A MULTIVARIATE EXTENSION OF FRIEDMAN'S χ_r^2-TEST

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Abstract

This paper deals with a multivariate extension of Friedman's χ_r^2-test. A rank permutation distribution and the large sample properties of the criterion are studied. The asymptotic relative efficiency (A.R.E.) for a sequence of translation alternatives is studied and bounds are given for certain special cases. It is shown that, under specified conditions, the A.R.E. of this test with respect to the likelihood ratio test is largest when the block dispersion matrices differ and can be greater than unity when the differences are large.

1. INTRODUCTION

Suppose we are given p-variate data which form a complete two-way layout. Let $\underset{\sim}{X}_{ij} = (X_{ij}^1, X_{ij}^2, \ldots, X_{ij}^p)$ be the response from the plot in the i^{th} block that received the j^{th} treatment; $i=1,2,\ldots,n$; $j=1,2,\ldots,k$; $(k \geq 2)$. Assume that the cumulative distribution function of $\underset{\sim}{X}_{ij}$ is $F_{ij}(\underset{\sim}{x})$, $\underset{\sim}{x} \varepsilon R^p$.

We wish to test the null hypothesis

$$H_o: \quad F_{i1} \equiv F_{i2} \equiv \cdots \equiv F_{ik} \equiv F_i, \quad \text{for } i=1,2,\ldots,n, \tag{1.1}$$

against translation type alternatives of the form

$$H_A: \quad F_{ij}(\underset{\sim}{x}) = F_i(\underset{\sim}{x} - \underset{\sim}{\alpha}_j), \quad \text{for } j=1,2,\ldots,k \text{ and } i=1,2,\ldots,n, \tag{1.2}$$

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where $\underset{\sim}{\alpha}_j = (\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^p)$. Under this type of alternative, we can write the null hypothesis as

$$H_o: \underset{\sim}{\alpha}_1 = \underset{\sim}{\alpha}_2 = \ldots = \underset{\sim}{\alpha}_k. \tag{1.3}$$

In standard parametric tests of this hypothesis, it is usually assumed that

$$\underset{\sim}{X}_{ij} = \underset{\sim}{\mu} + \underset{\sim}{\alpha}_j + \underset{\sim}{\beta}_i + \underset{\sim}{e}_{ij},$$

where $\underset{\sim}{\mu}$, $\underset{\sim}{\alpha}_j$ and $\underset{\sim}{\beta}_i$ are the constant vectors of mean effects, treatment effects and block effects. The vectors $\underset{\sim}{e}_{ij}$ are nk independent random vectors following a common multinormal distribution with zero mean vector and a common dispersion matrix $\underset{\sim}{\Sigma} = ((\text{cov}(e_{ij}^s, e_{ij}^{s'})))$. Several such tests are discussed in Anderson [1], in Rao [7], and elsewhere.

Frequently, some or all of these assumptions cannot be justified. The subject of this investigation is a non-parametric test which allows the assumptions of block additivity and multinormality to be relaxed. Friedman [5] proposed the χ_r^2-test for the univariate case and its asymptotic efficiency was later studied by Elteren and Noether [4]. A class of non-parametric tests, which do not require the multinormality or independence assumptions, has been proposed and studied by Sen [8].

A brief outline of the rank permutation principle, which was studied in depth by Chatterjee [3] for k=p=2 and by Sen [8], will be presented in the next section.

2. PERMUTATION ARGUMENT

The test to be proposed is based upon intrablock rankings. Ranking will be carried out over the k treatments within each block and done individually for each variable. Thus, R_{ij}^s will be the rank of X_{ij}^s among the observations $\{X_{it}^s,\ t=1,2,\ldots,k\}$.

To illustrate the permutation argument, form the n matrices

$$\underset{\sim}{R}_i = \begin{bmatrix} R^1_{i1} & \cdots & R^1_{ik} \\ \vdots & & \vdots \\ R^p_{i1} & \cdots & R^p_{ik} \end{bmatrix}, \quad i=1,2,\ldots,n.$$

Each row of $\underset{\sim}{R}_i$ will be some permutation of the numbers 1,2,...,k. Define $\underset{\sim}{R}^*_i$ to be the matrix derived from $\underset{\sim}{R}_i$ by permuting the columns in such a way that the numbers 1,2,...,k appear in sequence in the first row. We say that two matrices, $\underset{\sim}{A}$ and $\underset{\sim}{B}$, are permutationally equivalent if $\underset{\sim}{A}$ can be obtained from $\underset{\sim}{B}$ by a finite number of permutations of the columns of $\underset{\sim}{B}$. Let $S(\underset{\sim}{R}^*_i)$ be the set of matrices which are permutationally equivalent to $\underset{\sim}{R}^*_i$. Hence, $S(\underset{\sim}{R}^*_i)$ contains k! elements.

The distribution of $\underset{\sim}{R}^*_i$ over all its $(k!)^{p-1}$ realizations will depend upon the parent distributions, even under H_o. However, given a particular realization of $\underset{\sim}{R}^*_i$, the distribution of $\underset{\sim}{R}_i$ over $S(\underset{\sim}{R}^*_i)$ will be uniform under H_o. In fact, if $\underset{\sim}{R}_o \in S(\underset{\sim}{R}^*_i)$, then $P\{\underset{\sim}{R}_i = \underset{\sim}{R}_o | S(\underset{\sim}{R}^*_i), H_o\} = 1/k!$, no matter what be the parent distributions.

Finally, if $\underset{\sim}{R}_{o_i} \in S(\underset{\sim}{R}^*_i)$, i=1,2,...,n, then

$$\begin{aligned} P\{\underset{\sim}{R}_i = \underset{\sim}{R}_{o_i},\ i=1,2,\ldots,n \,|\, S(\underset{\sim}{R}^*_i),\ i=1,2,\ldots,n;\ H_o\} \\ = \prod_{i=1}^{n} P\{\underset{\sim}{R}_i = \underset{\sim}{R}_{o_i} \,|\, S(\underset{\sim}{R}^*_i), H_o\} = \left(\frac{1}{k!}\right)^n \end{aligned} \tag{2.1}$$

because intrablock rankings are used and complete independence is assumed from block to block.

We are now in a position to select a test function which depends upon $\underset{\sim}{R}_i$, i=1,2,...,n. Such a test will be completely specified by the conditional probability law (2.1) and, hence, will be a similar test of H_o.

Henceforth, we will let $\mathcal{P}_n$ denote the probability law given by (2.1).

3. THE PERMUTATION TEST

Define $T_j^s = (1/n)\Sigma_{i=1}^n R_{ij}^s$ for $s=1,2,\ldots,p$; $j=1,2,\ldots,k$. Note that T_j^s depends upon n.

