NECESSARY CONDITIONS FOR ALMOST SURE EXTINCTION OF A BRANCHING PROCESS WITH RANDOM ENVIRONMENT

by

Walter L. Smith

University of North Carolina

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DEPARTMENT OF STATISTICS
University of North Carolina
Chapel Hill, N. C.

1. Introduction

In this note we shall be considering what may be called a branching process with random environment (B.P.R.E.); it will be a sequence $\{Z_n\}$ of random variables forming a particular sort of Markov chain.

Let $\{\zeta_n\}$ be a sequence of independent and identically distributed random variables, to be known as environment variables, taking values in some (possibly abstract) space Ξ. Suppose that every point $\zeta\epsilon\Xi$ has associated with it a probability generating function $\phi_\zeta(s)$, $0 \leq s \leq 1$, of an integer-valued random variable.

The B.P.R.E. develops as follows: $Z_0 = 1$; Z_{n+1} is the total number of offspring resulting from Z_n parents, each such parent having a random number of offspring governed by the p.g.f. $\phi_{\zeta_{n+1}}(s)$ independently of other parents and of the environment variables other than ζ_{n+1}. This model is clearly the same as the classical Galton-Watson branching process except that we allow family-size distributions to vary stochastically from generation to generation. However, all families of a given generation are governed by the same distribution of family size. Thus the separate family trees springing from different parents in a given generation, which are independent in the classical Galton-Watson process, a fact which renders the classical Galton-Watson process so tractable, are dependent in the B.P.R.E.

The B.P.R.E. will be discussed more fully elsewhere (Smith and Wilkinson, 1967). In the present note we are concerned with proving one theorem which, taken with Theorem A below (whose proof will be given in the aforementioned reference), settles at an acceptable level of generality the question: under what circumstances will the B.P.R.E. almost surely become extinct?

At this point we need some definitions. We shall suppose the means of the family size distributions

$$\xi_n =_{\text{def}} \lim_{s\uparrow 1} \frac{1 - \phi_{\zeta_{n+1}}(s)}{1 - s}, \quad n = 1,2,\cdots,$$

are independent and identically distributed proper random variables (i.e. $P\{\xi_n < \infty\} = 1$). We shall also write

$$\eta_n =_{\text{def}} \phi_{\zeta_{n+1}}(0)$$

for the probability that a parent of the nth generation shall have no offspring. We shall additionally suppose that $\{\eta_n\}$ constitutes a sequence of independent and identically distributed random variables. To avoid triviality we shall assume $P\{\eta_n = 0\} < 1$, since the alternative possibility obviously implies $P\{Z_n > 0\} = 1$ for all n.

We can now quote:

Theorem A Suppose that $\mathcal{E}|\log \xi_n| < \infty$. Then

a) The B.P.R.E. $\{Z_n\}$ will almost surely become extinct, i.e. $P\{Z_n > 0\} \rightarrow 0$ as $n \rightarrow \infty$, if $\mathcal{E} \log \xi_n \leq 0$;

b) If $\mathcal{E} \log \xi_n > 0$ and if, additionally,

(A1) $$\mathcal{E}|\log(1 - \eta_n)| < \infty,$$

then $P\{Z_n > 0\}$ tends to some strictly positive limit as $n \rightarrow \infty$, i.e. extinction is not almost sure.

The theorem we prove in this paper is as follows:

Theorem 1 If there is a $c > 0$ such that $P\{Z_n > 0\} \geq c$ for all n, i.e. if extinction is not almost certain, and if $\mathcal{E}|\log \xi_n| < \infty$, then it is necessary that the following two conditions both hold:

$$\text{(i)} \quad \mathcal{E} \log \xi_n > 0$$

$$\text{(ii)} \quad \mathcal{E}|\log(1 - \eta_n)| < \infty.$$

The method of proof in the present note is entirely different from that used by Smith and Wilkinson (1967) to prove Theorem A. It will be noted that both theorems prove that extinction of the B.P.R.E. is almost sure if $\mathcal{E} \log \xi_n \leq 0$; thus two different methods are available for proving this partial result.

