SIMULTANEOUS TEST PROCEDURES FOR ONE-WAY ANOVA AND MANOVA BASED ON RANK SCORES*

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SUMMARY. A MANOVA statistic based on rank scores is proposed and its distribution derived along the lines of Puri and Sen (1966). Using the simultaneous inference approach of Gabriel (1967a) this statistic is shown to provide a procedure which simultaneously tests all subgroups of samples on all subsets of variables, and to set simultaneous confidence bounds on all location differences. The formulae for actual ranks, i.e., Wilcoxon type scores, are worked out explicitly.

1. INTRODUCTION

The resolution of rejections of overall hypotheses into significant detail, allowing simultaneous inference on subhypotheses, has received a good deal of attention in the literature. (For a survey see Miller (1966)). The original work was principally on univariate normal ANOVA, but this was more recently extended to non-parametric set-ups (see Nemenyi (1963) and Sen (1966) which also cite other references), and to the multivariate normal model (see Gabriel 1967b), which also cites earlier work). No multivariate non-parametric technique seems to have been available hitherto, and the present paper is intended to fill this gap. Joining the general approach to simultaneous inference being developed by Gabriel (1967a) with the method of derivation of

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statistics used by Puri and Sen (1966), a technique is obtined which has known asymptotic distributions and allows a permutation approach for small samples.

The present approach relaxes the assumption of multi-normality (as in Gabriel (1967b)). However, it still requires that all the distributions compared be equal except for shifts in location. Denoting the k-th distribution function by $F_k(\underset{\sim}{x})$, the model under which the present approach is valid is

$$F_k(\underset{\sim}{x}) = F(\underset{\sim}{x} + \underset{\sim}{\theta}_k) \text{ for all } k, \tag{1.1}$$

where $\underset{\sim}{\theta}_k$ is any vector of constants. A further extension to provide methods for the purely distribution-free set-up is not available at present.

Definitions are set out in section 2 of the paper, followed by the distribution theory in section 3. Section 4 describes the simultaneous inference procedures and their properties. The special case of actual ranks, rather than rank scores, is worked out in detail in section 5.

2. DEFINITIONS

Consider c_o (≥ 2) independent samples of sizes $n_1,\ldots,n_{c_o}$, each of the $N=n_1+\ldots+n_{c_o}$ observations being made on p_o variables. Denote the α-th observation of the k-th sample on the i-th variable by $X_{k\alpha}^{(i)}$, $\alpha=1,\ldots,n_k$, $k=1,\ldots,c_o$, $i=1,\ldots,p_o$.

For any pair (k,q) of samples, $k\neq q=1,\ldots,c_o$, rank the $n_{kq} = n_k + n_q$ observations separately on each variable $i(=1,\ldots,p_o)$. Write $R_{k\alpha(q)}^{(i)}$, $\alpha=1,\ldots,n_k$, for the rank of $X_{k\alpha}^{(i)}$ among $X_{k1}^{(i)},\ldots,X_{kn_k}^{(i)},X_{q1}^{(i)},\ldots,X_{qn_q}^{(i)}$. Further define the rank scores

$$J_{n_{kq}}^{(i)}\left(\frac{R_{k\alpha(q)}^{(i)}}{n_{kq}+1}\right)$$

for each $X_{k\alpha}^{(i)}$ by means of functions $J_n^{(i)}(\frac{R}{n+1})$, $i=1,\ldots,p_o$, which depend on n and R, being $\uparrow$ in $R(1 \leq R \leq n)$ and satisfies the Chernoff-Savage (1958) conditions (See also Puri and Sen (1966) for further detail of these regularity conditions). Now obtain pairwise mean rank scores differences

$$U_{kq}^{(i)} = \frac{1}{n_k} \sum_{\alpha=1}^{n_k} J_{n_{kq}}^{(i)} \left(\frac{R_{k\alpha(q)}^{(i)}}{n_{kq}+1}\right) - \frac{1}{n_q} \sum_{\alpha=1}^{n_q} J_{n_{kq}}^{(i)} \left(\frac{R_{q\alpha(k)}^{(i)}}{n_{kq}+1}\right) \qquad (2.1a)$$

and similarly define

$$U_{kk}^{(i)} = 0. \qquad (2.1b)$$

for $k=1,\ldots,c_o$ and $i=1,\ldots,p_o$.