It is easily seen that

$$E\{T_j^s|\mathcal{P}_n\} = (1/n)\Sigma_{i=1}^n E\{R_{ij}^s|\mathcal{P}_n\} = (1/n)\Sigma_{i=1}^n\{(1/k)\Sigma_{j=1}^k R_{ij}^s\} = (k+1)/2,$$

and, after some essentially simple steps, that

$$\mathrm{Cov}\{R_{ij}^s,R_{i'j'}^{s'}|\mathcal{P}_n\} = \begin{cases} 0, & \text{for } i\neq i' \\ -(1/k)[\Sigma_{t=1}^k R_{it}^s R_{it}^{s'}-(k+1)^2/4], & \text{for } i=i',\ j\neq j' \\ \{(k-1)/k\}[\Sigma_{t=1}^k R_{it}^s R_{it}^{s'} - (k+1)^2/4], & \text{for } i=i',\ j=j'. \end{cases}$$

Finally, we find that

$$\mathrm{Cov}\{T_j,T_{j'}|\mathcal{P}_n\} = (1/n)(\delta_{jj'}-1/k)\sigma_{[s,s']}(\underset{\sim}{R}^*), \tag{3.1}$$

where $\sigma_{[s,s']}(\underset{\sim}{R}^*) = (1/n(k-1))\Sigma_{i=1}^n\Sigma_{t=1}^k R_{it}^s R_{it}^{s'}-(k(k+1)^2/4(k-1))$, and $\delta_{jj'}$ is the Kronecker delta. Again, we note that $\sigma_{[s,s']}(\underset{\sim}{R}^*)$ depends upon n.

Define $\underset{\sim}{\Sigma}(\underset{\sim}{R}^*) = ((\sigma_{[s,s']}(\underset{\sim}{R}^*)))$ and

$$\underset{\sim}{T}'_{(1\times p(k-1))} = (T_1^1,\ldots,T_1^p,T_2^1,\ldots,T_2^p,\ldots,T_{k-1}^1,\ldots,T_{k-1}^p).$$

Thus,

$$\mathrm{Var}(\underset{\sim}{T}) = \underset{\sim}{\Delta}\otimes\underset{\sim}{\Sigma}(\underset{\sim}{R}^*) = \underset{\sim}{V}(\underset{\sim}{R}^*) \text{ say,}$$

where $\underset{\sim}{\Delta} = (((\delta_{jj'}k-1)/nk))$ and $\otimes$ denotes the direct product of the two matrices as defined in [1, p. 347]. If we assume that $\underset{\sim}{\Sigma}(\underset{\sim}{R}^*)$ is positive definite, then

$$V^{-1}(R^*) = \Lambda^{-1} \otimes \Sigma^{-1}(R^*).$$

Finally, using the well-known formula

$$(\Lambda - u\,u')^{-1} = \Lambda^{-1} + \frac{1}{(1-u'\Lambda^{-1}u)} \Lambda^{-1}u\,u'\Lambda^{-1}$$

where u' is $(1\times r)$ and Λ is $(r\times r)$, it follows that

$$V^{-1}(R^*) = n \begin{bmatrix} 21 & \dots & 1 \\ 12 & \dots & 1 \\ \vdots & & \\ 11 & \dots & 2 \end{bmatrix} \otimes \Sigma^{-1}(R^*).$$

Now, for our test function we take the quadratic form

$$\mathcal{L}_n = (T-\bar{T})'V^{-1}(R^*)(T-\bar{T}), \tag{3.2}$$

where $\bar{T}(1\times p(k-1)) = ((k+1)/2,\dots,(k+1)/2)$. After some simple steps, we obtain

$$\mathcal{L}_n = n \sum_{s=1}^{p} \sum_{s'=1}^{p} \sigma^{[s,s']}(R^*) \sum_{j=1}^{k} (T_j^s - \frac{k+1}{2})(T_j^{s'} - \frac{k+1}{2}), \tag{3.3}$$

where $((\sigma^{[s,s']}(R^*))) = ((\sigma_{[s,s']}(R^*)))^{-1} = \Sigma^{-1}(R^*)$.

It will be demonstrated later that, for large n and with some mild restrictions on F_i, $i=1,2,\dots,n$, $\Sigma(R^*)$ is positive definite with high probability. If, in practice, it is found to be singular, then elimination of the proper variables will remedy the problem.

Following the arguments of Section 2, we can now compute the exact permutation distribution of $\mathcal{L}_n$ and thereby, determine a randomized test of H_o which is strictly distribution-free and has exact size α. However, when p, k and n are large, these computations may be too lengthly to be practical. In the next section, we will study the asymptotic permutation distribution of $\mathcal{L}_n$. This will provide us with a large sample approximation to the permutation distribution of $\mathcal{L}_n$ and, thereby, increase the usefulness of the test.

4. ASYMPTOTIC PERMUTATION DISTRIBUTION OF $\mathcal{L}_N$

Define $F_{ij[s]}(x)$ and $F_{ij[s,s']}(x,y)$ to be the marginal cdf's of X_{ij}^s and $(X_{ij}^s, X_{ij}^{s'})$ for $s,s'=1,2,\ldots,p$; $i=1,2,\ldots,n$; $j=1,2,\ldots,k$.

Define

$$\sigma_{[s,s']}(\underset{\sim}{F}_{io}) = \frac{1}{k-1}\left\{\sum_{\substack{j\neq t\neq t'=1\\ j\neq t'}}^{k}\sum^{k}\sum^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{it[s]}(x)F_{it'[s']}(y)\,dF_{ij[s,s']}(x,y)\right.$$

$$\left. + \sum_{j\neq t=1}^{k}\sum^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{it[s,s']}(x,y)\,dF_{ij[s,s']}(x,y) - \frac{k(k-1)^2}{4}\right\}$$

for $s\neq s'=1,2,\ldots,p$; $i=1,2,\ldots,n$

$$= k(k+1)/12 \text{ for } s=s'=1,2,\ldots,p;\ i=1,2,\ldots,n. \tag{4.1}$$

Let

$$\underset{\sim}{\Sigma}(\underset{\sim}{F}_{io}) = ((\sigma_{[s,s']}(\underset{\sim}{F}_{io}))),\ s,s'=1,2,\ldots,p. \tag{4.2}$$

<u>Theorem 4.1</u>: <u>If F_{ij} is continuous for $i=1,2,\ldots,n$; $j=1,2,\ldots,k$, then</u>

$$\underset{\sim}{\Sigma}(\underset{\sim}{R}^*)-(1/n)\Sigma_{i=1}^{n}\{\underset{\sim}{\Sigma}(\underset{\sim}{F}_{io})\}$$

<u>converges in probability to the null matrix as $n\uparrow\infty$.</u>

Proof: Define the function

$$\Phi(x,y) = \begin{cases} 1 & \text{for } x > y \\ 0 & \text{for } x \leq y. \end{cases} \tag{4.3}$$

