EIGENVALUES OF THE ADJACENCY MATRIX OF TETRAHEDRAL GRAPHS.

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Institute of Statistics Mimeo Series No. 571

March 1968

This research was supported by the National Science Foundation Grant No. GP-5790, and the Army Research Office (Durham) Grant No. DA-ARO-D-31-12-G910.

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ABSTRACT

A tetrahedral graph is defined to be a graph G, whose vertices are identified with the $\binom{n}{3}$ unordered triplets on n symbols, such that vertices are adjacent if and only if the corresponding triplets have two symbols in common. If $n_2(x)$ denotes the number of vertices y, which are at distance 2 from x and A(G) denotes the adjacency matrix of G, then G has the following properties: P_1) the number of vertices is $\binom{n}{3}$. P_2) G is connected and regular. P_3) $n_2(x) = \frac{3}{2}(n-3)(n-4)$ for all x in G. P_4) the distinct eigenvalues of A(G) are -3, 2n-9, n-7, 3(n-3). We show that, if $n > 16$, then any graph G (with no loops and multiple edges) having the properties P_1) - P_4) must be a tetrahedral graph. An alternative characterization of tetrahedral graphs has been given by the authors (J. Comb. Theory, Vol 3, No 4, December 1967, 366-385).

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1. Introduction

We shall consider only finite undirected graphs without loops or multiple edges. A tetrahedral graph with characteristic n is defined to be a graph whose vertices are identified with the $\binom{n}{3}$ unordered triplets on n symbols, with two vertices adjacent if and only if their corresponding triplets have two symbols in common. If $d(x,y)$ denotes the distance between two vertices x and y and $\Delta(x,y)$, the number of vertices adjacent to both x and y, then it has been shown by Bose and Laskar [1] that for $n > 16$, the following properties characterize the tetrahedral graph with characteristic n:

b_1) The number of vertices is $\binom{n}{3}$.

b_2) The graph is connected and regular of degree $3(n-3)$.

b_3) If $d(x,y) = 1$, then $\Delta(x,y) = n-2$.

b_4) If $d(x,y) = 2$, then $\Delta(x,y) = 4$.

The adjacency matrix $A(G)$ of a graph G is a square (0,1) matrix whose rows and columns correspond to the vertices of G, and $a_{ij} = 1$ if and only if i and j are adjacent. Let $n_2(x)$ denote the number of vertices y at distance 2 from x.

A tetrahedral graph G with characteristic n has the following properties:

P_1) The number of vertices is $\binom{n}{3}$

P_2) G is connected and regular.

P_3) $n_2(x) = \frac{3}{2}(n-3)(n-4)$ for all x in G.

P_4) The distinct eigenvalues of A(G) are -3, 2n-9, n-7, 3(n-3).

(P_1), (P_2), (P_3) are obvious. (P_4) is proved in paragraph 2. We go on to show that (P_1), (P_2), (P_3), (P_4) characterize a tetrahedral graph with characteristic n.

2. Determination of the eigenvalues of A(G).

Given v objects, a relation satisfying the following conditions is said to be an association scheme with m classes:

a) Any two objects are eigher 1st, 2nd, ... or mth associates, the relation of association being symmetrical.

b) Each object α has n_i ith associates, the number n_i being independent of α.

c) If any two objects α and β are ith associates, then the number of objects which are jth associates of α, and kth associates of β, is p^i_{jk} and is independent of the pair of ith associates α and β. The numbers v, n_i $(i = 1, 2, \ldots, m)$ and p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$ are the parameters of the association scheme.

The concept of an association scheme was first introduced by Bose and Shimamoto [3].