Next, rank the observations within each single sample. For variable i and sample k denote by $R_{k\alpha}^{(i)}$ the rank of $X_{k\alpha}^{(i)}$ among $X_{k1}^{(i)},\ldots,X_{kn_k}^{(i)}$. Now define $(p_o \times p_o)$ matrix V_N so that its (i,j)th element is

$$v_{Nij} = \frac{1}{N} \sum_{k=1}^{c_o} \left\{ \sum_{\alpha=1}^{n_k} \left[J_{n_k}^{(i)} \left(\frac{R_{k\alpha}^{(i)}}{n_{k+1}}\right) J_{n_k}^{(j)} \left(\frac{R_{k\alpha}^{(j)}}{n_{k+1}}\right) \right] - \left[\sum_{\alpha=1}^{n_k} J_{n_k}^{(i)} \left(\frac{\alpha}{n_k+1}\right) \right] \left[\sum_{\alpha=1}^{n_k} J_{n_k}^{(j)} \left(\frac{\alpha}{n_k+1}\right) \right] / n_k \right\} \qquad (2.2)$$

(Note that $\sum_{\alpha=1}^{n_k} \left[J_{n_k}^{(i)} \left(\frac{R_{k\alpha}^{(i)}}{n_k+1}\right) \right]^r = \sum_{\alpha=1}^{n_k} \left[J_{n_k}^{(i)} \left(\frac{\alpha}{n_k+1}\right) \right]^r$ for all r, which for r=1 explains the negative term in v_{Nij}. Also, this shows, for r=2, the diagonals

$$v_{Nii} = \frac{1}{N} \sum_{k=1}^{c_o} \left\{ \sum_{\alpha=1}^{n_k} \left[J_{n_k}^{(i)} \left(\frac{\alpha}{n_k+1}\right) \right]^2 - \left[\sum_{\alpha=1}^{n_k} J_{n_k}^{(i)} \left(\frac{\alpha}{n_k+1}\right) \right]^2 / n_k \right\} \qquad (2.3)$$

to be independent of the within sample rankings).

Denote the group of all c_o samples, or corresponding populations, by

G_o and any subgroup by G_e, containing c_e $(1 < c_e \leq c_o)$ samples, or distributions. Similarly, denote the set of all p_o variables by S_o and any subset by S_a, containing p_a $(1 \leq p_a \leq p_o)$ variables. Denote the sum of sizes of samples in G_e by $N_e = \sum_{k \in G_e} n_k$, and let $v^{ij(a)}$ be the (i,j)th element of the inverse of the principal minor of V_N with rows and columns corresponding to the variables of S_a. To test hypothesis H_e^a that c_e populations of subgroup G_e are equally located on the p_a variables of set S_a, one may use the statistic

$$L_e^a = \sum_{k \in G_e} \sum_{q \in G_e} n_k n_q \sum_{i \in S_a} \sum_{j \in S_a} U_{kq}^{(i)} U_{kq}^{(j)} v_N^{ij(a)}/2N. \tag{2.4}$$

In particular, to test the overall hypothesis H_o^o that all c_o populations are equally located on all p_o variables one would use statistic L_o^o.

Note that the statistic L_e^a is completely unaffected by the within sample variabilities. The diagonal terms of V_N are constants (2.3) depending only on sample sizes and not on variability. However, the off-diagonal terms of V_N (2.2) are affected by within sample correlations -- as expressed through rank scores. Hence, the statistic L_e^a may be affected by the rank-score correlation within the c_o samples, and its usefulness as a test statistic for H_e^a depends on the equality of the correlation matrices of the distributions $F_1(\underset{\sim}{x}),\ldots,F_{c_o}(\underset{\sim}{x})$, which is, however, implied by the location model (1.1).

