EXTENDED TABLES OF ZONAL POLYNOMIALS

by

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1. Introduction

Zonal polynomials are defined by James [2] in a manner similar to the following:

Let V_k be the vector space of homogeneous polynomials $\Phi(S)$ of degree k in the $\frac{1}{2}m(m + 1)$ different elements of a symmetrix $m \times m$ matrix S. Let G(m) be the group of non-singular $m \times m$ matrices. For $L \in G(m)$, define an induced linear transformation T_L of the polynomials $\Phi(S)$ by

$$T_L: \Phi(S) \to (L\Phi)(S) = \Phi(L^{-1} SL^{-1'})$$

A representation of G(m) is then given by the homomorphism T where $T:G(m) \to V_k$ and is given by $T(L) = T_L$. A formal definition of a representation may be found on page 312 of Hewitt and Ross [1].

A subspace $V' \subset V_k$ is called <u>invariant</u> if and only if $LV' \subset V'$ for all $L \in G(m)$. V' is an <u>irreducible invariant subspace</u> if and only if it has no proper invariant subspace.

Let P(k,m) be the set of partitions $p = (k_1, k_2, \ldots, k_q)$ of k into q non-zero parts $k = k_1 + k_2 + \ldots + k_q$ where $k_i \geq k_j$ for $i < j$ and where $q \leq m$.

Using theorem 3 of Thrall [3], James concludes that V_k can be decomposed into a direct sum of irreducible invariant subspaces $V_{k,p}$, one corresponding to each element of P(k,m), i.e.

$$V_k = \bigoplus_{p \in P(k,m)} V_{k,p}$$

The zonal polynomial $C_{k,p}(S)$ is then defined to be the component of $(\text{tr } S)^k$ which belongs to the subspace $V_{k,p}$, so that

$$(\text{tr } S)^k = \sum_{p \in P(k,m)} C_{k,p}(S).$$

Let us now define

$$d_{k,p} = \frac{(2k)! \prod_{i<j}^{q} (2k_i - 2k_j - i + j)}{\prod_{i=1}^{q} (2k_i + q - i)!}$$

and

$$Z_{k,p}(S) = \left[d_{k,p} \frac{2^k (k!)}{(2k)!} \right]^{-1} C_{k,p}(S) \qquad (1.1)$$

Then, the $Z_{k,p}(S)$ differ from $C_{k,p}(S)$ only by a multiplicative constant and are also called zonal polynomials. As is the case in this paper, the $Z_{k,p}$, which have leading coefficient unity, are the polynomials which are usually tabulated. They are usually written as functions of

$$s_j = \sum_{i=1}^{m} t_i^{\,j} \qquad (1.2)$$

where $\{t_i\}_{i=1}^{m}$ are the eigenroots of S.

2. Tables.

The following tables give the zonal polynomials $Z_{k,p}(S)$ defined in (1.1) for k = 1 through k = 9. The polynomials for each k are given in the table identified as "Table k." Each table presents the polynomial corresponding to the partition of k, listed at the left of the table, by means of coefficients of terms in s_j, listed at the top of the table, where s_j is defined by (1.2).

The tables for k = 1 through k = 6 are taken from James [2] and are included here for the sake of completeness. The tables for k = 7 through k = 9 were obtained using a computer program written by the author and based on the numerical technique outlined by James [2]. In addition, a procedure was included within the program to check the results for one of the 'orthogonality' conditions given by James. The program was tested by generating the polynomials for k = 2 through k = 6 and comparing the results with those of James. The polynomials for k = 7 through k = 9 were then generated.

TABLES OF ZONAL POLYNOMIALS

Table 1

Z	1
1	1

Table 2

Z	s_1^2	s_2
2	1	2
1^2	1	-1

Table 3

Z	s_1^3	s_1s_2	s_3
3	1	6	8
21	1	1	-2
1^3	1	-3	2

Table 4

Z	s_1^4	$s_1^2s_2$	s_2^2	s_1s_3	s_4
4	1	12	12	32	48
31	1	5	-2	4	-8
2^2	1	2	7	-8	-2
21^2	1	-1	-2	-2	4
1^4	1	-6	3	8	-6

