ASYMPTOTIC EXPANSIONS OF THE POWER FUNCTIONS OF THE LIKELIHOOD RATIO TESTS FOR MULTIVARIATE LINEAR HYPOTHESIS AND INDEPENDENCE

by

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Institute of Statistics Mimeo Series No. 563

January, 1968

This research was supported by the National Science Foundation Grant No. GU-2059 and the Sakko-kai Foundation.

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SUMMARY. Asymptotic non-null distribution of the likelihood ratio criterion for testing the linear hypothesis in multivariate analysis is obtained up to the order N^{-1}, where N means the sample size, by using the characteristic function expressed by the hypergeometric function of matrix argument. This result holds without any assumption on the rank of noncentrality matrix. Asymptotic non-null distribution of the likelihood ratio test for independence between two sets of variates is also obtained up to the order $N^{-\frac{1}{2}}$.

1. Expansion of the test criterion for the multivariate linear hypothesis. Let each column vector of $p \times N$ matrix X be distributed independently according to p-variate normal distribution with the common covariance matrix Σ and $E(X) = \Theta A$, where A is a known $s \times N$ matrix of rank s and Θ is unknown $p \times s$ matrix. Then the multivariate linear hypothesis is defined by asking, under this model, whether the parameters Θ satisfies the hypothesis H: $\Theta B=0$, where B is a known $s \times b$ matrix ($b \leq s$) and ΘB is assumed to be estimable. By making an appropriate orthogonal transformation from X to Y by $Y = XT$, we can obtain the canonical form of the linear hypothesis. Each column vector of $Y = (Y_1, \ldots, Y_N)$ is distributed independently according to the normal distribution with the common covariance matrix Σ. The hypothesis H and alternatives K are specified by

$$H\colon E(Y_j) = 0 \quad j=1,2,\ldots,b$$

(1.1)

$$K\colon E(Y_j) = 0 \quad j=s+1,\ldots,N.$$

The likelihood ratio test for this problem is expressed by

(1.2) $$\lambda = \{|S_e| \;/\; |S_e+S_h|\}^{N/2}$$

where $S_e = \Sigma_{\alpha=s+1}^{N} Y_\alpha Y'_\alpha$ and $S_h = \Sigma_{\alpha=1}^{b} Y_\alpha Y'_\alpha$. The matrix S_e is the sum of square due to error and the matrix S_h is the sum of square due to the hypothesis. Hence under the alternative K, S_e is distributed according to the Wishart distribution with N-s degrees of freedom and S_h is distributed according to the noncentral Wishart distribution with b degrees of freedom and noncentrality matrix of $\Omega = (\frac{1}{2})\Lambda\Lambda'\Sigma^{-1}$ where $\Lambda = E(Y_1, \ldots, Y_b)$, as in Constantine [4].

Posten and Bargmann [8] obtained the asymptotic expansion of the power function $P(-2\rho \log \lambda < x)$ up to the order N^{-2} under the assumption that the noncentrality matrix Ω is of rank 2, where ρ is a correction factor such that under H the first remainder term of lowest degree will disappear if we approximate the statistic $-2\rho \log \lambda$ by χ^2 variate with bp degrees of freedom, that is, ρ is determined by $\rho N = N-s + (b-p-1)/2$ (Anderson [1, p. 208]).

On the other hand Constantine [4] showed that the hth moment of the ratio of the determinants $|S_e| \;/\; |S_e+S_h|$ under K could be expressed by the hypergeometric function of matrix argument. His result can be expressed by our notation as follows.