A statistic of the form L_e^a was chosen because its distribution is unaffected by the location of samples not belonging to G_e and variables outside S_a. Thus, given the location model (1.1) the distribution of L_e^a is completely specified under H_e^a. To see this note again that the matrix V_N is essentially a "within samples" rank-score variance matrix, and is clearly unaffected by any shifts in the location of the samples on any of the variables. Next, the rank-scores used in obtaining $U_{kq}^{(i)}, U_{kq}^{(j)}$ and $v_N^{ij(a)}$, $i,j \varepsilon S_a$, involve only the variables of S_a. Further, the rank scores used for $U_{kq}^{(i)}, U_{kq}^{(j)}$, $k,q \varepsilon G_e$, involve only pairs of samples from G_e. Hence none of the elements which go into the formula for L_e^a are affected by changes in location of samples outside G_e and/or variables not in S_a. Thus, since H_e^a specifies the relative locations of the relevant samples on the relevant variables, it completely specifies the distribution of L_e^a. This may be summarized (as in Gabriel (1967a)) by saying that the collection of hypotheses H_e^a implied by H_o^o and their corresponding statistics L_e^a forms a testing family.

3. NULL DISTRIBUTION OF L_e^a

Under the null hypothesis H_e^a, the distribution of L_e^a may depend on the parent distribution $F(\underset{\sim}{x})$ (unless we are essentially dealing with univaraite or independent multivariate distributions). Thus, unlike the univariate rank tests, the proposed STP stes will not, in general, be strictly distribution-free. However, the Chatterjee-Sen permutation arguments considered in detail by Puri and Sen (1966) may be used to construct permutationally distribution-free tests. We shall consider the permutation distribution of L_o^o in some detail, while the case of general L_e^a will follow on the same lines.

A set of $N^* = (N!/\prod_1^{c_o} n_k!)$ realizations of L_o^o can be obtained from a given set of c_o samples by considering all possible N^* allocations of the N vectors $\underset{\sim}{X}_{k\alpha} = (X_{k\alpha}^{(1)},\ldots,X_{k\alpha}^{(p_o)})$, $\alpha=1,\ldots,n_k$, $k=1,\ldots,c_o$, into c_o samples of sizes $n_1,\ldots,n_{c_o}$, respectively. Since under H_o^o, the $\underset{\sim}{X}_{k\alpha}$'s are independent and identically distributed stochastic vectors, all these N^* allocations will be conditionally equally probable, each having the conditional probability $1/N^*$.

This completely specified conditional probability law generates the permutation distribution of L_o^o. If $n_1,\ldots,n_c$ are all small, this distribution may be used to find the critical value of L_o^o. If the sample sizes are large, one can either proceed as in Puri and Sen (1966) to find the asymptotic permutation distribution of L_o^o, or alternatively, one can take a sample of several hundred allocations from the universe of all possible allocations to give a consistent estimate of the critical value of L_o^o. Since, it has been shown by Puri and Sen (1966) that the asymptotic permutation distribution and the asymptotic null distribution of rank statistics do coincide, one need only consider the following large sample approach.

Recall that according to the model (1.1) $F_k(\underset{\sim}{x}) = F(\underset{\sim}{x}+\underset{\sim}{\theta}_k)$, $k=1,\ldots,c_o$, where $\underset{\sim}{\theta}_1,\ldots,\underset{\sim}{\theta}_{c_o}$ are all p_o vectors. Denote the ith marginal cdf of F by $F_{[i]}$ and the bivariate joint cdf of the (i,j)th variates by $F_{[i,j]}(x,y)$, for $i\neq j=1,\ldots,p_o$. Next, define

$$J^{(i)}(u) = \lim_{n\to\infty} J_n^{(i)}(u): \ 0 < u < 1, \ i=1,\ldots,p_o, \tag{3.1}$$

where n stands for n_{kq}, for any $k\neq q=1,\ldots,c_o$. Then, let

$$\mu_i = \int_{-\infty}^{\infty} J^{(i)}(F_{[i]}(x))dF_{[i]}(x), \ i=1,\ldots,p_o; \tag{3.2}$$

$$\nu_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{(i)}(F_{[i]}(x))J^{(j)}(F_{[j]}(y))dF_{[i,j]}(x,y)-\mu_i\mu_j, \tag{3.3}$$

for $i,j=1,\ldots,p_o$.