All this work, it should be noted, assumes the family size distributions have, almost surely, finite mean values. Work remains to be done for cases when this assumption is invalid. In this connection we close this note with a specific example of such a situation, and show that, for this special example, the necessary and sufficient conditions for almost certain extinction are quite different from those obtained in Theorems A and 1.

2. Proof of Theorem 1.

For every $\lambda > 0$, let us write

$$\psi(\lambda) = \mathcal{E} \eta_n^{\lambda} .$$

We need two preparatory lemmas.

<u>Lemma 2.1 If $0 < c < \infty$, $0 < \mu_1 < \infty$, then</u>

$$\sum_{r=0}^{\infty} \psi(c^{-1}\mu_1^r) < \infty \tag{2.1}$$

<u>if and only if $\mu_1 > 1$ and $\mathcal{E}|\log(1 - \eta_n)| < \infty$.</u>

<u>Proof</u> The function $\psi(\lambda)$ is non-increasing and so the series (2.1) must obviously diverge if $\mu_1 \leq 1$. Therefore suppose $\mu_1 > 1$.

Now, if $0 < \eta \leq 1$, $\mu_1 > 1$,

$$\int_1^{\infty} \eta^{c^{-1}\mu_1^r}\, dr \leq \sum_{r=1}^{\infty} \eta^{c^{-1}\mu_1^r} \leq \int_0^{\infty} \eta^{c^{-1}\mu_1^r}\, dr.$$

If we set $x = c^{-1}\mu_1^r \log(1/\eta)$ we find

$$\int_{c^{-1}\mu_1\log(1/\eta)}^{\infty} e^{-x}\,\frac{dx}{x} \leq (\log \mu_1) \sum_{r=1}^{\infty} \eta^{c^{-1}\mu_1^r} \leq \int_{c^{-1}\log(1/\eta)}^{\infty} e^{-x}\,\frac{dx}{x}$$

Let us write $G(u) = P\{\eta_n \leq u\}$ for the d.f. of η_n. From the above inequalities and the fact that

$$\int_y^{\infty} e^{-x}\,\frac{dx}{x} \sim \log(1/y)$$

as $y \downarrow 0$, it follows on taking expectations that (2.1) converges if and only if

$$\int_{1-\varepsilon}^{1} \log\{\frac{1}{\log(1/u)}\}\, dG(u) < \infty \tag{2.2}$$

for some small $\varepsilon > 0$. But if $1 - \varepsilon \leq u \leq 1$, we can find $\delta > 0$ such that

$$(1 - u) \leq \log(1/u) \leq (1 + \delta)(1 - u).$$

Thus (2.2) will hold if and only if

(2.3) $$\int_{1-\varepsilon}^{1} \log\{\frac{1}{(1-u)}\}dG(u) < \infty,$$

and this result establishes the lemma.

<u>Lemma 2.2</u> <u>If $\rho = \mathcal{E}\log \xi_n$ and we choose any $\mu_1 > e^{\rho}$ then</u>

$$\mathcal{E}\, e^{-\lambda\xi_1\xi_2\cdots\xi_n} > \frac{1}{2}\, e^{-\lambda\mu_1^n}$$

<u>for all sufficiently large n</u>.

<u>Proof</u> By the weak law of large numbers,

$$\frac{1}{n}\sum_{r=1}^{n} \log \xi_n \longrightarrow \rho, \quad n \longrightarrow \infty$$

in probability. Thus if $\rho_1 > \rho$,

$$P\{\sum_{r=1}^{n} \log \xi_n < n\rho_1\} \longrightarrow 1, \quad n \longrightarrow \infty$$

and so we can find $n_0(\rho_1)$ such that

$$P\{\sum_{r=1}^{n} \log \xi_n < n\mu_1\} > \frac{1}{2}, \quad n \geq n_0.$$

This implies that, for any $\lambda > 0$, if we set $\mu_1 = e^{\rho_1}$,

$$\mathcal{E}\, e^{-\lambda\xi_1\xi_2\cdots\xi_n} > \frac{1}{2}\, e^{-\lambda\mu_1^n}, \quad n \geq n_0.$$

Thus the lemma is proved.