(1.3) $$E\left[\left\{\frac{|S_e|}{|S_e+S_h|}\right\}^h\right] = \frac{\Gamma_p(h+\frac{N-s}{2})\Gamma_p(\frac{N-s+b}{2})}{\Gamma_p(\frac{N-s}{2})\Gamma_p(h+\frac{N-s+b}{2})} \; {}_1F_1(h;\, h+\frac{N-s+b}{2};\, -\Omega),$$

where $\Gamma_p(x)$ and the hypergeometric function ${}_1F_1$ are defined by

(1.4) $$\Gamma_p(x) = \pi^{p(p-1)/4} \,\Pi_{\alpha=1}^{p}\, \Gamma(x-(\alpha-1)/2)$$

$$ {}_1F_1(a;b;Z) = \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(a)_\kappa}{(b)_\kappa} \frac{C_\kappa(Z)}{k!} $$

$$ (a)_\kappa = \Pi_{\alpha=1}^{p} a(a+1)\dots(a+k_\alpha-1). $$

The function $C_\chi(Z)$ is called a zonal polynomial of symmetric matrix Z corresponding to the partition $\kappa = \{k_1, k_2, \dots, k_p\}, k_1+k_2+\dots+k_p=k$, $k_i \geq 0$, $i = 1, \dots, p$ and $\sum_{(\kappa)}$ means the sum of all such partitions. It is a kth degree homogeneous symmetric polynomial of p characteristic roots of Z.

By using this result we shall show the asymptotic expansion of $P_K(-2\rho \log \lambda < x)$ up to the order N^{-1} without restricting the rank of Ω. We further require some formula for zonal polynomials in Constantine [4].

$$ (1.5) \qquad |I-Z|^{-a} = \sum_{k=0}^{\infty} \sum_{(\kappa)} (a)_\kappa \; C_\kappa(Z) / k! $$

$$ (1.6) \qquad {}_0F_0(Z) = \sum_{k=0}^{\infty} \sum_{(\kappa)} C_\kappa(Z) / k! $$

$$ = \operatorname{etr} Z. $$

Put $m = \rho N = N-s+(b-p-1)/2$ and let m tend to infinity instead of N as in Posten and Bargmann [8], we can express the characteristic function of $-2\rho \log \lambda$ as

$$ C(t) = E\left[e^{-2it\,\rho \log \lambda}\right] $$
$$ = E\left[|S_e|^{-itm} / |S_e+S_h|^{-itm}\right] $$

$$(1.7) = \frac{\Gamma_p(\frac{m}{2}(1-2it) - \frac{b-p-1}{4})\, \Gamma_p(\frac{m}{2} + \frac{b+p+1}{4})}{\Gamma_p(\frac{m}{2} - \frac{b-p-1}{4})\, \Gamma_p(\frac{m}{2}(1-2it) + \frac{b+p+1}{4})} \; {}_1F_1(-itm;\ \tfrac{m}{2}(1-2it) + \tfrac{b+p+1}{4}; -\Omega)$$

$$= C_1(t)\ C_2(t).$$

Under the hypothesis H, the noncentrality matrix $\Omega = 0$ and the hypergeometric function ${}_1F_1$ is equal to unity. So the first four Γ functions give us the characteristic function under H, which we shall denote by $C_1(t)$ and ${}_1F_1$ by $C_2(t)$. $C_1(t)$ can be expanded by the usual manner due to Box [3] (Anderson [1, p. 204]). Applying the formula

$$(1.8)\ \log \Gamma(x+h) = \log \sqrt{2\pi} + (x+h-\tfrac{1}{2}) \log x - x - \sum_{r=1}^{m} \frac{(-1)^r B_{r+1}(h)}{r(r+1)x^{r+1}} + O(|x|^{-m-1})$$

which holds for large $|x|$ and fixed h with the Bernoulli polynomial $B_r(h)$ of defree r, $B_2(h) = h^2 - h + (1/6)$, to $C_1(t)$, we have

$$\log C_1(t) = -\frac{bp}{2} \log (1-2it) - \frac{1}{m} \sum_{\alpha=1}^{p} \left\{ B_2\left(\frac{b+p+3-2\alpha}{4}\right) - B_2\left(-\frac{b-p-3+2\alpha}{4}\right) \right\} \frac{2it}{1-2it} + O(m^{-2}).$$

The second term of the above expression vanishes to get

$$(1.9)\quad C_1(t) = (1-2it)^{-bp/2}\ [1 + O\ (m^{-2})].$$

This is the reason why we use the correction factor ρ. Now we shall consider the second term $C_2(t)$ of (1.7).