Let $\sigma_{[s,s']}(\underset{\sim}{R}^*_{io}) = [1/(k-1)]\Sigma_{j=1}^{k}R_{ij}^{s}R_{ij}^{s'} - k(k+1)^2/4(k-1)$, then $\sigma_{[s,s']}(\underset{\sim}{R}^*) = (1/n)\Sigma_{i=1}^{n}\sigma_{[s,s']}(\underset{\sim}{R}^*_{io})$. Now using (4.3) we can write $R_{ij}^{s} = 1+\Sigma_{t=1}^{k}\Phi(X_{ij}^{s}, X_{it}^{s})$

and, hence,

$$\sigma_{[s,s']}(\underset{\sim}{R}^*_{io}) = \frac{1}{k-1}\sum_{t=1}^{k}\Phi(X^s_{ij},X^s_{it}) + \frac{1}{k-1}\sum_{t=1}^{k}\Phi(X^{s'}_{ij},X^{s'}_{it'})$$

$$+\frac{1}{k-1}\sum_{t=1}^{k}\sum_{t'=1}^{k}\Phi(X^s_{ij},X^s_{it})\Phi(X^{s'}_{ij},X^{s'}_{it'}) - \frac{k(k+1)^2}{4(k-1)} + \frac{k}{k-1}.$$

From (4.3), it follows that

$$E\{\Phi(X^s_{ij},X^s_{it})\} = \begin{cases} \int_{-\infty}^{\infty} F_{it[s]}(x)\,dF_{ij[s]}(x), & \text{for } j\neq t \\ 0, & \text{for } j=t, \end{cases}$$

that

$$\sum_{j\neq t=1}^{k}\sum^{k}\int_{-\infty}^{\infty} F_{it[s]}(x)\,dF_{ij[s]}(x) = \frac{k(k-1)}{2}$$

and that, if $s\neq s'$, then

$$E\{\Phi(X^s_{ij},X^s_{it})\Phi(X^{s'}_{ij},X^{s'}_{it'})\}$$

$$= \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{it[s]}(x)F_{it'[s']}(y)\,dF_{ij[s,s']}(x,y), & \text{for } t\neq t',\ t\neq j,\ t'\neq j \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{it[s,s']}(x,y)\,dF_{ij[,s']}(x,y), & \text{for } t=t',\ t\neq j \\ 0, & \text{otherwise.} \end{cases}$$

Using the above results, we can show that

$$E\{\sigma_{[s,s']}(\underset{\sim}{R}^*_{io})\} = \frac{1}{k-1}\left\{\sum_{\substack{j\neq t\neq t'=1 \\ j\neq t'}}^{k}\sum^{k}\sum^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{it[s]}(x)F_{it'[s']}(y)\,dF_{ij[s,s']}(x,y)\right.$$

$$\left.+\sum_{j\neq t=1}^{k}\sum^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{it[s,s']}(x,y)\,dF_{ij[s,s']}(x,y) - \frac{k(k-1)^2}{4}\right\} = \sigma_{[s,s']}(\underset{\sim}{F}_{io}),$$

and hence, that $E\{\sigma_{[s,s']}(\underset{\sim}{R}^*)\} = (1/n)\Sigma_{i=1}^{n}\sigma_{[s,s']}(\underset{\sim}{F}_{io})$.

Since R_{ij}^{s} is bounded above and below for $s=1,2,\ldots,p$; $j=1,2,\ldots,k$; $i=1,2,\ldots,n$, it follows that $\text{Var}\{\sigma_{[s,s']}(\underset{\sim}{R}^*_{io})\}$ has an upper bound that is independent of i and n. Thus, $(1/n^2)\Sigma_{i=1}^{n}\ \text{Var}\{\sigma_{[s,s']}(R^*_{io})\}\to 0$ as $n\uparrow\infty$ and so, by the weak law of large numbers $\sigma_{[s,s']}(\underset{\sim}{R}^*)-(1/n)\Sigma_{i=1}^{n}\sigma_{[s,s']}(\underset{\sim}{F}_{io}) \overset{p}{\to} 0$ as $n\uparrow\infty$, which completes the proof.

<u>Theorem 4.2</u>: <u>Under</u> H_o, $(1/n)\Sigma_{i=1}^{n}\sigma_{[s,s']}(\underset{\sim}{F}_{io}) = \sigma_{[s,s']}$, <u>where</u>

$$\sigma_{[s,s']} = \begin{cases} (1/n)\Sigma_{i=1}^{n}\{(k/4)\tau_{i[s,s']} + (k(k-2)/12)\rho_{gi[s,s']}\}, & \text{for } s\neq s' \\ k(k+1)/12, & \text{for } s=s', \end{cases} \tag{4.4}$$

<u>with</u>

$$\tau_{i[s,s']} = 4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{i[s,s']}(x,y)-\tfrac{1}{4}]dF_{i[s,s']}(x,y);$$

$$\rho_{gi[s,s']} = 12 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_{i[s]}(x)F_{i[s']}(y)-\tfrac{1}{4}]dF_{i[s,s']}(x,y),$$

<u>and where</u> $F_{i[s]}(x)$, $F_{i[s,s']}(x,y)$ <u>are the</u> s^{th} <u>and</u> $(s,s')^{th}$ <u>marginals of</u> $F_i(\underset{\sim}{x})$ <u>given by (1.2)</u>.

Proof: The theorem follows from Theorem 4.1 after setting $F_{ij}(\underset{\sim}{x}) = F_i(\underset{\sim}{x})$.

<u>Theorem 4.3</u>: <u>If</u> $F_1,F_2,\ldots,F_n$ <u>are continuous and</u> $\underset{\sim}{\Sigma} = ((\sigma_{[s,s']}))$ <u>is positive definite, then the joint distribution of</u> $\{n^{\frac{1}{2}}(T_j^s - \frac{k+1}{2});\ s=1,2,\ldots,p;j=1,2,\ldots,k-1\}$ <u>is asymptotically p(k-1)-variate normal with mean vector zero and dispersion matrix</u> $\underset{\sim}{\Delta}\otimes\underset{\sim}{\Sigma}$ <u>where</u> $\underset{\sim}{\Delta} = ((\delta_{jj'}-1/k))$.