We are now ready to prove the theorem. We begin by observing that

$$(2.4)\quad \begin{aligned} P\{Z_{n+1} > 0\} &= P\{Z_n > 0\}\mathcal{E}\{1 - [\phi_{\zeta_{n+1}}(0)]^{Z_n} | Z_n > 0\} \\ &= P\{Z_n > 0\}\mathcal{E}\{1 - \psi(Z_n) | Z_n > 0\}. \end{aligned}$$

It is an easy exercise to verify that

$$\mathcal{E}\{Z_n | \xi_1, \xi_2, \cdots, \xi_n\} = \xi_1 \xi_2 \cdots \xi_n$$

Thus

$$P\{Z_n > 0\}\mathcal{E}\{Z_n | \xi_1, \xi_2, \cdots, \xi_n;\ Z_n > 0\} = \xi_1 \xi_2 \cdots \xi_n.$$

Plainly $P\{Z_n > 0\}$ is a non-increasing function of n. Let us assume that <u>extinction is not almost sure</u>. Then there must be a $c > 0$ such that $P\{Z_n > 0\} \geq c$ for all n. It then follows that

$$(2.5)\quad \mathcal{E}\{Z_n | \xi_1, \xi_2, \cdots, \xi_n;\ Z_n > 0\} \leq c^{-1} \xi_1 \xi_2 \cdots \xi_n.$$

Let $\lambda > 0$; by familiar convexity arguments

$$\begin{aligned} \mathcal{E}\{e^{-\lambda Z_n} | \xi_1, \xi_2, \cdots, \xi_n;\ Z_n > 0\} &\geq \exp[-\lambda \mathcal{E}\{Z_n | \xi_1, \cdots, \xi_n;\ Z_n > 0\}] \\ &\geq \exp[-\lambda c^{-1} \xi_1 \xi_2 \cdots \xi_n], \quad \text{by (2.5).} \end{aligned}$$

If we now take expectations with respect to $\xi_1, \xi_2, \cdots, \xi_n$ and use Lemma 2.2 we find

$$(2.6)\quad \mathcal{E}\{e^{-\lambda Z_n} | Z_n > 0\} \geq \frac{1}{2} e^{-\lambda c^{-1} \mu_1^n}$$

for all large n and any $\mu_1 > e^{\rho}$.

The proof of (2.6) has depended only on the environment variables $\zeta_1, \zeta_2, \cdots, \zeta_n$ and variables related thereto. These are all independent of $\eta_n = \phi_{\zeta_{n+1}}(0)$. Thus in (2.6) we may take $e^{-\lambda} = \eta_n$ and deduce

(2.7) $$\mathcal{E}\{\psi(Z_n) | Z_n > 0\} \geq \frac{1}{2}\, \psi(c^{-1}\mu_1^n)$$

for all $n \geq n_0$. But (2.4) implies

$$P\{Z_{n+1} > 0\} = \prod_{r=1}^{n} \mathcal{E}\{1 - \psi(Z_r) | Z_r > 0\},$$

so we discover that

$$0 < c \leq \prod_{r=n_0}^{n} \{1 - \frac{1}{2}\, \psi(c^{-1}\mu_1^n)\},$$

for all n. By a well-known result on infinite products this implies the convergence of

(2.8) $$\sum_{r=1}^{\infty} \psi(c^{-1}\mu_1^r).$$

If $\rho < 0$ we can choose $\mu_1 < 1$ and obtain a contradiction with Lemma 2.1. Thus if $\rho < 0$ extinction must be almost sure.