$$(1.10)\quad C_2(t) = \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(-imt)_\kappa\ C_\kappa(-\Omega)}{([m(1-2it)]/2 + (b+p+1)/4)_\kappa \cdot k!} \ .$$

This follows from the definition (1.4). The coefficient of each term can be arranged by the order of m.

$$(1.11)\quad (-itm)_{\kappa} = \prod_{\alpha=1}^{p} (-itm - \frac{\alpha-1}{2})(-itm - \frac{\alpha-1}{2} + 1)\ldots(-itm - \frac{\alpha-1}{2} + k_{\alpha} - 1)$$

$$= \prod_{\alpha=1}^{p} (-itm)^{k_{\alpha}} \left[1 + \frac{1}{itm} \left\{ \frac{\alpha-1}{2} k_{\alpha} - \frac{k_{\alpha}(k_{\alpha}-1)}{2} \right\} + o(\frac{1}{m^2}) \right]$$

$$= (-itm)^{k} \left[1 + \frac{1}{2itm} \sum_{\alpha=1}^{p} k_{\alpha}(\alpha - k_{\alpha}) + o(\frac{1}{m^2}) \right]$$

$$(1.12)\quad ([m(1-2it)/2] + (b+p+1)/4)_{\kappa}$$

$$= \left\{ \frac{m(1-2it)}{2} \right\}^{k} \left[1 + \frac{2}{m(1-2it)} \left\{ \frac{b+p+1}{4} k - \sum_{\alpha=1}^{p} \frac{k_{\alpha}(\alpha-k_{\alpha})}{2} + o(\frac{1}{m^2}) \right\} \right] .$$

Hence we can write $C_2(t)$ as

$$(1.13)\quad \sum_{k=0}^{\infty} \sum_{(\kappa)} (\frac{-2it}{1-2it})^{k} \left[1 + \frac{1}{m} \left\{ \frac{(b+p+1)k}{2(1-2it)} + \frac{1}{(1-2it)2it} \sum_{\alpha=1}^{p} k_{\alpha}(\alpha - k_{\alpha}) \right\} + 0\ (m^{-2}) \right] \{C_{\kappa}(-\Omega)/k!\} .$$

Now we shall evaluate the each term of the above infinite series. By (1.6) we can see $\exp(x \operatorname{tr} Z) = \sum_{k=0}^{\infty} \sum_{(\kappa)} x^k C_{\kappa}(Z)/k!$, differentiating this equality with respect to x to get

$$(1.14)\qquad (\operatorname{tr} Z) \operatorname{etr} Z = \sum_{k=1}^{\infty} \sum_{(\kappa)} C_{\kappa}(Z)/(k-1)! .$$

This formula is used by Fujikoshi [5], in deriving the asymptotic expansion of generalized variance under the noncentral case. By (1.5) we have

$$(1.15)\quad |I - n^{-1}Z|^{-na} = \sum_{k=0}^{\infty} \sum_{(\kappa)} (na)_\kappa\, C_\kappa(Z)/(k!\, n^k)$$

$$= \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{a^k C_\kappa(Z)}{k!} \left[1 + \frac{1}{na} \sum_{\alpha=1}^{p} \frac{k_\alpha(k_\alpha - \alpha)}{2} + O(\frac{1}{n^2}) \right],$$

which holds for any number n and a, Z is a $p \times p$ positive definite matrix such that all characteristic root of Z/n lies in the interval (0,1). This is satisfied, if we consider n morderately large. The left handside can be expanded asymptotically in another way by the formula $-\log |I - n^{-1}Z| = tr(Z/n) + tr(Z/n)^2/2 + O(n^{-3})$.

$$(1.16)\quad |I - n^{-1}Z|^{-na} = \exp\{-na \log |I - n^{-1}Z|\}$$
$$= (\mathrm{etr}\ a\ Z)\ \{1 + (a/2n)tr\ Z^2 + O(n^{-2})\}.$$