Proof: Let a_{sj}; $s=1,2,\ldots,p$; $j=1,2,\ldots,k$ be any sp constants such that $\sum_{j=1}^{k} a_{sj}=0$ for $s=1,2,\ldots,p$. Then write

$$Y_n = n^{\frac{1}{2}} \sum_{s=1}^{p} \sum_{j=1}^{k} a_{sj} T_j^s = n^{-\frac{1}{2}} \sum_{i=1}^{n} \left\{ \sum_{s=1}^{p} \sum_{j=1}^{k} a_{sj} R_{ij}^s \right\} = n^{-\frac{1}{2}} \sum_{i=1}^{n} U_i,$$

where

$$U_i = \sum_{s=1}^{p} \sum_{j=1}^{k} a_{sj} R_{ij}^s.$$

Now, we find that $E\{U_i | \mathcal{P}_n\} = 0$ and that

$$E\{U_i^2 | \mathcal{P}_n\} = E\left\{ \sum_{s=1}^{p} \sum_{s'=1}^{p} \sum_{j=1}^{k} \sum_{j'=1}^{k} a_{sj} a_{s'j'} R_{ij}^s R_{ij'}^{s'} | \mathcal{P}_n \right\}$$

$$= \sum_{s=1}^{p} \sum_{s'=1}^{p} \left\{ \sum_{j \neq j'=1}^{k} \sum^{k} a_{sj} a_{s'j'} \left[\frac{k(k+1)^2}{4(k-1)} - \frac{1}{k(k-1)} \sum_{t=1}^{k} R_{it}^s R_{it}^{s'}\right] \right.$$

$$\left. + \sum_{j=1}^{k} a_{sj} a_{s'j} \left[\frac{1}{k} \sum_{t=1}^{k} R_{it}^s R_{it}^{s'}\right] \right\}$$

$$= \sum_{s=1}^{p} \sum_{s'=1}^{p} \left\{ \frac{1}{k-1} \sum_{j=1}^{k} a_{sj} a_{s'j} \sum_{t=1}^{k} R_{it}^s R_{it}^{s'} - \sum_{j=1}^{k} a_{sj} a_{s'j} \frac{k(k+1)^2}{4(k-1)} \right\}$$

$$= \frac{1}{k-1} \sum_{s=1}^{p} \sum_{s'=1}^{p} \left\{ \sum_{j=1}^{k} a_{sj} a_{s'j} \right\} \left\{ \sum_{t=1}^{k} R_{it}^s R_{it}^{s'} - \frac{k(k+1)^2}{4} \right\}$$

$$= \sum_{s=1}^{p} \sum_{s'=1}^{p} \left\{ \sum_{j=1}^{k} a_{sj} a_{s'j} \right\} \sigma_{[s,s']}(\underset{\sim}{R}^*_{io}).$$

Thus,

$$\frac{1}{n} \sum_{i=1}^{n} E\{U_i^2 | \mathcal{P}_n\} = \sum_{s=1}^{p} \sum_{s'=1}^{p} \left\{ \sum_{j=1}^{k} a_{sj} a_{s'j} \right\} \sigma_{[s,s']}(\underset{\sim}{R}^*).$$

But,

$$-\frac{k(k+1)}{12} \leq \sigma_{[s,s']}(\underset{\sim}{R}^*) \leq \frac{k(k+1)}{12},$$

so that

$$\left[\sum_{i=1}^{n} E\{U_i^2|\mathcal{P}_n\}\right]^{3/2} = O(n^{3/2}).$$

By a similar argument, we can show that $\Sigma_{i=1}^{n} E\{|U_i|^3|\mathcal{P}_n\} = O(n)$ and hence,

$$\lim_{n\to\infty} \sum_{i=1}^{n} E\{|U_i^3||\mathcal{P}_n\} / \left[\sum_{i=1}^{n} E\{U_i^2|\mathcal{P}_n\}\right]^{3/2} = 0.$$

Thus, the conditions for the Liapounoff central limit theorem are satisfied, and we may conclude that Y_n is asymptotically normal. An application of some well-known results from multivariate analysis completes the proof.

Theorem 4.4: Under the conditions of Theorem 4.3, the permutation distribution of $\mathcal{S}_n$ is asymptotically chi-square with p(k-1) degrees of freedom.

Proof: The theorem follows as an immediate consequence of Theorems 4.2 and 4.3.

Using Theorem 4.4, we can now write the asymptotic permutation test of H_o, given by (1.1), as

$$\begin{aligned} &\text{Reject } H_o \quad \text{if} \quad \mathcal{S}_n \geq \chi^2_{p(k-1),\alpha} \\ &\text{Accept } H_o \quad \text{if} \quad \mathcal{S}_n < \chi^2_{p(k-1),\alpha}, \end{aligned} \tag{4.5}$$

where

$$P\{\chi_t^2 \leq \chi_{t,\alpha}^2\} = 1-\alpha, \quad 0<\alpha<1.$$

We need to know conditions under which $\underset{\sim}{\Sigma}$ is positive definite (pd). If we examine $\sigma_{[s,s']}$ in (4.4), we see that we can write

$$\underset{\sim}{\Sigma} = C_1(1/n)\Sigma_{i=1}^{n}\underset{\sim}{T}_i + C_2(1/n)\Sigma_{i=1}^{n}\underset{\sim}{G}_i \tag{4.6}$$

where C_1 and C_2 are constants, $\underset{\sim}{T}_i = ((\tau_{i[s,s']}))$ and $\underset{\sim}{G}_i = ((\rho_{gi[s,s']}))$.

Since $\underset{\sim}{T}_i$ and $\underset{\sim}{G}_i$ are positive semi-definite, $\underset{\sim}{\Sigma}$ will be pd when either terms on the right hand side of (4.6) is pd. If we let $(X_i^1, X_i^2, \ldots, X_i^p)$ be a random vector whose cdf is $F_i(\underset{\sim}{x})$, then it can be shown that $\underset{\sim}{G}_i$ is the dispersion matrix of $(F_{i[1]}(X_i^1), \ldots, F_{i[p]}(X_i^p))$. Thus, $\underset{\sim}{G}_i$ is pd unless there exists a linear relationship between the variables $F_{i[s]}(X_i^s)$, $s=1,2,\ldots,p$ that holds a.s. Now, if there are $m=m(n)$ values of i for which this relationship fails to hold and if $m(n)/n \to \lambda$, where $0<\lambda\leq 1$ as $n\uparrow\infty$, then $(1/n)\Sigma_{i=1}^n \underset{\sim}{G}_i$ is pd a.s. and so is $\underset{\sim}{\Sigma}$.