If $\rho > 0$ then (2.8) converges for some $\mu_1 > 1$. Lemma 2.1 then shows that we must have $\mathcal{E}|\log(1-\eta_n)| < \infty$.

Thus the theorem is proved except for the case $\rho = 0$. In this case we set $S_n = \sum_{r=1}^{n} \log \xi_r$, and note that

$$\mathcal{E} e^{-\lambda \xi_1 \cdots \xi_n} = \mathcal{E} e^{-\lambda e^{S_n}}$$

$$\geq e^{-\lambda} P\{S_n \leq 0\}.$$

In place of (2.6) we then have

$$\mathscr{E}\{e^{-\lambda Z_n} | Z_n > 0\} \geq e^{-\lambda} P\{S_n \leq 0\},$$

and in place of (2.7) we obtain

$$\mathscr{E}\{\psi(Z_n) | Z_n > 0\} \geq \psi(1) P\{S_n \leq 0\}.$$

The infinite product argument then shows that $P\{Z_n > 0\} \geq c > 0$ for all n requires the convergence of

$$\sum_{n=1}^{\infty} P\{S_n \leq 0\}. \tag{2.9}$$

But it is known from renewal theory that (2.9) can only converge if $\mathscr{E}\log \xi_r > 0$. Thus the proof is completed.

3. An example with infinite mean family sizes.

Suppose α is a real constant, $0 < \alpha < 1$, and that the environment variables $\{\zeta_n\}$ are real and such that, almost surely, $0 < \zeta_n < 1$ for all n. Let us then set

$$\phi_{\zeta_n}(s) = 1 - \zeta_n (1 - s)^{\alpha}, \ 0 \leq s \leq 1.$$

It is not difficult to show that $\phi_{\zeta_n}(s)$ is indeed a p.g.f. and that the mean of the associated probability distribution is infinite. Furthermore, elementary computation will show that these generating functions have a very handy property:

$$\phi_{\zeta_2}(\phi_{\zeta_1}(s)) = 1 - \zeta_2 \zeta_1^{\alpha} (1 - s)^{\alpha^2}.$$

By repeated use of this algorithm we have that

$$\phi_{\zeta_n}(\phi_{\zeta_{n-1}}(\cdots\phi_{\zeta_1}(s)\ \ldots)) = 1 - \zeta_n\zeta_{n-1}^{\alpha}\ \zeta_{n-2}^{\alpha^2}\cdots\zeta_1^{\alpha^{n-1}}(1 - s)^{\alpha^n}.$$

Thus, if $\Pi_n(s)$ is the p.g.f. of Z_n, we have

$$\Pi_n(s) = 1 - (1 - s)^{\alpha^n}\,\mathcal{E}\,\zeta_n\zeta_{n-1}^{\alpha}\cdots\zeta_1^{\alpha^{n-1}}.$$

Let us set $\phi(\lambda) = \mathcal{E}\zeta^{\lambda}$. Then we see from the last equation that $\Pi_n(s) \longrightarrow 1$ if and only if the infinite product

$$\prod_{n=1}^{\infty} \phi(\alpha^n)$$

diverges to zero. This will happen if and only if

$$\sum_{n=1}^{\infty} [1 - \phi(\alpha^n)] = \infty.$$

Arguments similar to the ones employed in proving Lemma 2.1 then show that extinction is almost sure if and only if

$$\mathcal{E}\log^{+}\log(1/\zeta_1) = \infty$$

or, in other words, if and only if

$$\mathcal{E}\log^{+}|\log(1 - \phi_{\zeta_1}(0)| = \infty.$$

Thus, for almost sure extinction, the probabilities $\phi_{\zeta_n}(0)$ of no offspring from a given parent have got to be "much nearer" unity, in a probabilistic sense, than would be the case in the situation covered by Theorem 1.

Reference

W. L. Smith and W. E. Wilkinson, (1967), "On branching processes with random environments," _to appear_.