Comparing the coefficients of each term of the order n^{-1} in (1.15) and (1.16), we can get

$$(1.17) \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{a^k C_\kappa(Z)}{k!} \left\{ \sum_{\alpha=1}^{p} k_\alpha(k_\alpha - \alpha) \right\} = a^2 tr\ Z^2 (\mathrm{etr}\ a\ Z).$$

Applying the formula (1.14) and (1.17) to the expression of $C_2(t)$ in (1.13), we have

$$(1.18)\quad C_2(t) = \exp [\{2it / (1-2it)\}\ tr\ \Omega]$$
$$\times \left[1 - \frac{1}{m} \left\{ \frac{(b+p+1)it}{(1-2it)^2}\ tr\ \Omega + \frac{2it}{(1-2it)^3}\ tr\ \Omega^2 \right\} + O(\frac{1}{m^2}) \right].$$

Combining this with the expression of $C_1(t)$ in (1.9), we can obtain asymptotic expansion of the characteristic function C(t) in terms of noncentral χ^2 distribution.

$$(1.19)\quad C(t) = (1-2it)^{-bp/2} \exp[\{2it/(1-2it)\} \operatorname{tr} \Omega]$$

$$\times \left[1 + \frac{1}{m} \left\{ \frac{(b+p+1)\operatorname{tr}\Omega}{2(1-2it)} - \frac{1}{(1-2it)^2}\left(\frac{b+p+1}{2} \operatorname{tr}\Omega - \operatorname{tr}\Omega^2\right) - \frac{\operatorname{tr}\Omega^2}{(1-2it)^3} \right\} + O\left(\frac{1}{m^2}\right)\right].$$

By inverting this characteristic function with the well known result that $(1-2it)^{-f/2} \exp[2it\,\delta^2/(1-2it)]$ is the characteristic function of the noncentral χ^2 distribution with f degrees of freedom and noncentrality parameter δ^2, we can finally obtain the following theorem.

Theorem 1.1. The non-null distribution of the likelihood ratio criterion (1.2) for multivariate linear hypothesis defined by (1.1) can be approximated asymptotically up to the order m^{-1} by

$$P(-2\rho \log \lambda < x) = P(\chi_f^2(\delta^2) < x) + \frac{1}{m}\left\{ \frac{b+p+1}{2} \operatorname{tr}\Omega\, P(\chi_{f+2}^2(\delta^2) < x) - \left(\frac{b+p+1}{2} \operatorname{tr}\Omega - \operatorname{tr}\Omega^2\right) P(\chi_{f+4}^2(\delta^2) < x) - \operatorname{tr}\Omega^2 P(\chi_{f+6}^2(\delta^2) < x)\right\} + O(m^{-2}),$$

where

$$m = \rho N = N - s + (b-p-1)/2,\quad f = bp/2,\quad \delta^2 = \operatorname{tr}\Omega = \operatorname{tr}\Lambda\Lambda'\Sigma^{-1}/2$$

and $\chi_f^2(\delta^2)$ means the noncentral χ^2 variate with f degrees of freedom and noncentrality parameter δ^2.

If we specialize the rank of Ω to two, we can easily recognize the agreement of the result by Posten and Bargmann [8] and ours, after minor change of notation.

2. Expansion of the test criterion for independence.

There is a close connection between multivariate linear hypothesis and the test for indepence between two sets of variates. Because we can reduce the test for independence to that of the linear hypothesis by taking conditional distribution of one set of variate for given another set. Monotonicity property of the power function of the test criteria was proved by this fact in Anderson and Gupta [2]. Thus we can expect that the same is true in asymptotic expansion.

Let $p \times 1$ vectors $X_1, \ldots, X_N$ be a random sample from multivariate normal distribution with mean vector μ and covariance matrix Σ. Put $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$, $\bar{X} = N^{-1} \sum_{\alpha=1}^{N} X_\alpha$ and let us partition Σ and S into p_1 and p_2 rows and columns $(p_1 + p_2 = p)$ as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} .$$

Without loss of generality we can assume $p_1 \leq p_2$. The likelihood ratio test for the hypothesis of independence $H : \Sigma_{12} = 0(p_1 \times p_2)$ against all alternatives $K : \Sigma_{12} \neq 0$ is given by

$$(2.1) \qquad \lambda = (|S| / |S_{11}| |S_{22}|)^{N/2} = |I - S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}|^{N/2}.$$

This can be also expressed as $\Pi_{j=1}^{p_1} (1 - r_j^2)^{N/2}$ by using the sample canonical correlations defined by the p_1 characteristic roots of $S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}$. First we shall show the moment of the statistic λ under K in the convenient form for the expansion.