Finally, we can show that the sequence of tests given by (4.5) is consistent, since,

$$\mathcal{L}_n \geq n \max_{s,j}\{T_j^s-(k+1)/2\}^2/\sigma_{[s,s]}(\underset{\sim}{R}^*) = (12n/k(k+1))\max_{s,j}\{T_j^s - (k+1)/2\}^2 = 0_p(n),$$

and so

$$P\{\mathcal{L}_n \geq \chi^2_{p(k-1),\alpha} | H_o \text{ not true}\} \to 1 \text{ as } n\uparrow\infty \ .$$

5. THE NON-NULL ASYMPTOTIC DISTRIBUTION OF $\{T_j^s\}$

Theorem 5.1: *If F_{ij} is continuous for $i=1,2,\ldots,n$; $j=1,2,\ldots,k$, then $\{n^{\frac{1}{2}}(T_j^s - \mu_j^s), j=1,2,\ldots,k; s=1,2,\ldots,p\}$ is asymptotically normal with mean vector zero and dispersion matrix $\underset{\sim}{\Sigma}^* = ((\sigma^*_{jj'[s,s']}))$, where*

$$\mu_j^s = \frac{1}{n}\sum_{i=1}^{n}\left\{\tfrac{1}{2} + \sum_{t=1}^{k}\int_{-\infty}^{\infty} F_{it[s]}(x)\,dF_{ij[s]}(x)\right\},$$

and

$$\sigma^*_{jj[s,s]} =$$

$$\frac{1}{n}\sum_{i=1}^{n}\left\{\sum_{\substack{t\neq t'=1 \\ t\neq j,\, t'\neq j}}^{k}\sum^{k}\left[\iint_{-\infty<y<x<\infty} A_{ij}^s(x,y)\,dF_{it[s]}(x)\,dF_{it'[s]}(y)\right.\right.$$

$$\left. + \iint_{-\infty<x<y<\infty} A_{ij}^s(x,y)\,dF_{it[s]}(x)\,dF_{it'[s]}(y)\right] +$$

$$+ \sum_{\substack{t=1\\t\neq j}}^{k} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{ij}^{s}(x,y)\,dF_{it[s]}(x)\,dF_{it[s]}(y) \Bigg\},$$

$$\sigma^{*}_{jj'[s,s]} = \frac{1}{n}\sum_{i=1}^{n}\Bigg\{\sum_{\substack{t=1\\t\neq j\\t\neq j'}}^{k}\Bigg[\iint_{-\infty<x<y<\infty} A_{it}^{s}(x,y)\,dF_{ij[s]}(x)\,dF_{ij'[s]}(x)$$

$$+ \iint_{-\infty<y<k<\infty} A_{it}^{s}(y,x)\,dF_{ij[s]}(x)\,dF_{ij'[s]}(y)\Bigg]$$

$$- \sum_{\substack{t=1\\t\neq j\\t\neq j'}}^{k}\Bigg[\iint_{-\infty<x<y<\infty} A_{ij}^{s}(x,y)\,dF_{it[s]}(x)\,dF_{ij'[s]}(y)$$

$$+ \iint_{-\infty<y<x<\infty} A_{ij}^{s}(y,x)\,dF_{it[s]}(x)\,dF_{ij'[s]}(y)\Bigg]$$

$$- \sum_{\substack{t=1\\t\neq j\\t\neq j'}}^{k}\Bigg[\iint_{-\infty<x<y<\infty} A_{ij'}^{s}(x,y)\,dF_{ij[s]}(x)\,dF_{it[s]}(y)$$

$$+ \iint_{-\infty<y<x<\infty} A_{ij'}^{s}(y,x)\,dF_{ij[s]}(x)\,dF_{it[s]}(y)\Bigg]$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{ij}^{s}(x,y)\,dF_{ij'[s]}(x)\,dF_{ij'[s]}(y)\Bigg\} \quad \text{for} \quad j\neq j',$$

$$\sigma^{*}_{jj[s,s']} = \frac{1}{n}\sum_{i=1}^{n}\Bigg\{\sum_{\substack{t=1\\t\neq j}}^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{it}^{s,s'}(x,y)\,dF_{ij[s,s']}(x,y)$$

$$+ \sum_{\substack{t=1\\t\neq j}}^{k}\sum_{\substack{t'=1\\t'\neq j}}^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{ij}^{s,s'}(x,y)\,dF_{it[s]}(x)\,dF_{it'[s]}(y)\Bigg\}$$

for $s\neq s'$,

$$\sigma^*_{jj'[s,s']} = \frac{1}{n}\sum_{i=1}^{n}\left\{\sum_{\substack{t=1\\t\neq j\\t\neq j'}}^{k}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{it}^{s,s'}(x,y)dF_{ij[s]}(x)dF_{ij'[s']}(y)\right.\right.$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{ij}^{s,s'}(x,y)dF_{it[s]}(x)dF_{ij'[s']}(y)$$

$$\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{ij'}^{s,s'}(x,y)dF_{ij[s]}(x)dF_{it[s']}(y)\right]$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{ij'}^{s,s'}(x,y)dF_{ij[s,s']}(x,y)$$

$$\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{ij}^{s,s'}(x,y)dF_{ij'[s]}(x)dF_{ij'[s']}(y)\right\} \text{ for } s\neq s' \text{ and } j\neq j',$$

where

$$A_{ij}^{s}(x,y) = F_{ij[s]}(x)[1-F_{ij[s]}(y)],$$

and

$$B_{ij}^{s,s'}(x,y) = [F_{ij[s,s']}(x,y)-F_{ij[s]}(x)F_{ij[s']}(y)].$$

Proof: Let a_{sj}, $s=1,2,\ldots,p$; $j=1,2,\ldots,k$ be pk arbitrary constants. Write

$$Y_n = n^{\frac{1}{2}}\sum_{s=1}^{p}\sum_{j=1}^{k} a_{sj}\{T_j^s-\mu_j^s\} = n^{-\frac{1}{2}}\left\{\sum_{i=1}^{n}\sum_{s=1}^{p}\sum_{j=1}^{k} a_{sj}[R_{ij}^s-\mu_{ij}^s]\right\} = n^{-\frac{1}{2}}\sum_{i=1}^{n} U_i,$$

where

$$\mu_{ij}^s = \tfrac{1}{2} + \sum_{t=1}^{k}\int_{-\infty}^{\infty} F_{it[s]}(x)dF_{ij[s]}(x).$$

Now, it is easily seen that $EU_i=0$ and that

$$EU_i^2 = \sum_{s=1}^{p}\sum_{s'=1}^{p}\sum_{j=1}^{k}\sum_{j'=1}^{k} a_{sj}a_{sj'}\,\mathrm{Cov}\{R_{ij}^s,R_{ij'}^{s'}\} .$$

Since R_{ij}^s, $i=1,2,\ldots,n$; $s=1,2,\ldots,p$, $j=1,2,\ldots,k$ are bounded above and below by 1 and k, there exists upper and lower bounds for $\mathrm{Cov}\{R^s_{ij}, R^{s'}_{ij'}\}$ which do not depend upon i, j, j', s, s' nor n. Thus, $\sum_{i=1}^{n} E\{U_i^2\} = O(n)$ and similarly $\Sigma_{i=1}^{n} E\{|U_i|^3\} = O(n)$, so that

$$\lim_{n\to\infty} \Sigma_{i=1}^{n} E\{|U_i|^3\}/[\Sigma_{i=1}^{n} E\{U_i^2\}]^{3/2} = 0.$$

Hence, by the Liapounoff central limit theorem and some well-known results from multivariate analysis, the asymptotic normality of $\{n^{\frac{1}{2}}(T^s_j - \mu^s_j)\}$ is established.