LEMMA 3.1. _If (1.1) holds and the conditions of Theorem 4.2 of Puri and Sen (1966) hold,_

$$v_{N,ij} \overset{p}{\to} \nu_{ij}, \ v_{N,ii} \to \nu_{ii}, \ i\neq j=1,\ldots,p_o,$$

uniformly in $\underset{\sim}{\theta}_1,\ldots,\underset{\sim}{\theta}_{c_o}$.

As the proof follows along the lines of Theorem 4.2 of Puri and Sen (1966), hence the details are omitted.

Now, define

$$\underset{\sim}{\nu} = ((\nu_{ij})) \tag{3.4}$$

and assume that

$$\underset{\sim}{\nu} \text{ is positive-definite} \tag{3.5}$$

<u>THEOREM 3.1</u>. <u>If the conditions of Theorem 5.1 of Puri and Sen (1966) hold, then under</u> H_e^a <u>and (3.5)</u> $(N/N_e)L_e^a$ <u>has asymptotically a chi-square distribution with</u> $p_a(c_e-1)$ <u>degrees of freedom</u>.

<u>Proof</u>. It suffices to consider the case of L_o^o, since the others follow along the same lines. Define

$$L_o^{*o} = \sum_{i=1}^{p_o} \sum_{j=1}^{p_o} \nu^{ij} \sum_{k=1}^{c_o} \sum_{q=1}^{c_o} (n_k n_q/2N) U_{kq}^{(i)} U_{kq}^{(j)}. \tag{3.6}$$

It follows from (2.4), lemma 3.1 and (3.6) that

$$L_o^o \underset{\sim}{\overset{p}{}} L_o^{*o}. \tag{3.7}$$

So, it is enough to prove the theorem for L_o^{*o}.

Now write

$$B(X_{k\alpha}^{(i)}) = \int_{-\infty}^{\infty} [c(x-X_{k\alpha}^{(i)})-F_{[i]}(x)]J^{(i)'}[F_{[i]}(x)]dF_{[i]}(x) \tag{3.8}$$

for $\alpha=1,\ldots,n_k$, $i=1,\ldots,p_o$, $k=1,\ldots,c_o$, where

$$c(u) = \begin{cases} 1 & u > 0 \\ 0 & u \leq 0. \end{cases} \tag{3.9}$$

Then, from the definition of $U_{kq}^{(i)}$ and Theorem 5.1 of Puri and Sen (1966), it follows that

$$U_{kq}^{(i)} = \frac{1}{n_q}\sum_{\alpha=1}^{n_q} B(X_{q\alpha}^{(i)}) - \frac{1}{n_k}\sum_{\alpha=1}^{n_k} B(X_{k\alpha}^{(i)}) + o_p(n_{kq}^{-\frac{1}{2}}).$$

$$= \bar{B}_q^{(i)} - \bar{B}_k^{(i)} + o_p(n_{kq}^{-\frac{1}{2}}), \text{ say.} \tag{3.10}$$

Consequently,

$$\frac{1}{2N}\sum_k\sum_q n_k n_q U_{kq}^{(i)} U_{kq}^{(j)}$$

$$= \frac{1}{2N}\sum_k\sum_q n_k n_q [\bar{B}_q^{(i)}-\bar{B}_k^{(i)}][\bar{B}_q^{(j)}-\bar{B}_k^{(j)}] + o_p(1) \tag{3.11}$$

$$= \sum_k n_k B_k^{(i)} B_k^{(j)} - N(\frac{1}{N}\sum_k n_k \bar{B}_k^{(i)})(\frac{1}{N}\sum n_k \bar{B}_k^{(i)}) + o_p(1).$$