Table 5

Z	s_1^5	$s_1^3s_2$	$s_1s_2^2$	$s_1^2s_3$	s_2s_3	s_1s_4	s_5
5	1	20	60	80	160	240	384
41	1	11	6	26	-20	24	-48
32	1	6	11	-4	20	-26	-8
31^2	1	3	-10	2	-4	-8	16
2^21	1	0	5	-10	-10	10	4
21^3	1	-4	-3	2	10	6	-12
1^5	1	-10	15	20	-20	-30	24

Table 6

Z	s_1^6	$s_1^4 s_2$	$s_1^2 s_2^2$	s_2^3	$s_1^3 s_3$	$s_1 s_2 s_3$	s_3^2	$s_1^2 s_4$	$s_2 s_4$	$s_1 s_5$	s_6
6	1	30	180	120	160	960	640	720	1440	2304	3840
51	1	19	48	-12	72	80	-64	192	-144	192	-384
42	1	12	27	30	16	24	-8	-18	108	-144	-48
41^2	1	9	-12	-12	22	-60	16	12	-24	-48	96
3^2	1	9	33	-27	-8	120	136	-78	-114	-48	-24
321	1	4	3	-2	-8	0	-24	-18	-4	32	16
31^3	1	0	-21	6	4	12	16	-6	12	24	-48
2^3	1	0	15	30	-20	-60	40	30	-60	24	0
$2^2 1^2$	1	-3	3	-9	-8	0	4	24	24	-24	-12
21^4	1	-8	3	6	12	20	-16	-6	-36	-24	48
1^6	1	-15	45	-15	40	-120	40	-90	90	144	-120

Table 7(a)

Z	s_1^7	$s_1^5 s_2$	$s_1^3 s_2^2$	$s_1 s_2^3$	$s_1^4 s_3$	$s_1^2 s_2 s_3$	$s_2^2 s_3$	$s_1 s_3^2$
7	1	42	420	840	280	3360	3360	4480
61	1	29	160	60	150	760	-280	320
52	1	20	79	114	60	148	368	-184
51^2	1	17	16	-84	66	-56	-136	-64
43	1	15	69	39	10	228	-42	376
421	1	10	9	14	10	-42	-52	-64
41^3	1	6	-39	-18	22	-78	84	112
$3^2 1$	1	7	21	-49	-14	84	14	56
32^2	1	4	15	50	-20	-60	80	-40
321^2	1	1	-9	-13	-8	12	-16	-28
31^4	1	-4	-29	42	12	52	-16	32
$2^3 1$	1	-3	15	15	-20	-60	-60	100
$2^2 1^3$	1	-7	7	-21	0	28	56	-28
21^5	1	-13	25	15	30	-20	-70	-40
1^7	1	-21	105	-105	70	-420	210	280

Table 7(b)

Z	$s_1^3 s_4$	$s_1 s_2 s_4$	$s_3 s_4$	$s_1^2 s_5$	$s_2 s_5$	$s_1 s_6$	s_7
7	1680	10080	13440	8064	16128	26880	46080
61	640	720	-1120	1824	-1344	1920	-3840
52	118	180	-112	-120	816	-1104	-384
51^2	160	-432	224	96	-192	-384	768
43	-102	90	588	-360	-504	-264	-144
421	-22	100	-112	-100	-24	176	96
41^3	6	-36	48	-36	72	144	-288
$3^2 1$	-70	-182	-196	56	168	56	48
32^2	-10	-140	80	104	-144	80	0
321^2	-4	64	68	44	12	-76	-48
31^4	-14	-36	-112	24	-48	-96	192
$2^3 1$	60	0	-60	-36	108	-60	0
$2^2 1^3$	28	0	-28	-84	-84	84	48
21^5	-50	-90	140	24	168	120	-240
1^7	-210	630	-420	504	-504	-840	720

Table 8(a)