Finally, we obtain $\sigma^*_{jj'[s,s']} = n\,\mathrm{Cov}\{T^s_j, T^{s'}_{j'}\} = n^{-1}\Sigma_{i=1}^{n}\mathrm{Cov}\{R^s_{ij}, R^{s'}_{ij'}\}$. Using (4.3), it can be shown that

$$\mathrm{Cov}\{R^s_{ij}, R^{s'}_{ij'}\} = \sum_{\substack{t=1\\t\neq j}}^{k} \sum_{\substack{t'=1\\t'\neq j}}^{k} E\{\Phi(X^s_{ij}, X^s_{it})\Phi(X^{s'}_{ij'}, X^{s'}_{it'})\}$$

$$- \sum_{\substack{t=1\\t\neq j}}^{k} \sum_{\substack{t'=1\\t'\neq j'}}^{k} \left[\int_{-\infty}^{\infty} F_{it[s]}(x)\,dF_{ij[s]}(x)\right]\left[\int_{-\infty}^{\infty} F_{it'[s']}(y)\,dF_{ij'[s']}(y)\right]. \quad (5.1)$$

After some very tedious calculations, the first term on the right hand side of 5.1 can be evaluated, and the expressions for $\sigma^*_{jj'[s,s']}$, given in the theorem, can be obtained. This completes the proof.

It is useful to note that, under H_o (1.1), we find that $\sigma^*_{jj'[s,s']} = (\delta_{jj'} - 1/k)\sigma_{[s,s']}$ where $\sigma_{[s,s']}$ is given by (4.4).

The rank of the asymptotic multinormal distribution can be at most p(k-1), since $\sum_{j=1}^{k} T^s_j = k(k+1)/2$ for s=1,2,...,p. In fact, when H_o (1.1) holds and $F_1, F_2, \ldots, F_n$ are continuous and are such that $\underset{\sim}{\Sigma}$ is pd, the rank is p(k-1) and

$$\mathcal{L}_n^* = n \sum_{s=1}^{p} \sum_{s'=1}^{p} \sigma^{[s,s']} \sum_{j=1}^{k} (T_j^s - \frac{k+1}{2})(T_j^{s'} - \frac{k+1}{2}), \tag{5.2}$$

where $((\sigma^{[s,s']})) = ((\sigma_{[s,s']}))^{-1}$, is asymptotically chi-square with p(k-1) degrees of freedom.

6. ASYMPTOTIC DISTRIBUTIONS OF $\mathcal{L}_n$ AND $\mathcal{L}_n^*$ UNDER A SEQUENCE OF TRANSLATION ALTERNATIVES

For the purpose of studying asymptotic power efficiency, we shall concern ourselves with the sequence of translation alternatives, $\{H_n\}$, given by

$$F_{ij}(x^1,x^2,\ldots,x^p) = F_i(x^1_{nj},x^2_{nj},\ldots,x^p_{nj}) \tag{6.1}$$

where $x^s_{nj} = x^s + n^{-\frac{1}{2}}\alpha^s_j$, for s=1,2,...,p; j=1,2,...,k and where $\underset{\sim}{\alpha}_1=\underset{\sim}{\alpha}_2=\ldots=\underset{\sim}{\alpha}_k$ fails to hold.

Throughout this section, we will write $\underset{\sim}{F}\varepsilon\mathfrak{F}$ to mean that $F_1,F_2,\ldots,F_n$ are absolutely continuous and are such that $\underset{\sim}{\Sigma}$ is pd.

We note that if $\underset{\sim}{\alpha}_1=\underset{\sim}{\alpha}_2=\ldots=\underset{\sim}{\alpha}_k$, then $F_{ij}(\underset{\sim}{x}) = F_i(\underset{\sim}{x})$ for i=1,2,...,n; j=1,2,...,k.

<u>Theorem 6.1</u>: <u>If $\underset{\sim}{F}\varepsilon\mathfrak{F}$, then, under</u> $\{H_n\}$, $\{n^{\frac{1}{2}}(T^s_j-\mu^s_j);\ j=1,2,\ldots,k;\ s=1,2,\ldots,p\}$ <u>is asymptotically normal with mean vector zero and dispersion matrix</u> $\underset{\sim}{\Delta}\otimes\underset{\sim}{\Sigma}$ <u>where</u> $\underset{\sim}{\Delta} = ((\delta_{jj'},-1/k))$ <u>and</u> $\underset{\sim}{\Sigma}$ <u>is given in Theorem 4.3.</u>

Proof: Using 6.1, Taylor's theorem and some routine analysis, the theorem follows from Theorem 5.1.

<u>Theorem 6.2</u>: <u>If $\underset{\sim}{F}\varepsilon\mathfrak{F}$, then, under</u> $\{H_n\}$, <u>the asymptotic distribution of</u> $\mathcal{L}_n$, <u>defined by (3.3), is non-central chi-square with p(k-1) degrees of freedom and non-centrality parameter</u>

$$\lambda^2_{\mathcal{L}_n} = \sum_{s=1}^{p} \sum_{s'=1}^{p} \sigma^{[s,s']} A^s A^{s'} \sum_{j=1}^{k} \alpha_j^s \alpha_j^{s'}, \tag{6.2}$$

where

$$A^s = (k/n)\Sigma_{i=1}^{n} \int_{-\infty}^{\infty} [f_{i[s]}(x)]^2 dx, \ s=1,2,\ldots,p,$$

$$\int_{-\infty}^{y} f_{i[s]}(x)dx = F_{i[s]}(y), \text{ and } ((\sigma^{[s,s']})) = ((\sigma_{[s,s']}))^{-1}. \tag{6.3}$$

Proof: Under $\{H_n\}$ and $\underset{\sim}{F} \epsilon \mathfrak{F}$, it can be shown that $(1/n)\Sigma_{i=1}^{n}\{\underset{\sim}{\Sigma}(\underset{\sim}{F}_{io})\} - \underset{\sim}{\Sigma}$ tends to the null matrix as $n\uparrow\infty$. Thus, we have, by Theorem 4.1, that $\underset{\sim}{\Sigma}(\underset{\sim}{R}^*)-\underset{\sim}{\Sigma}$ converges in probability to the null matrix as $n\uparrow\infty$. Also, under $\{H_n\}$, $|E\{n^{\frac{1}{2}}(T_j^s-(k+1)/2)\}-\alpha_j^s A^s| \overset{p}{\to} 0$. The theorem now follows from Theorem 6.1.