Theorem 2.1. Under the alternative K, the moment of likelihood ratio statistic can be expressed as

$$(2.2)\quad E\left[\left(\frac{|S|}{|S_{11}||S_{22}|}\right)^h\right] = \frac{\Gamma_{p_1}(h + \frac{N-p_2-1}{2})\Gamma_{p_1}(\frac{N-1}{2})}{\Gamma_{p_1}(\frac{N-p_2-1}{2})\Gamma_{p_1}(h+ \frac{N-1}{2})}$$

$$\prod_{j=1}^{p_1} (1-\rho_j^2)^h \; {}_2F_1(h,h;h+ \tfrac{N-1}{2};P^2),$$

where $P^2 = \text{diag}\,(\rho_1^{\,2},\ldots,\rho_{p_1}^{\,2})$ is a $p_1 \times p_1$ diagonal matrix, diagonal element ρ_j^2 means the polulation canonical correlation and the hypergeometric function ${}_2F_1(a_1,a_2;b;Z)$ is defined by

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} (a_1)_\kappa (a_2)_\kappa \, C_\kappa(Z) \;/\; \{(b)_\kappa k!\}\,.$$

Proof. Considering the conditional distribution of the first p_1 components of the sample for given p_2 second components in the canonical set up of Σ, Constantine [4] showed that the joint distribution of $1 - r_1^{\,2},\ldots,\, 1 - r_{p_1}^{\,2}$ is the same as that of p_1 characteristic roots of $\mathcal{W}'(\mathcal{U}\mathcal{U}'+ \mathcal{W}\mathcal{W}')^{-1}$, where $p_1 \times p_1$ matrices $\mathcal{W}\mathcal{W}'$ and $\mathcal{U}\mathcal{U}'$ have the Wishart distribution for given $Y(p_2 \times (N-1))$ with common covariance matrix $\Gamma= \text{diag}\,(1-\rho_1^{\,2},\ldots,1-\rho_{p_1}^{\,2})$ such that $\mathcal{W}\mathcal{W}'$ is central with $N-p_2-1$ degrees of freedom, $\mathcal{U}\mathcal{U}'$ is noncentral with p_2 degrees of freedom and noncentrality matrix $\Omega = \Gamma^{-1}\Lambda YY'\Lambda'/2$ and that they are distributed independently for given $YY'(p_2 \times p_2)$, which has the Wishart distribution with N-1 degrees of freedom, covariance matrix I, and the $p_1 \times p_2$ matrix Λ is given by

$$\Lambda = \begin{pmatrix} \rho_1 & & 0 & \\ & \ddots & & \\ 0 & & \rho_{p_1} & 0 \end{pmatrix}.$$

Hence we can express the conditional moment of $|S|/(|S_{11}||S_{22}|)$ for given YY' from (1.3) by simple change of notation.

$$(2.3)\quad E\left[\,|VV'|^h/|UU'+VV'|^h\,|YY'\right]$$
$$= \frac{\Gamma_{p_1}\left(h+\frac{N-p_2-1}{2}\right)\Gamma_{p_1}\left(\frac{N-1}{2}\right)}{\Gamma_{p_1}\left(\frac{N-p_2-1}{2}\right)\Gamma_{p_1}\left(h+\frac{N-1}{2}\right)}\ {}_1F_1\left(h;h+\frac{N-1}{2};\ -\tfrac{1}{2}\Gamma^{-1}\Lambda YY'\Lambda'\right).$$

