_2\ldots,a_{m-2},\alpha,\beta)$. If K is any one of the other m-2 grand cliques containing x, then K intersects both K_0 and K_1, and therefore the (m-1)-plet corresponding to K must have m-2 symbols in common with both $(a_1,a_2,\ldots,a_{m-2},\alpha)$ and $(a_1,a_2,\ldots,a_{m-2},\beta)$. Hence it must either be of the form $(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{m-2},\alpha,\beta)$ for some i, $1 \leq i \leq m-2$, or else of the form $(a_1,a_2,\ldots,a_{m-2},\gamma)$ where $\gamma \neq \alpha,\beta$. Suppose first that it is of the latter form. The total number of (m-1)-plets which contain the (m-2)-plet $(a_1,a_2,\ldots,a_{m-2})$ is n-m+2, and corresponding to each of these is a grand clique in G. Now there are only m grand cliques containing x and $n-m+2 > m$ since $n > 2m(m-1)+4$, and thus there exists a grand clique L not containing x whose corresponding (m-1)-plet has the form $(a_1,a_2,\ldots a_{m-2}, \xi)$, where $\xi \neq \alpha,\beta,\gamma$. Then L is a grand clique which intersects K_0,K_1, and K. Thus x is a vertex not in L which is adjacent to at least three vertices in L, which contradicts Lemma 5. It follows that if K is any grand clique containing x other than K_0 and K_1, then the (m-1)-plet corresponding to K must be of the form $(a_1,a_2,\ldots,a_{i-1}, a_{i+1},\ldots,a_{m-2}, \alpha,\beta)$. Hence the m-plet $(a_1,a_2,\ldots a_{m-2},\alpha,\beta)$ associated with x is unambiguously determined by any two of the m grand cliques intersecting in x, and to each of the $\binom{n}{m}$ vertices in G we can associate a unique m-plet.

If two vertices x and y of G are adjacent, then there is a unique grand clique K containing x and y. If $(a_1,a_2,\ldots,a_{m-1})$ is the (m-1)-plet corresponding to K, then the m-plets corresponding to x and y must contain the symbols $a_1,a_2,\ldots,a_{m-1}$. Conversely, if the m-plets corresponding to x and y contain the symbols $a_1,a_2,\ldots,a_{m-1}$, then x and y are contained in the grand clique K corresponding to the (m-1)-plet $(a_1,a_2,\ldots,a_{m-1})$. Thus two vertices of G are adjacent if and only if the corresponding m-plets have m-1 symbols in common. Hence G must be the graph of a T_m association scheme.

Acknowledgements

My thanks are offered to Professor R. C. Bose and Dr. Renu Laskar for the problem considered which is implicitly suggested by their paper [1] on tetrahedral graphs. I'm also very grateful for the time and effort they devoted to discussing the problem with me.

References

[1] Bose, R. C. and Renu Laskar, A characterization of tetrahedral graphs, Inst. Stat., UNC, Mimeo Ser. No 509. To be published in J. Comb. Theory.

[2] Connor, W. S, The uniqueness of the triangular association scheme, Ann. Math. Stat., 29 (1958), 262-266.

[3] Hoffman, A. J., On the uniqueness of the triangular association scheme, Ann. Math. Stat., 31 (1960), 492-497.

[4] Hoffman, A. J., On the exceptional case in a characterization of the arcs of a complete graph, IBM J., 4 (1960), 487-496.

[5] Li-Chien, Chang, The uniqueness and non-uniqueness of the triangular association schemes, Science Record, Math. New Ser., 3 (1959), 604-613.

[6] Li-Chien, Chang, Association schemes of partially balanced designs with parameters $v = 28$, $n_1 = 12$, $n_2 = 15$, and $p_{11}^2 = 4$, Science Record, Math. New Ser., 4 (1960), 12-18.

[7] Ogasawara, M., A necessary condition for the existence of regular and symmetrical PBIB designs of T_m type, Inst. Stat., UNC, Mimeo Ser. No. 418.

[8] Shrikhande, S. S., On a characterization of the triangular association scheme, Ann. Math. Stat., 30 (1959), 39-47.