Z	s_1^8	$s_1^6 s_2$	$s_1^4 s_2^2$	$s_1^2 s_2^3$	s_2^4	$s_1^5 s_3$	$s_1^3 s_2 s_3$	$s_1 s_2^2 s_3$
8	1	56	840	3360	1680	448	8960	26880
71	1	41	390	660	-120	268	2960	1680
62	1	30	203	396	276	136	848	1504
61^2	1	27	110	-180	-120	142	440	-1400
53	1	23	147	291	-102	52	512	1140
521	1	18	47	36	-12	52	-28	40
51^3	1	14	-33	-180	36	64	-184	-48
4^2	1	20	138	156	321	16	608	-336
431	1	13	47	-19	-22	2	132	-210
42^2	1	10	23	116	116	-4	-132	24
421^2	1	7	-19	-19	-34	8	-72	-48
41^4	1	2	-69	36	36	28	-52	312
$3^2 2$	1	7	35	35	-70	-28	0	420
$3^2 1^2$	1	4	2	-100	41	-16	96	-48
$32^2 1$	1	1	5	35	-10	-22	-60	-30
321^3	1	-3	-19	-9	6	-2	68	22
31^5	1	-9	-25	135	-30	28	80	-260
2^4	1	-4	30	60	165	-32	-160	-480
$2^3 1^2$	1	-7	21	-21	-42	-14	-28	42
$2^2 1^4$	1	-12	26	-36	33	16	32	112
21^6	1	-19	75	-15	-30	58	-220	-210
1^8	1	-28	210	-420	105	112	-1120	1680

Table 8(b)

Z	$s_1^2 s_3^2$	$s_2 s_3^2$	$s_1^4 s_4$	$s_1^2 s_2 s_4$	$s_2^2 s_4$	$s_1 s_3 s_4$	s_4^2	$s_1^3 s_5$
8	17920	35840	3360	40320	40320	107520	80640	21504
71	3520	-2560	1560	7920	-2880	6720	-5760	7104
62	-176	1312	526	1320	3192	-3136	-480	1120
61^2	160	-320	580	-480	-1200	-1120	960	1504
53	496	1256	-6	522	-1260	2184	-144	-672
521	-344	-464	94	-48	-120	-256	96	-152
51^3	160	512	138	-792	360	960	-288	48
4^2	1504	-1888	-204	360	2412	4704	5580	-960
431	216	16	-106	-18	-80	-196	-664	-232
42^2	-216	112	-34	0	232	-256	-160	-40
421^2	-36	64	-16	216	-32	8	272	-64
41^4	304	-256	-6	-144	-72	-192	-288	-24
$3^2 2$	0	280	-70	-630	-140	-280	560	224
$3^2 1^2$	0	-128	-52	-72	100	-160	92	128
$32^2 1$	0	-140	20	0	-20	320	-160	104
321^3	-56	124	4	96	-12	8	-48	16
31^5	40	-80	-50	-210	60	-280	240	64
2^4	400	560	120	0	-720	-480	540	-96
$2^3 1^2$	112	-28	84	0	252	-336	0	-168
$2^2 1^4$	-128	-128	4	-120	-228	224	60	-128
21^6	40	320	-150	90	360	420	-360	264
1^8	1120	-1120	-420	2520	-1260	-3360	1260	1344

Table 8(c)

Z	$s_1s_2s_5$	s_3s_5	$s_1^2s_6$	s_2s_6	s_1s_7	s_8
8	129024	172032	107520	215040	368640	645120
71	8064	-12288	21120	-15360	23040	-46080
62	1728	-1024	-1056	7872	-10752	-3840
61^2	-4032	2048	960	-1920	-3840	7680
53	720	4128	-2568	-3552	-2016	-1152
521	720	-832	-768	-192	1344	768
51^3	-288	384	-288	576	1152	-2304
4^2	-4032	-4224	-1056	-1248	-1152	-720
431	-280	-192	232	768	304	288
42^2	-592	576	544	-768	448	0
421^2	104	48	244	72	-416	-288
41^4	144	-192	144	-288	-576	1152
3^22	-112	-672	280	0	160	0
3^21^2	512	384	-128	-384	-128	-144
32^21	-16	72	-140	240	-200	0
321^3	-216	-232	-156	-48	264	192
31^5	144	512	-120	240	480	-960
2^4	864	-768	-240	240	0	0
2^31^2	0	168	84	-336	216	0
2^21^4	0	128	384	384	-384	-240
21^6	504	-768	-120	-960	-720	1440
1^8	-4032	2688	-3360	3360	5760	-5040

Table 9(a)