7. ASYMPTOTIC POWER EFFICIENCY OF $\mathcal{L}_n$

For the purpose of studying asymptotic relative efficiency (A.R.E.), we will confine our attention to the well-known likelihood ratio test (l.r.t.) discussed in Anderson [1, pp. 187-210]. Unlike many of the parametric tests of H_o, the non-null asymptotic properties of the l.r.t. are known. In fact, under the sequence of alternatives $\{H_n\}$ given by (6.1), the l.r.t. statistic, U_n, can be shown to be asymptotically non-central chi-square with $p(k-1)$ degrees of freedom and non-centrality parameter $\lambda^2_{U_n} = \Sigma_{j=1}^{k} \underset{\sim}{\alpha}'_j \bar{\underset{\sim}{\Lambda}}^{-1} \underset{\sim}{\alpha}_j$, where $\underset{\sim}{\alpha}'_j$, $j=1,2,\ldots,k$ is given by (1.2), $\bar{\underset{\sim}{\Lambda}}^{-1} = (n^{-1}\Sigma_{i=1}^{n}\underset{\sim}{\Lambda}_i)^{-1}$, and $\underset{\sim}{\Lambda}_i = \mathrm{Var}(\underset{\sim}{x}_{ij})$.

Thus, the A.R.E. of $\mathcal{L}_n$ with respect to U_n can be taken as

$$e_{\mathcal{L}_n, U_n} = \lambda^2_{\mathcal{L}_n} / \lambda^2_{U_n} \tag{7.1}$$

In the univariate case, Sen [9] has shown that $e_{\mathcal{L}_n, U_n} \geq 0.864k/(k+1)$ uniformly in $F_1, F_2, \ldots, F_n$, for absolutely continuous F_i, $i=1,2,\ldots,n$, with finite, non-zero variances.

In the multivariate case, $e_{\mathcal{L}_n,U_n}$ depends upon $\underset{\sim}{\alpha}_j$, $\bar{\underset{\sim}{\Lambda}}$, $\underset{\sim}{\Sigma}$ and A^s, and, hence, a bound would be very difficult to establish.

For the special case when $\underset{\sim}{\Lambda}_i = \lambda_i^2 \underset{\sim}{I}$ and $\underset{\sim}{x}_{ij}$ has identically distributed components, it is easily shown that $e_{\mathcal{L}_n,U_n}$ reduces to that of the univariate case and, hence, the results given by Sen [9] apply.

Sen [9] studied the A.R.E. of Friedman's χ_r^2-test for the case $F_i(x)=F(x/\sigma_i)$, $i=1,2,\ldots,n$. These results may be extended to the multivariate case. We have $\lambda_{\mathcal{L}_n}^2 = \Sigma_{j=1}^k \underset{\sim}{\alpha}_j' \underset{\sim}{\Gamma}^{-1} \underset{\sim}{\alpha}_j$, where $\underset{\sim}{\Gamma} = ((\sigma_{[s,s']}/A^s A^{s'}))$ and, under $F_{i[s]}(x)=F_{[s]}(x/\lambda_{i[s]})$, we find that

$$\underset{\sim}{\Gamma} = ((\sigma_{[s,s']}[B^s B^{s'}(n^{-1}\Sigma_{i=1}^n(1/\lambda_{i[s]}))(n^{-1}\Sigma_{i=1}^n(1/\lambda_{i[s']}))]^{-1})),$$

where

$$B^s = k\int_{-\infty}^{\infty}[f_{[s]}(x)]^2dx \text{ and } \lambda_{i[s]}^2 = (\underset{\sim}{\Lambda}_i)s,s.$$

Now, if $\underset{\sim}{\Lambda}_i = c_i\underset{\sim}{\Lambda}$, $c_i > 0$, then $\lambda_{i[s]}^2 = c_i\lambda_s^2$ and $\bar{\underset{\sim}{\Lambda}}^{-1} = \bar{c}^{-1}\underset{\sim}{\Lambda}^{-1}$, where $\lambda_s^2 = (\underset{\sim}{\Lambda})_{s,s}$ and $\bar{c} = n^{-1}\Sigma_{i=1}^n c_i$. Thus,

$$e_{\mathcal{L}_n,U_n} = \bar{c}[n^{-1}\Sigma_{i=1}^n(1/\sqrt{c_i})]^2\Sigma_{j=1}^k \underset{\sim}{\alpha}_j'\underset{\sim}{\Gamma}^{*-1}\underset{\sim}{\alpha}_j/\Sigma_{j=1}^k\underset{\sim}{\alpha}_j'\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\alpha}_j,$$

where

$$\underset{\sim}{\Gamma}^* = ((\sigma_{[s,s']}\lambda_{[s]}\lambda_{[s']}/B^s B^{s'})).$$

But

$$n^{-1}\Sigma_{i=1}^n(1/\sqrt{c_i}) \geq [n^{-1}\Sigma_{i=1}^n \sqrt{c_i}]^{-1} \geq [n^{-1}\Sigma_{i=1}^n c_i]^{-\frac{1}{2}} = \sqrt{\bar{c}^{-1}}$$

and hence,

$$\bar{c}[n^{-1}\Sigma_{i=1}^n(1/\sqrt{c_i})]^2 \geq 1,$$

with equality holding only when $c_1=c_2=\ldots=c_n$. Thus, under the stated conditions, $e_{\mathcal{L}_n,U_n}$ is a minimum when $c_1=c_2=\ldots=c_n$ and can be made larger than one by a suitable choice of $c_1,c_2,\ldots,c_n$.

Turning attention to another aspect of interest, suppose we make the assumption of normality and equal block dispersion matrices, then for p=2 we have

$$e_{\mathfrak{L}_n, U_n} = \frac{3k(1-\rho^2)}{\pi(k+1)(1-\rho^{*2})}\left[\frac{\Sigma_{j=1}^{k}\{\delta_{j1}^2 + \delta_{j2}^2 - 2\delta_{j1}\delta_{j2}\rho^*\}}{\Sigma_{j=1}^{k}\{\delta_{j1}^2 + \delta_{j2}^2 - 2\delta_{j1}\delta_{j2}\rho\}}\right] ,$$

where

$$\delta_{js} = \alpha_j^s/\lambda_s, \quad \rho^* = \frac{6}{\pi(k+1)}\,[\sin^{-1}\rho + (k-2)\sin^{-1}\rho/2], \text{ and}$$

$$\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2\rho \\ \lambda_1\lambda_2\rho & \lambda_2^2 \end{bmatrix}$$

is the common dispersion matrix. It can be shown that the extreme values of $e_{\mathfrak{L}_n, U_n}$ are obtained when $\delta_{j1} = \delta_{j2}$ for j=1,2,...,k or when $\delta_{j1} = -\delta_{j2}$ for j=1,2,...,k. Thus

$$\max_{\substack{\delta_{j1},\delta_{j2} \\ j=1,2,\ldots,k}} e_{\mathfrak{L}_n, U_n} = e_M(\rho,k) = \begin{cases} 3k(1+\rho)/\pi(k+1)(1+\rho^*), & \text{for } \rho \geq 0 \\ 3k(1-\rho)/\pi(k+1)(1-\rho^*), & \text{for } \rho \leq 0, \end{cases}$$

and

$$\min_{\substack{\delta_{j1},\delta_{j2} \\ j=1,2,\ldots,k}} e_{\mathfrak{L}_n, U_n} = e_m(\rho,k) = \begin{cases} 3k(1-\rho)/\pi(k+1)(1-\rho^*), & \text{for } \rho \geq 0 \\ 3k(1+\rho)/\pi(k+1)(1+\rho^*), & \text{for } \rho \leq 0. \end{cases}$$