Applying Kummer transformation formula ${}_1F_1(a;b;Z)=(\mathrm{etr}\ Z)\ {}_1F_1(b-a;b;-Z)$ by Hertz [6] to the second expression and taking expectation by YY' with the Wishart distribution $W_{p_2}(N-1,I)$, we can write

$$E_{YY'}\left[{}_1F_1\left(h;h+\frac{N-1}{2};\ -\tfrac{1}{2}\Gamma^{-1}\Lambda YY'\Lambda'\right)\right] = \sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{((N-1)/2)_\kappa}{(h+(N-1)/2)_\kappa\ k!}$$

$$(2.4)\qquad \times\frac{1}{2^{(N-1)p_2/2}\Gamma_{p_2}\left(\frac{N-1}{2}\right)}\int_{YY'>0}\{\mathrm{etr}-(I+\Lambda'\Gamma^{-1}\Lambda)YY'/2\}|YY'|^{(N-p_2-2)/2}$$
$$C_\kappa\left(\tfrac{1}{2}\Lambda'\Gamma^{-1}\Lambda YY'\right)d(YY')\ .$$

By the formula $\int_{S>0}\{\mathrm{etr}(-RS)\}|S|^{t-(p+1)/2}C_\kappa(ST)dS=\Gamma_p(t)(t)_\kappa|R|^{-t}C_\kappa(TR^{-1})$, which holds for $p\times p$ positive definite matrix R, S and $p\times p$ any symmetric matrix T, established by Constantine [4], we have

$$(2.5)\quad \sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{((N-1)/2)_\kappa((N-1)/2)_\kappa\ \ C_\kappa(P^2)}{(h+(N-1)/2)_\kappa\ k!\ \ |I+\Lambda'\Gamma^{-1}\Lambda|^{(N-1)/2}}\ .$$

Applying again the Kummer transformation formula ${}_2F_1(a_1,a_2;b;Z) = |I-Z|^{b-a_1-a_2}\ {}_2F_1(b-a_1,b-a_2;b;Z)$ due to Hertz [6] and the identity $|I+\Lambda'\Gamma^{-1}\Lambda| = |I-P^2|^{-1}$ to the above expression, we have

$$(2.6)\quad \sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(h)_\kappa(h)_\kappa}{(h+(N-1)/2)_\kappa}\frac{C_\kappa(P^2)}{k!}\ \ |I-P^2|^h.$$

Combining this result with (2.3), we can get the theorem.

From this theorem, we shall expand the non-null distribution of the likelihood ratio statistic $-2 \log \lambda$ asymptotically. It is known by Olkin and Siotani [7] that the limiting distribution of $\sqrt{N}\{(|S|/|S_{11}||S_{22}|) - |\Sigma|/|\Sigma_{11}||\Sigma_{22}|\}$ is normal with mean 0 and variance $4(|\Sigma|/|\Sigma_{11}||\Sigma_{22}|)^2 \mathrm{tr}\, \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, so $-2N^{-\frac{1}{2}}(\log \lambda - \log \Pi_{j=1}^{p_1} (1-\rho_j^2)^{N/2})$ has the same limiting distribution with mean 0 and variance $4\sum_{j=1}^{p_1} \rho_j^2$. This situation is somewhat different from the previous section. Under the hypothesis the test statistic $-2\rho \log \lambda$ is recommended instead of $-2 \log \lambda$, where the correction factor ρ is determined so that the first remainder term disappears in the asymptotic expansion. We have $\rho N = N-(3/2)-(p_1+p_2)/2$ (Anderson [1, p. 239]).