It also follows from Theorem 5.1 of Puri and Sen (1966) that

$$E(B(X_{k\alpha}^{(i)})|H_o^o) = 0,$$

$$E\{B(X_{k\alpha}^{(i)})B(X_{q\beta}^{(j)})|H_o^o\} = \delta_{\alpha\beta}\, \nu_{ij}, \tag{3.12}$$

for all α,β and $i,j=1,\ldots,p_o$. Hence,

$$E(\bar{B}_k^{(i)}|H_o^o) = 0,$$

$$E(\bar{B}_k^{(i)}\bar{B}_k^{(j)}|H_o^o) = \frac{1}{n_k}\nu_{ij}. \tag{3.13}$$

Consider now the c_o vectors $n_k^{\frac{1}{2}}\bar{\underset{\sim}{B}}_k = (n_k\bar{B}_k^{(1)},\ldots,n_k^{\frac{1}{2}}\bar{B}_k^{(p_o)})$, $k=1,\ldots,c_o$. As they involve summation over independent and identically distributed random variables $B(X_{k\alpha}^{(i)})$, by the vector valued central limit theorem,

$$(n_1^{\frac{1}{2}}\bar{\underset{\sim}{B}}_1,\ldots,n_{c_o}^{\frac{1}{2}}\bar{\underset{\sim}{B}}_{c_o}) \to \mathcal{N}(0,\ I\otimes\underset{\sim}{\nu}) \tag{3.14}$$

Thus, it follows from the well-known results on the distributions associated with quadratic forms in multinormal vairables that

$$\sum_i\sum_j \nu^{ij}\{\sum_k n_k\bar{B}_k^{(i)}\bar{B}_k^{(j)} - (\sum_k n_k\bar{B}_k^{(i)})(\sum_k n_k\bar{B}_k^{(j)})/N\} \tag{3.15}$$

has asymptotically a chi-square distribution with $p_o(c_o-1)$ d.f..

Now, (3.6) and (3.11) imply that L_o^{*o} has asymptotically a chi-square distribution with $p_o(c_o-1)$ d.f.. The rest of the proof follows from (3.7).

The corresponding derivation for any L_e^a would require the subtotal N_e in place of the total N. Hence the result holds for $(N/N_e)L_e^a$.

4. SIMULTANEOUS TEST PROCEDURE AND CONFIDENCE BOUNDS

A Simultaneous Test Procedure (STP) of level α for the testing family considered in this paper is to accept or reject each H_e^a according as

$$L_e^a \leq \zeta_\alpha \quad \text{or} \quad L_e^a > \zeta_\alpha \tag{4.1}$$

where ζ_α is the upper α point of the distribution of L_o^o under H_o^o.

This STP has the following properties:

(I) Its decisions are always coherent in the sense that if H_e^a implies H_f^b then H_e^a cannot be accepted if H_f^b is rejected;

(II) Its decisions are not always consonant in the sense that if a hypothesis H_e^a is rejected it does not necessarily follow that some other H_f^b implied by H_e^a is also rejected;

(III) The probability of any type I error in all its decisions is no more than the level α, being exactly α if the overall hypothesis H_o^o is true. The probability of a type I error on any particular true H_e^a is less than the above.

Property (I) follows readily from the fact that the testing family is monotone in the sense that if H_e^a implies H_f^b then $L_e^a \geq L_f^b$. (See Gabriel (1967a), Theorem 1). To prove this monotonicity note first that the implication relation between H_e^a and H_p^b is equivalent to the pair of containment relations $S_a \supseteq S_b$ and $G_e \supseteq G_f$.

Next, note that since $S_a \supseteq S_b$

$$\sum_{i\varepsilon S_a} \sum_{j\varepsilon S_a} U_{kq}^{(i)} U_{kq}^{(j)} v^{ij(a)}/2N \geq \sum_{i\varepsilon S_b} \sum_{j\varepsilon S_b} U_{kq}^{(i)} U_{kq}^{(j)} v_N^{ij(b)}/2N \qquad (4.2)$$

for any k and q. This is so since the matrix V_N is symmetric positive definite (See Lemma in Appendix). Finally, note that L_e^a sums terms $n_k n_q \sum_{i\varepsilon S_a} \sum_{j\varepsilon S_a} U_{kq}^{(i)} U_{kq}^{(j)} v^{ij(a)}/2N$ over G_e whereas L_f^b sums smaller terns $n_k n_q \sum_{i\varepsilon S_b} \sum_{j\varepsilon S_b} U_{kq}^{(i)} U_{kq}^{(j)} v^{ij(b)}/2N$ over G_f. where $G_f \subseteq G_e$. Clearly, then $L_e^a \geq L_f^b$, as was to be proved.

Property (II) is essentially negative and is established by any counter-example.

