

ON THE RANK OF INCIDENCE MATRICES
IN FINITE GEOMETRIES

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Abstract

The incidence matrix N of points and d -flats in $PG(t, q)$ is a $(0,1)$ -matrix which may be considered a matrix with entries from the field $GF(q)$, where q is a prime power, $q = p^n$. Let $r_d(t, q)$ denote the rank of N over $GF(q)$. In this paper, a formula for $r_{t-1}(t, q)$ and an upper bound for $r_d(t, q)$, $1 \leq d \leq t-1$, are given. In the case $q = p$, the bound on $r_d(t, p)$ is attained. Similar results are given for the rank of the incidence matrix of points and d -flats not passing through the origin in $EG(t, q)$. Proofs of the results stated in this paper are omitted and are given in the author's Ph. D. dissertation.

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1. Introduction

Much attention has been given recently to the application of incidence matrices of incomplete block designs to the construction of q -ary linear error-correcting codes. Of particular importance in this connection is the determination of the rank of the incidence matrix of the design over the field $\text{GF}(q)$. In this paper, we shall state some results on the rank of the incidence matrix of a configuration of points and d -flats in the finite projective and affine geometries $\text{PG}(t, q)$ and $\text{EG}(t, q)$. The proofs of these results and their applications to the theory of error-correcting codes are given in [5].

2. Points and d -flats in $\text{PG}(t, q)$

The $v = (q^{t+1} - 1)/(q - 1)$ points in $\text{PG}(t, q)$ may be represented by the non-zero elements of $\text{GF}(q^{t+1})$. Here, q is a prime power, say $q = p^n$. Let γ be a primitive element of $\text{GF}(q^{t+1})$. The elements γ^{u_1} and γ^{u_2} represent the same point if and only if $u_1 \equiv u_2 \pmod{v}$. We denote the point represented by γ^u by P_u , $u = 0, 1, \dots, v-1$.

For $1 \leq d \leq t-1$, let $\gamma^{e_0}, \gamma^{e_1}, \dots, \gamma^{e_d}$ be linearly independent over $\text{GF}(q)$. A d -flat consists of those points represented by the elements

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γ^u , where

$$\gamma^u = a_0 \gamma^{e_0} + a_1 \gamma^{e_1} + \dots + a_d \gamma^{e_d},$$

and where a_0, a_1, \dots, a_d run independently over the elements of $\text{GF}(q)$ and are not simultaneously zero. Regarding $\text{GF}(q^{t+1})$ as a $(t+1)$ -dimensional vector space over $\text{GF}(q)$, a d -flat corresponds to a $(d+1)$ -dimensional subspace of $\text{GF}(q^{t+1})$ with the null vector omitted. There are $k = \phi(d, 0, q)$ points on a d -flat and the number of d -flats in $\text{PG}(t, q)$ is $b = \phi(t, d, q)$, where

$$(2.2) \quad \phi(N, m, q) = \frac{(q^{N+1}-1)(q^N-1)\dots(q^{N-m+1}-1)}{(q^{m+1}-1)(q^m-1)\dots(q-1)}.$$

For $d=t-1$, each t -dimensional subspace of $\text{GF}(q^{t+1})$ is the kernel of a non-zero linear functional from $\text{GF}(q^{t+1})$ onto $\text{GF}(q)$, which we may express in terms of the trace from $\text{GF}(q^{t+1})$ to $\text{GF}(q)$. To each $(t-1)$ -flat there corresponds a non-zero element μ of $\text{GF}(q^{t+1})$ such that the $(t-1)$ -flat is the set of points represented by those elements γ^u such that

$$(2.3) \quad \text{T}(\mu \gamma^u) = 0,$$

where

$$(2.4) \quad \text{T}(x) = x + x^q + \dots + x^{q^t}.$$

Let Σ be a given d -flat, where $1 \leq d \leq t-1$. Suppose the k points on Σ are $P_{u_1}, P_{u_2}, \dots, P_{u_k}$. We may assume $0 \leq u_1 < u_2 < \dots < u_k \leq v-1$. We define the incidence polynomial of the d -flat Σ as the polynomial

$I_\Sigma(x)$, given by

$$(2.5) \quad I_\Sigma(x) = x^{u_1} + x^{u_2} + \dots + x^{u_k}.$$

Suppose we order the d -flats in some manner and denote them by $\Sigma_0, \Sigma_1, \dots, \Sigma_{b-1}$. We define the incidence matrix N of points and d -flats in $\text{PG}(t, q)$ as the $b \times v$ matrix given by $N = (n_{ij})$, where

$$(2.6) \quad n_{ij} = \begin{cases} 1, & \text{if the point } P_j \text{ is incident with the flat } \Sigma_i; \\ 0, & \text{otherwise, } i=0,1,\dots,b-1; j=0,1,\dots,v-1. \end{cases}$$

We may regard N as a $(0,1)$ -matrix with entries in $\text{GF}(q)$.