Z	s_1^9	$s_1^7 s_2$	$s_1^5 s_2^2$	$s_1^3 s_2^3$	$s_1 s_2^4$	$s_1^6 s_3$
9	1	72	1512	10080	15120	672
81	1	55	798	2940	840	434
72	1	42	447	1380	1620	252
71^2	1	39	318	60	-1080	258
63	1	33	303	993	378	126
621	1	28	153	148	108	126
61^3	1	24	33	-540	-180	138
54	1	28	258	708	633	56
531	1	21	111	141	-270	42
52^2	1	18	63	228	468	36
521^2	1	15	-3	-99	-126	48
51^4	1	10	-93	-204	324	68
$4^2 1$	1	18	108	18	243	6
432	1	13	63	133	98	-14
431^2	1	10	12	-134	-61	-2
$42^2 1$	1	7	-3	85	50	-8
421^3	1	3	-51	-15	-54	12
41^5	1	-3	-93	225	90	42
3^3	1	9	63	105	-630	-42
$3^2 21$	1	4	18	-20	-55	-32
$3^2 1^3$	1	0	-18	-144	189	-12
32^3	1	0	18	120	225	-36
$32^2 1^2$	1	-3	-3	15	-90	-18
321^4	1	-8	-18	40	45	12
31^6	1	-15	3	285	-270	54
$2^4 1$	1	-8	42	0	105	-28
$2^3 1^3$	1	-12	42	-84	-63	0
$2^2 1^5$	1	-18	72	-90	135	42
21^7	1	-26	168	-210	-105	98
1^9	1	-36	378	-1260	945	168

Table 9(b)

Z	$s_1^4 s_2 s_3$	$s_1^2 s_2^2 s_3$	$s_2^3 s_3$	$s_1^3 s_3^2$	$s_1 s_2 s_3^2$	s_3^3
9	20160	120960	80640	53760	322560	143360
81	8260	21000	-5040	15680	17920	-8960
72	3060	6960	7440	2160	3360	-640
71^2	2340	-3000	-3120	2880	-7680	1280
63	1440	5754	-948	720	11496	6704
621	470	-96	-568	-640	-1344	-1536
61^3	90	-2400	1080	480	1920	1280
54	1240	2424	2232	2720	-224	-3776
531	288	366	-204	144	168	-304
52^2	-180	144	1392	-840	-1584	704
521	-180	-96	-360	-276	-96	128
51^4	-260	624	240	704	1024	-512
4^21	450	-1026	-558	1200	-2304	944
432	0	294	-28	0	1176	-336
431^2	90	-426	170	144	-96	48
42^21	-180	-120	-280	-180	0	0
421^3	-36	204	240	60	240	-64
41^5	0	390	-540	600	-1680	320
3^3	0	1890	-420	0	2520	3920
3^221	0	240	80	0	-480	-480
3^21^3	180	-36	-180	-96	144	368
32^3	-180	-540	420	240	0	560
32^21^2	0	30	-60	0	-60	80
321^4	140	-60	20	-160	240	-240
31^6	0	-1110	300	120	240	320
2^41	-140	-420	-420	560	1120	-560
2^31^3	0	336	336	0	-672	224
2^21^5	-90	90	-450	-240	0	80
21^7	-770	210	630	560	1120	-560
1^9	-2520	7560	-2520	3360	-10080	2240

Table 9(c)

Z	$s_1^5 s_4$	$s_1^3 s_2 s_4$	$s_1 s_2^2 s_4$	$s_1^2 s_3 s_4$	$s_2 s_3 s_4$	$s_1 s_4^2$
9	6048	120960	362880	483840	967680	725760
81	3192	35280	20160	84000	-60480	40320
72	1398	8760	15480	-3360	26880	-15840
71^2	1464	4560	-14400	3360	-6720	-5760
63	354	2478	3924	708	5784	-2448
621	474	248	1704	-4032	-4256	1632
61^3	534	-2040	-1800	960	5280	1440
54	-156	1368	2124	14448	3024	19692
531	-30	150	-2412	588	-1176	-2736
52^2	54	-72	504	-1152	-576	-288
521^2	84	-168	144	492	1464	1008
51^4	114	-1368	1224	1632	-3456	-2592
4^21	-216	108	2484	2268	-756	2052
432	-126	-602	84	-1092	2184	-1008
431^2	-96	172	-36	-468	-516	-492
42^21	-12	280	240	300	-840	240
421^3	-12	312	-288	-132	408	432
41^5	-42	-510	-180	-1020	600	-720
3^3	-126	-1890	-1260	-1260	-7560	5040
3^221	-36	-440	-60	240	240	540
3^21^3	-36	72	324	-288	144	108
32^3	72	0	-1440	720	0	-180
32^21^2	48	120	360	240	240	-720
321^4	-12	-40	-420	0	-560	300
31^6	-138	-390	1260	-420	840	720
2^41	168	0	0	-1680	0	1260
2^31^3	84	-168	252	-336	-336	252
2^21^5	-72	-180	-540	1260	1260	-540
21^7	-336	1260	1260	-420	-3780	-1260
1^9	-756	7560	-11340	-15120	15120	11340