Property (III) consists of several statements. (1) That the probability of any type I error at all is exactly α under H_o^o. This follows readily from monotonicity which entails that any type I error at all be made if and only if such an error is made on H_o^o. (2) That the probability of a particular error is no more than that of any error. This is obvious. (3) That the probability of any type I error at all is no more than α even if H_o^o is not true. This follows from the fact that the family of hypotheses is closed under intersection (See Theorem 2 of Gabriel (1967a), where a derivation of these and other properties is given for STPs in general).

Particular type I error probabilities may be evaluated by means of the approximate null distributions. Thus, the probability of falsely rejecting H_e^a by the STP with critical value ζ_α is

$$\alpha_{(a,e)} = P_{H_e^a}(L_e^a > \zeta_\alpha) \qquad (4.3)$$

which may be obtained from the distribution of L_e^a under H_e^a. Thus, for large samples, $\zeta_\alpha N/N_e$ will be the upper $\alpha_{(a,e)}$ point of the chi-square distribution with $p_a(c_e-1)$ d.f..

It is interesting to compare the STP proposed here with the only other non-parametric multiple comparison method available for unequal sample sizes, and restricted to univariate ANOVA. Nemenyi (1963) (See also Miller (1966), Chapter 3, section 6) essentially tests H_e (the superscript is superflous in the univariate case) by means of the sum of squares of rank scores "between" the samples of subgroup G_e, though he specifically mentions only G_es which are pairs of samples (and contrasts). In particular, he uses the Kruskal-Wallis rank ANOVA statistic whose asymptotic distribution for any H_e is chi-square with c_e-1 d.f., just as is that of $L_e N/N_e$ in the present STP. The α level critical value ζ_α is therefore the same in both techniques, but the type I error probability on H_e with Nemenyi's method, $\alpha^N_{(e)}$, exceeds that, $\alpha_{(e)}$, with the STP. To see this note that ζ_α is the upper $\alpha^N_{(e)}$ point of chi-square with (c_o-1) d.f. whereas $\zeta_\alpha N/N_e$ is the upper $\alpha_{(e)}$ point of that distribution. Since corresponding inequalities must also hold for power comparisons it is clear that at a given level Nemenyi's technique offers a higher probability of obtaining significant detail than the present STP.

This advantage of Nemenyi's technique must be set against the disadvantage that its decision on any sybhypothesis H_e is influenced not only by the location of the samples in G_e but also by the location of all other samples, which is irrelevant to H_e. This is so because Nemenyi computes all his statistics from the overall ranking of all c_o samples.

Finally, the STP might be adjusted to accepting H_e^a only if $(N/N_e)L_e^a \leq \zeta_\alpha$, with the same critical value ζ_α as before. This would increase the type I error probabilities but destroy the coherence of the decisions. However, the decision on each H_e^a would still depend only on the location of the samples in G_e on the variables of S_a. In the univariate case the type I errors would be asymptotically equal to those of Nemenyi's technique. Thus this adjusted STP has the advantage that the decision on each H_e^a depends only on the truth of H_e^a itself whereas Nemenyi's technique has the advantage of coherence. The unadjusted STP has both advantages, but at the expense of lower chances of rejection.

The STP (4.1) is readily extended to a simultaneous confidence statement about bounds on the location differences

$$\Delta_{kq}^{(i)} = \theta_k^{(i)} - \theta_q^{(i)} , \tag{4.4}$$

$i=1,\ldots,p_o$, $k\neq q=1,\ldots,c_o$, where $\theta_k^{(i)}$ is the i-th co-ordinate of the vector $\underset{\sim}{\theta}_k$ of (1.1). Start by defining $U_{kq}^{i}(d)$ as $U_{kq}^{(i)}$ computed upon replacing each $X_{k\alpha}^{(i)}$, $\alpha=1,\ldots,n_k$, by $X_{k\alpha}^{(i)} - d$, the $X_{q\beta}^{(i)}$s being unaltered. Next define

$$D_{kq}^{(i)} = \sup\{d: U_{kq}^{(i)}(d) \geq [\zeta_\alpha v_{Nii} \frac{N}{n_k n_q}]^{\frac{1}{2}}\} \tag{4.5a}$$

and

$$d_{kq}^{(i)} = \inf\{d: U_{kq}^{(i)}(d) \leq [\zeta_\alpha v_{Nii} \frac{N}{n_k n_q}]^{\frac{1}{2}}\}. \tag{4.5b}$$

Then

$$P\{d_{kq}^{(i)} \leq \Delta_{kq}^{(i)} \leq D_{kq}^{(i)}, \forall\, i,k,q\} \geq 1-\alpha \tag{4.6}$$

provides $1-\alpha$ simultaneous confidence bounds for all $\Delta_{kq}^{(i)}$s.