From equations (2.4) and (2.5), we have

$$I_{\Sigma_i}(x) = \sum_{j=0}^{v-1} n_{ij} x^j, \quad i=0,1,\dots,b-1.$$

The polynomial $I_{\Sigma_i}(x)$ will be regarded as a polynomial in $\text{GF}(q)[x]$.

Let $H_d(t,q)$ be the number of integers m , $1 \leq m \leq v-1$, such that $I_{\Sigma}(\beta^m) = 0$ for every d -flat Σ , where $\beta = \gamma^{q-1}$ and γ is a primitive element of $\text{GF}(q^{t+1})$.

Theorem 2.1 [5] Over $\text{GF}(q)$, the rank of the incidence matrix N of points and d -flats in $\text{PG}(t,q)$ is equal to $v - H_d(t,q)$.

For $d=t-1$, it is shown in [5] that for any $(t-1)$ -flat Σ and for $1 \leq m \leq v-1$, $I_{\Sigma}(\beta^m) = 0$ if and only if $G(\gamma^{-m(q-1)}) = 0$, where

$$G(x) = \sum_{j=0}^{t+1} \left\{ 1 - [T(\gamma^j)]^{q-1} \right\} x^j.$$

Expanding $1 - [T(x)]^{q-1} = 1 - (x + x^q + \dots + x^{q^t})^{q-1}$ as a polynomial in x of degree less than $q^{t+1}-1$, it can be shown that for $0 \leq u \leq q^{t+1}-2$, $G(\gamma^{-u}) \neq 0$ if and only if, in the p -adic representation of u as

$$u = c_0 + c_1 p + \dots + c_{n-1+tn} p^{n-1+tn},$$

the equations

$$c_j + c_{j+n} + \dots + c_{j+tn} = p-1$$

hold for each $j = 0, 1, \dots, n-1$. From this it follows that

$$H_{t-1}(t,q) = v - 1 - \binom{p+t-1}{t}^n.$$

Combining this with Theorem 2.1, we have

Theorem 2.2 [5] The rank of the incidence matrix N of points and $(t-1)$ -flats in $\text{PG}(t,q)$, where $q=p^n$, is equal to

$$\binom{p+t-1}{t}^n + 1.$$

This result was proved by Graham and MacWilliams [2] in the case $t=2$. For general t , it was conjectured by Rudolph [4] and was proved independently by Goethals and Delsarte [1] and by Smith [5], each using different methods. A further proof of this result using the theory of group characters has recently appeared by MacWilliams and Mann [3].

For the case $1 \leq d \leq t-1$, it is shown in [5] that, for $1 \leq m \leq v-1$, $I_{\Sigma}(\beta^m) = 0$ for every d -flat Σ if and only if, for some $j=0,1,\dots,n-1$,

$$D_q(p^j m(q-1)) = s(q-1), \quad 1 \leq s \leq d,$$

where $D_q(u)$ denoted the sum of the digits in the q -adic representation of u . That is, if $u = A_0 + A_1 q + \dots + A_t q^t$, where $0 \leq A_i \leq q-1$, $i=0,1,\dots,t$, then $D_q(u) = A_0 + A_1 + \dots + A_t$. This result is obtained in [5] by an extension of the method used by Graham and MacWilliams [2] for the case $t=2$.

Define the functions $L_s(q)$, $B_s(t,q)$ by

$$L_s(q) = \left[\frac{s(q-1)}{q} \right],$$

$$B_s(t,q) = \sum_{i=0}^{L_s(q)} (-1)^i \binom{t+1}{i} \binom{t+s(q-1)-iq}{t},$$

$$R_d(t,q) = \sum_{s=1}^d B_s(t,q).$$

It is shown that $R_d(t,q)$ is the number of integers m , $1 \leq m \leq v-1$, such that

$$D_q(m(q-1)) = s(q-1), \quad 1 \leq s \leq d.$$

Hence

$$H_d(t,q) \geq R_d(t,q),$$

and in the case $q = p$,

$$H_d(t,p) = R_d(t,p).$$

¹⁾ The notation $[x]$ denotes the greatest integer less than or equal to x .

We have the following theorem.

Theorem 2.3 [5] With $R_d(t, q)$ defined as above, the rank over $GF(q)$ of the incidence matrix N of points and d -flats in $PG(t, q)$ is at most equal to $v - R_d(t, q)$. For $q = p$, the rank of N is equal to $v - R_d(t, p)$.