It is easily seen that, for fixed k, $e_M(\rho,k)$ is symmetric about $\rho=0$ and is concave for $-1 \leq \rho \leq 0$ and $0 \leq \rho \leq 1$. It can be shown that $3k/\pi(k+1) \leq e_M(\rho,k) \leq .966$. The lower bound is obtained when $|\rho|\to 1$ or $\rho=0$. Using the inequality $-1 \leq 3\tau-2\rho_g \leq 1$, given in Kendall [6, p. 12], it can be shown that $e_M(\rho,k)$ is monotonic increasing in k. In view of this, and after some numerical calculation, we find that $\max_{\rho} \lim_{k\to\infty} e_M(\rho,k) = .966$ and occurs when $\rho=.52$. The following table gives the quantity $\max_{\rho} e_M(\rho,k)$ for some values of k and gives the corresponding A.R.E. for the univariate χ_r^2-test.

k	2	3	4	5	6	7	10	20	100	∞
$\max_{\rho} e_M(\rho,k)$	.725	.790	.828	.852	.869	.881	.905	.934	.960	.966
ρ	.69	.68	.67	.67	.66	.66	.64	.61	.54	.52
$e_{\chi_r^2,F}$	.637	.716	.764	.796	.819	.836	.868	.909	.945	.955

It is worth noting that the A.R.E. for χ_r^2 is smaller than that for $\mathcal{L}_n$, under these assumptions.

Turning our attention to $e_m(\rho,k)$, we see that, as a function of ρ, it is symmetric about $\rho=0$. Also, we see that $e_m(\rho,k)$ is monotinically decreasing in $|\rho|$ and increasing in k. Finally, we find, by direct analysis that $0 \leq e_m(\rho,k) \leq 3k/\pi(k+1)$ and that $\sqrt{3}/4 \leq \lim_{k\to\infty} e_m(\rho,k) \leq 3/\pi$. The lower bounds are obtained when $|\rho|\to 1$ and the upper when $\rho=0$. In practice, having $|\rho|$ near unity will lead to degenerate test statistics. Hence, more meaningful bounds can be obtained by restricting the range of ρ. Chatterjee [3] has studied the case p=k=2 so we will confine our attention to $k \geq 3$. The following table gives values of ρ_o and $m(\rho_o)$ where $m(\rho_o) \leq e_m(\rho,k) \leq 3k/\pi(k+1)$, when $0 \leq |\rho| \leq \rho_o$ and $k \geq 3$.

ρ_o	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$m(\rho_o)$	.69	.67	.64	.61	.57	.52	.47	.40	.30	.00

These results follow from the stated monotonicity properties and some direct calculations.

Results similar to these have been obtained by Bickel [2].

Finally, if we assume that $\underset{\sim}{x}_{ij}$ follows a p-variate normal distribution with dispersion matrix

$$\underset{\sim}{\Lambda} = \lambda^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$$

then $e_{\mathcal{L}_n, U_n} = 3(1-\rho)/\pi(k+1)(1-\rho^*)$ and hence the bounds developed above may be applied.

8. NUMERICAL EXAMPLE

To illustrate the use of the test, bivariate normal data was generated and recorded in a two-way table with k=3 and n=10. Then 1 was added to each observation (both variables) in the second group and 2 was added to each observation in the third group. The results appear in the following table.

Groups

	1		2		3	
	Obs.	Rks.	Obs.	Rks.	Obs.	Rks.
1	0.547	1	1.811	2	2.561	3
	-0.575	1	1.840	2	2.399	3
2	1.706	2	2.509	3	1.414	1
	1.252	1	1.574	2	3.059	3
3	-0.288	1	2.524	2	3.310	3
	-0.310	1	1.553	3	0.560	2
4	1.417	3	0.703	1	0.961	2
	0.932	1	1.390	2	3.083	3
5	0.878	2	0.094	1	1.682	3
	0.819	2	0.045	1	3.348	3
6	-0.680	1	2.077	2	3.181	3
	0.497	1	1.747	3	1.355	2
7	0.056	1	0.542	2	2.983	3
	-0.285	1	0.760	2	2.332	3
8	0.711	2	0.269	1	1.662	3
	0.089	1	1.076	3	0.960	2
9	-1.335	1	1.545	2	2.920	3
	-0.349	1	1.471	2	4.121	3
10	1.635	2	0.200	1	2.065	3
	0.845	1	1.480	2	3.391	3

The parametric analysis gave $U_n = |S_E|/|S_E+S_H| = 0.3083$ and it is known that $[(1-U_n^{\frac{1}{2}})/U_n^{\frac{1}{2}}][(c_1-1)/c_2] = F_o$, where $c_1 = (n-1)(k-1)$ and $c_2 = (k-1)$, is distributed as $F_{2c_2,2(c_1-1)}$ when p=2. Thus, we find that $F_o = 6.807$ and is significant at the .0005 level.

To compute $\mathcal{L}_n$, we have

$$\Sigma(R^*) = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}, \Sigma^{-1}(R^*) = \begin{bmatrix} 1.1905 & -0.4762 \\ -0.4762 & 1.1905 \end{bmatrix}$$

and so $\mathcal{L}_n = 10[(1.1905)(2.08)-(2)(0.4762)(0.79)] = 17.238$. But $\mathcal{L}_n \sim \chi^2_{p(k-1)}$ so that $\mathcal{L}_n$ is significant at the .005 level.

Next, using the same bivariate data, the observations in each block were multiplied by a constant. In fact, the multiplier for block 1 was 1, for block 2 was 1.2, for block 3 was 1.4, and so on. Then constants were added to the observations of groups 2 and 3 as before. In this case, the parametric analysis gave $U_n = 0.4854$ and $F_o = 3.700$ which is significant at the 0.25 level, whereas the non-parametric analysis gave $\mathcal{L}_n = 16.044$ which is significant at the .005 level.

These figures verify the results obtained in Section 7 pertaining to the A.R.E. when $F_{i[s]}(x) = F_{[s]}(x/\lambda_{i[s]})$ and $\underset{\sim}{\Lambda}_i = c_i \underset{\sim}{\Lambda}$.

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