To prove (4.6) note first that $U_{kq}^{(i)}(c)$ is $\downarrow$ in d as $J_{n_{kq}}^{(i)}(u)$ is $\uparrow$ in u: $0 < u < 1$. Hence

$$d_{kq}^{(i)} \leq \Delta_{kq}^{(i)} \leq D_{kq}^{(i)} \tag{4.7}$$

is equivalent to

$$|U_{kq}^{(i)}(\Delta_{kq}^{(i)})| \le [\zeta_\alpha v_{Nii} \frac{N}{n_k n_q}]^{\frac{1}{2}}, \tag{4.8}$$

that is, to

$$2n_k n_q \{U_{kq}^{(i)}(\Delta_{kq}^{(i)})\}^2 / v_{Nii} 2N \le \zeta_\alpha \ . \tag{4.9}$$

Writing $L_e^a(\Delta)$ for the L_e^a statistic computed by means of $U_{kq}^{(i)}(\Delta_{kq}^{(i)})$ values in place of $U_{kq}^{(i)}$, (4.9) is equivalent to

$$L_e^a(\Delta) \le \zeta_\alpha, \text{ where } G_e = \{k,q\} \text{ and } S_a = \{i\}. \tag{4.10}$$

Now, by STP property (III), since Δ denotes the true location differences

$$P\{L_e^a(\Delta) \le \zeta_\alpha, \forall\, S_a, G_e\} = 1-\alpha \ . \tag{4.11}$$

Hence

$$P\{L_e^a(\Delta) \le \zeta_\alpha, \text{ where } S_a = \{k,q\} \text{ and } G_e = \{i\}, \forall i,k,q\} \ge 1-\alpha \ . \tag{4.12}$$

So that (4.6) follows from the set of equivalences of (4.7), (4.8), (4.9) and (4.10).

5. WILCOXON SCORES

A particularly simple as well as useful special case is that of Wilcoxon rank scores, i.e., of using the ranks themselves. In this case

$$J_n^{(i)}(\frac{\alpha}{n+1}) = \frac{\alpha}{n+1} \qquad 1 \le \alpha \le n. \tag{5.1}$$

Then $U_{kq}^{(i)}$ becomes $1/(n_{kq}+1)$ times the difference of the mean ranks of the k-th and q-th sample in the joint ranking of all n_{kq} observations of both these samples on the i-th variable. One notes that $U_{kq}^{(i)}$ is proportional to the two sample Wilcoxon statistic comparing the k-th and q-th samples on the i-th variable.

Further, one obtains from (2.2), (2.3) and (5.1) that

$$v_{Nii} = \frac{1}{12N}\sum_{k=1}^{c_o} n_k(n_k-1)/(n_k+1) = \frac{1}{12} + o(N^{-1}),\ i=1,\ldots,p_o \tag{5.2}$$

and

$$v_{Nij} = \frac{1}{N}\sum_{k=1}^{c}\sum_{\alpha=1}^{n_k}(R_{k\alpha}^{(i)} - \frac{n_k+1}{2})(R_{k\alpha}^{(j)} - \frac{n_k+1}{2})/(n_k+1)^2,\ i\neq j=1,\ldots,p_o. \tag{5.3}$$

A particularly notable simplification is achieved for the simultaneous confidence bound in (4.7). Let

$$Z_{kq,\alpha\beta}^{(i)} = X_{k\alpha}^{(i)} - X_{q\beta}^{(i)} \text{ for } \alpha=1,\ldots,n_k;\ \beta=1,\ldots,n_q,\ k\neq q=1,\ldots,c_o. \tag{5.4}$$