3. Points and d -flats in $EG(t, q)$

The q^t points in $EG(t, q)$ may be represented by elements of $GF(q^t)$; two distinct elements represent different points. We shall refer to the point represented by the zero element as the origin. Suppose γ is a primitive element of $GF(q^t)$. Denote the point represented by γ^u by P_u , $u=0, 1, \dots, q^t-2$.

For $1 \leq d \leq t-1$, let $\gamma^{e_1}, \gamma^{e_2}, \dots, \gamma^{e_d}$ be d elements of $GF(q^t)$ which are linearly independent over $GF(q)$. The set of points represented by the elements

$$a_1 \gamma^{e_1} + a_2 \gamma^{e_2} + \dots + a_d \gamma^{e_d}$$

as a_1, a_2, \dots, a_d run independently over the elements of $GF(q)$ constitute a d -flat in $EG(t, q)$ passing through the origin. Denote this d -flat by Σ . If γ^c does not represent a point on Σ , the set of points represented by the elements

$$\gamma^c + a_1 \gamma^{e_1} + \dots + a_d \gamma^{e_d}$$

constitute a d -flat passing through the point P_c and belonging to the same parallel bundle as Σ . Such a d -flat does not pass through the origin.

Because the structure of d -flats in $EG(t, q)$ which pass through the origin is essentially that of $(d-1)$ -flats in $PG(t-1, q)$, we shall henceforth consider only d -flats which do not pass through the origin. Also, unless otherwise stated, we shall not consider the origin as a point. There are $b' = (q^{t-d} - 1) \phi(t-1, d-1, q)$ d -flats in $EG(t, q)$ which do not pass through

the origin; each d-flat contains $k' = q^d$ points.

As in Section 2, we may consider a $(t-1)$ -flat in $EG(t, q)$ as the set of points represented by those elements γ^u such that

$$T(\mu\gamma^u) = 1,$$

where μ is a non-zero element of $GF(q^t)$ and $T(x) = x + x^q + \dots + x^{q^{t-1}}$.

For a given d-flat Σ , $1 \leq d \leq t-1$, suppose the k' points on Σ are

$P_{u_1}, P_{u_2}, \dots, P_{u_{k'}}$, where we may assume $0 \leq u_1 < u_2 < \dots < u_{k'} \leq q^t - 2$.

We define the incidence polynomial of Σ as the polynomial

$$I_{\Sigma}(x) = x^{u_1} + x^{u_2} + \dots + x^{u_{k'}}.$$

After ordering the b' d-flats in some manner, we define the incidence matrix N of points and d-flats as the matrix $N = (n_{ij})$, where

$$n_{ij} = \begin{cases} 1, & \text{if the point } P_j \text{ is incident with the flat } \Sigma_i; \\ 0, & \text{otherwise, } i=0,1,\dots,b'-1; j=0,1,\dots,q^t-2. \end{cases}$$

Again,

$$I_{\Sigma_i}(x) = \sum_{j=0}^{q^t-2} n_{ij} x^j, \quad i=0,1,\dots,b'-1.$$

Regarding N as a matrix with entries in $GF(q)$ and $I_{\Sigma_i}(x)$ as a polynomial in $GF(q)[x]$, let us define $K_d(t, q)$ as the number of integers u , $0 \leq u \leq q^t - 2$ such that $I_{\Sigma}(\gamma^u) = 0$ for every d-flat Σ in $EG(t, q)$ which does not pass through the origin.

Theorem 3.1 [5] Over $GF(q)$, the rank of the incidence matrix N of points and d-flats not passing through the origin in $EG(t, q)$ is equal to $q^t - 1 - K_d(t, q)$.

Using methods analogous to those used in Section 2, it is shown in [5] that

$$K_{t-1}(t, q) = q^t - \binom{p+t-1}{t}^n$$

and for $1 \leq d \leq t-1$,

$$K_d(t, q) \geq R_d(t, q) - R_d(t-1, q)$$

with equality holding for $q = p$, where $R_d(t, q)$ is defined in Section 2.

We state this as a theorem.

Theorem 3.2 [5] Over $\text{GF}(q)$, $q = p^n$ the rank of the incidence matrix of points and d -flats not passing through the origin in $\text{EG}(t, q)$ is

(i) for $d = t-1$, equal to

$$\binom{p+t-1}{t}^n - 1,$$

(ii) for $1 \leq d \leq t-1$, at most equal to

$$q^t - 1 - R_d(t, q) + R_d(t-1, q),$$

with equality holding for the case $q = p$.

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