Put $m = \rho N$ and let m tends to infinity instead of N. Then the characteristic function of the statistic $-2\rho m^{-\frac{1}{2}}(\log \lambda - \log \Pi_{j=1}^{p_1}(1-\rho_j^2)^{N/2})$ is obtained by putting $h = -it\sqrt{m}$ in (2.2).

$$(2.7)\quad C(t) = \frac{\Gamma_{p_1}(\frac{m}{2} - \sqrt{m}it + \frac{p_1-p_2+1}{4})\Gamma_{p_1}(\frac{m}{2} + \frac{p_1+p_2+1}{4})}{\Gamma_{p_1}(\frac{m}{2} + \frac{p_1-p_2+1}{4})\Gamma_{p_1}(\frac{m}{2} - \sqrt{m}it + \frac{p_1+p_2+1}{4})}.$$

$$\times \sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{(-\sqrt{m}it)_\kappa \quad (-\sqrt{m}it)_\kappa}{(\frac{m}{2} - \sqrt{m}it + \frac{p_1+p_2+1}{4})_\kappa} \cdot \frac{C_\kappa(P^2)}{k!}.$$

Applying the asymptotic formula for $\log \Gamma(x+h)$ in (1.8) to each of the four Γ functions, we get

(2.8) First Factor $= 1 + \frac{it}{\sqrt{m}} p_1 p_2 + O(\frac{1}{m^2})$.

By the same way as (1.11) and (1.12),

$$(-\sqrt{m}it)_\kappa = (-\sqrt{m}it)^k [1 + (\frac{1}{2\sqrt{m}it}) \sum_{\alpha=1}^{p_1} k_\alpha(\alpha - k_\alpha) + O(m^{-1})]$$

(2.9)

$$(\frac{m}{2} - \sqrt{m}it + \frac{p_1+p_2+1}{4})_\kappa = (\frac{m}{2})^k [1 - \frac{2it}{\sqrt{m}} k + O(\frac{1}{m})].$$

Hence we have $(-\sqrt{m}it)_\kappa(-\sqrt{m}it)_\kappa/((m/2) - \sqrt{m}it + (p_1+p_2+1)/4)_\kappa =$

$(-2t^2)^k [1 + m^{-\frac{1}{2}}\{2itk + (it)^{-1} \sum_{\alpha=1}^{p_1} k_\alpha(\alpha - k_\alpha)\} + O(m^{-1})]$, which gives,

with the formula (1.14) and (1.17), the second factor of (2.7).

(2.10) $${}_2F_1(-\sqrt{m}it, -\sqrt{m}it; (m/2) - \sqrt{m}it + (p_1+p_2+1)/4; P^2)$$
$$= \{\exp(-2t^2 \mathrm{tr}\, P^2)\} [1 + 4m^{-\frac{1}{2}}(it)^3 \{\mathrm{tr}\, P^2 - \mathrm{tr}\, P^4\} + O(m^{-1})].$$

Combining this expansion with (2.8), we have the following expansion of the characteristic function.

(2.11) $$C(t) = \{\exp(-2t^2 \mathrm{tr}\, P^2)\}[1 + \frac{1}{\sqrt{m}} \{it\, p_1 p_2 + 4(it)^3(\mathrm{tr}\, P^2 - \mathrm{tr}\, P^4)\} + O(m^{-1})],$$

which implies the following theorem.

Theorem 2.2. The power function of the likelihood ratio test for testing the independence between two sets of variates can be expanded asymptotically up to the order $m^{-\frac{1}{2}}$ in the following way. Put $\hat{\lambda} = -(m/\tau)\{\log |S|/(|S_{11}||S_{22}| - \log |\Sigma|/(|\Sigma_{11}||\Sigma_{22}|)\}$, where

$m = N - (3/2) - (p_1+p_2)/2$ and $\tau = 2\sqrt{\operatorname{tr} P^2} = 2(\sum_{j=1}^{p_1} \rho_j^2)^{\frac{1}{2}}$. Then we have

$$(2.12)\quad P(\tilde{\lambda} < x) = \Phi(x) - \frac{1}{\sqrt{m}}\left\{\frac{p_1 p_2}{\tau}\Phi'(x) + \frac{1}{\tau^3}\Phi'''(x)(\tau^2 - 4\sum_{j=1}^{p_1}\rho_j^4)\right\} + O(m^{-1}),$$

where $\Phi(x)$ means the distribution function of the standard normal distribution and $\Phi'(x)$, $\Phi'''(x)$ are its derivatives.

In case p = 2 the problem reduces to test that the correlation coefficient vanishes. It would be of interest to compare numerically with another formula in Ruben [9].

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