Arrange the $n_k n_q$ observations on $Z_{kq,\alpha\beta}^{(i)}$ in ascending order of magnitude and denote the rth smallest observation in this set by $Z_{kq(r)}^{(i)}$ for $r=1,\ldots,n_k n_q$, $i=1,\ldots,p_o$. Now define

$$C_{kq}^{(N)} = \frac{n_k n_q(n_k+n_q+1)}{(n_k+n_q)}\left\{\zeta_\alpha \frac{N}{n_k n_q} v_{Nii}\right\}^{\frac{1}{2}}, \tag{5.5}$$

(and note that $C_{kq}^{(N)}$ for large sample sizes reduces to $(n_k n_q \zeta_\alpha/12)^{\frac{1}{2}}(1+o(N^{-1}))$). Then let

$$m_{kq}^{(N)} = \tfrac{1}{2}n_k n_q - C_{kq}^{(N)} \quad\text{and}\quad M_{kq}^{(N)} = \tfrac{1}{2}n_k n_q + C_{kq}^{(N)}.$$

Proceeding then as in Miller (1966, pp. 146-149), one observes that (4.7) is equivalent to

$$Z_{kq(m_{kq}^{(N)})}^{(i)} \leq \Delta_{kq} \leq Z_{kq(M_{kq}^{(N)})}^{(i)} \tag{5.6}$$

which provides the desired confidence bounds. A graphical procedure due to Moses is explained in Miller's book and may also be adopted for the present purposes.

APPENDIX

LEMMA: Let $\underset{\sim}{x}$ be a $(p\times 1)$ vector and V a $(p\times p)$ symmetric positive definite matrix, and let $\underset{\sim}{x}$ be partitioned as $\begin{pmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2\end{pmatrix}$ and V, correspondingly, as $\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}$. Then $\underset{\sim}{x}'V^{-1}\underset{\sim}{x} \geq \underset{\sim}{x}_1'V_{11}^{-1}\underset{\sim}{x}_1$

Proof: Write

$$\underset{\sim}{x}'V^{-1}\underset{\sim}{x} = (\underset{\sim}{x}_1',\underset{\sim}{x}_2')\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}^{-1}\begin{pmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2\end{pmatrix}$$

$$= (\underset{\sim}{x}_1',\underset{\sim}{x}_2')\begin{pmatrix}I & -V_{11}^{-1}V_{12}\\ 0 & I\end{pmatrix}\left(\begin{pmatrix}I & 0\\ -V_{21}V_{11}^{-1} & I\end{pmatrix}\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}\begin{pmatrix}I & -V_{11}^{-1}V_{12}\\ 0 & I\end{pmatrix}\right)^{-1}\begin{pmatrix}I & 0\\ -V_{21}V_{11}^{-1} & I\end{pmatrix}\begin{pmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2\end{pmatrix}$$

$$= (\underset{\sim}{x}_1',\underset{\sim}{x}_2'-\underset{\sim}{x}_1'V_{11}^{-1}V_{12})\begin{pmatrix}V_{11} & 0\\ 0 & V_{22}-V_{21}V_{11}^{-1}V_{12}\end{pmatrix}^{-1}\begin{pmatrix}\underset{\sim}{x}_1\\ \underset{\sim}{x}_2-V_{21}V_{11}^{-1}\underset{\sim}{x}_1\end{pmatrix}$$

$$= \underset{\sim}{x}_1'V_{11}^{-1}\underset{\sim}{x}_1 + (\underset{\sim}{x}_2-V_{21}V_{11}^{-1}\underset{\sim}{x}_1)'(V_{22}-V_{21}V_{11}^{-1}V_{12})^{-1}(\underset{\sim}{x}_2-V_{21}V_{11}^{-1}\underset{\sim}{x}_1)$$

$$\geq \underset{\sim}{x}_1'V_{11}^{-1}\underset{\sim}{x}_1$$

since

$$V_{22}-V_{21}V_{11}^{-1}V_{12} = (-V_{21}V_{11}^{-1}|I)\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}\begin{pmatrix}-V_{11}^{-1}V_{12}\\ I\end{pmatrix}$$

is positive definite.

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