Table 9(d)

Z	$s_1^4 s_5$	$s_1^2 s_2 s_5$	$s_2^2 s_5$	$s_1 s_3 s_5$	$s_4 s_5$	$s_1^3 s_6$
9	48384	580608	580608	1548288	2322432	322560
81	19824	100800	-36288	86016	-145152	94080
72	5784	14688	34848	-33792	-10368	12960
71^2	6384	-5184	-13248	-12288	20736	17280
63	24	4968	-10944	19488	-2592	-5976
621	884	-432	-1104	-2432	1728	-1376
61^3	1284	-7344	3312	8832	-5184	480
54	-1776	-5472	5616	6528	31968	-5856
531	-600	1080	576	1824	-4320	-1656
52^2	-216	288	2016	-1152	-1152	-336
521^2	-156	972	-792	-1392	1728	-492
51^4	24	-288	288	768	-1152	-192
4^21	-756	-3672	-1404	-6912	-7992	144
432	-56	-1512	-1344	-672	1568	1064
431^2	-116	552	708	768	1928	368
42^21	4	-372	168	1488	-448	500
421^3	-60	-36	-216	-912	-1152	228
41^5	24	648	288	768	2592	120
3^3	504	-504	4032	-6048	2016	840
3^221	304	576	-48	192	-1024	-160
3^21^3	144	432	-432	288	-432	-432
32^3	144	720	-1008	-864	1008	-480
32^21^2	24	-432	288	-312	432	-300
321^4	-16	-432	48	-32	432	-80
31^6	264	1080	-288	1536	-2592	-360
2^41	-336	1008	1008	-672	-1008	0
2^31^3	-336	0	-1008	1344	0	672
2^21^5	-36	648	1188	-1152	-648	720
21^7	924	-504	-2268	-2688	4536	-1680
1^9	3024	-18144	9072	24192	-18144	-10080

Table 9(e)

Z	$s_1s_2s_6$	s_3s_6	$s_1^2s_7$	s_2s_7	s_1s_8	s_9
9	1935360	2580480	1658880	3317760	5806080	10321920
81	107520	-161280	288000	-207360	322560	-645120
72	20160	-11520	-11520	92160	-126720	-46080
71^2	-46080	23040	11520	-23040	-46080	92160
63	7200	38304	-24048	-33120	-19584	-11520
621	6720	-7936	-7488	-1920	13056	7680
61^3	-2880	3840	-2880	5760	11520	-23040
54	-23520	-23616	-7488	-8640	-8784	-5760
531	-2016	-1440	1584	5472	2304	2304
52^2	-4320	4224	4032	-5760	3456	0
521^2	792	384	1872	576	-3168	-2304
51^4	1152	-1536	1152	-2304	-4608	9216
4^21	4320	5904	432	2160	1296	1440
432	0	-2016	1392	-160	896	0
431^2	1152	144	-528	-1744	-688	-864
42^21	-360	0	-720	1280	-1120	0
421^3	-360	-192	-864	-288	1440	1152
41^5	-720	960	-720	1440	2880	-5760
3^3	0	-3360	720	1440	0	0
3^221	0	1440	-480	-160	-400	0
3^21^3	-1728	-1296	432	1296	432	576
32^3	2160	-1680	-720	720	0	0
32^21^2	0	-240	360	-720	720	0
321^4	960	1040	720	240	-1200	-960
31^6	-720	-2880	720	-1440	-2880	5760
2^41	-1680	1680	720	-720	0	0
2^31^3	0	-672	-288	1440	-1008	0
2^21^5	0	-720	-2160	-2160	2160	1440
21^7	-3360	5040	720	6480	5040	-10080
1^9	30240	-20160	25920	-25920	-45360	40320

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