If we define

$$B_i = (b^{\beta}_{\alpha i}) = \begin{pmatrix} b^1_{1i} & b^2_{1i} & \ldots & b^v_{1i} \\ b^1_{2i} & b^2_{2i} & \ldots & b^v_{2i} \\ \ldots & \ldots & \ldots & \ldots \\ b^1_{vi} & b^2_{vi} & \ldots & b^v_{vi} \end{pmatrix}, \quad i = 0, 1, \ldots, m,$$

where

$$b^{\beta}_{\alpha i} = 1, \text{ if the objects } \alpha \text{ and } \beta \text{ are ith associates}$$
$$= 0, \text{ otherwise,}$$

and

$$\rho_k = (p_{ik}^j) = \begin{pmatrix} p_{0k}^0 & p_{0k}^1 & \cdots & p_{0k}^m \\ p_{1k}^0 & p_{0k}^1 & \cdots & p_{1k}^m \\ \cdots & \cdots & \cdots & \cdots \\ p_{mk}^0 & p_{mk}^1 & \cdots & p_{mk}^m \end{pmatrix}, \quad k = 0, 1, \ldots, m,$$

then it has been shown by Bose and Mesner [2], that the matrices ρ_i, $i = 0, 1, \ldots, m$ are linearly independent and combine in the same way as the B's under addition as well as multiplication. It was further shown that if

$$B = \sum_{i=0}^{m} c_i B_i$$

$$\rho = \sum_{i=0}^{m} c_i \rho_i$$

then B and ρ have the same distinct eigenvalues. If in particular we take $c_0 = 0$, $c_1 = 1$, $c_2 = c_3 = \ldots = c_m = 0$, it follows that the distinct eigenvalues of B_1 are the same as those of ρ_1.

Consider a tetrahedral graph G with characteristic n. If a relation of association on the vertices of G is defined, such that two vertices are 1st, 2nd, or 3rd associates if they are at distances 1, 2 or 3 respectively, then it can be easily checked that G yields a three-class association scheme. It may be pointed out that the matrix A(G) is the matrix B_1, and thus the distinct eigenvalues of A(G) are given by those of the matrix

$$\rho_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ n_1 & p_{11}^1 & p_{11}^2 & p_{11}^3 \\ 0 & p_{12}^1 & p_{12}^2 & p_{12}^3 \\ 0 & p_{13}^1 & p_{13}^2 & p_{13}^3 \end{pmatrix}.$$

The parameters p_{jk}^i of the association scheme corresponding to G are easily calculated. They are given by

$$\begin{aligned} n_1 &= 3(n-3), \quad & p_{11}^1 &= n-2, & p_{11}^2 &= 4, & p_{11}^3 &= 0, \\ & & p_{12}^1 &= 2(n-4), & p_{12}^2 &= 2(n-4), & p_{12}^3 &= 9, \\ & & p_{13}^1 &= 0, & p_{13}^2 &= n-5, & p_{13}^3 &= 3(n-6). \end{aligned}$$

Substituting these values in the matrix ρ_1, the eigenvalues are easily calculated. They are found to be

$$-3,\ 2n-9,\ n-7,\ 3(n-3).$$

Thus, we have the following lemma:

Lemma 2.1. If G is a tetrahedral graph with characteristic n and if A(G) is the adjacency matrix of G, then the distinct eigenvalues of A(G) are

(2.1) $$-3,\ 2n-9,\ n-7,\ 3(n-3).$$

3. Some Preliminaries on Matrices.

Before stating the next lamma, we need the concept of the polynomial of a graph due to Hoffman [4]. Let J be the matrix all of whose entries are unity. Then for any graph G with adjacency matrix A = A(G), there exists a polynomial P(x) such that P(A) = J if and only if G is regular and connected. The unique polynomial of least degree satisfying this equation is called the polynomial of G, and is calculated as follows: if G has v vertices, regular of degree d, and the other distinct eigenvalues are $\alpha_1, \alpha_2, \ldots, \alpha_t$, then

(3.1) $$P(x) = \frac{v \prod_{i=1}^{t}(x-\alpha_i)}{\prod_{i=1}^{t}(d-\alpha_i)}.$$

Consider a regular, connected graph H (with no loops and multiple edges) on $v = \binom{n}{3}$ vertices such that the adjacency matrix A = A(H) has the distinct eigenvalues -3, 2n-9, n-7, 3(n-3).

Lemma 3.1. The matrix A satisfies the equation

$$(3.2) \qquad A^3 - (3n-19)\,A^2 + (2n^2 - 32n + 111)\,A + (6n^2 - 69n + 189)\,I = 36J,$$

where J is a v $\times$ v matrix all of whose entries are i, and I is the v $\times$ v identity matrix.

Proof: It follows immediately by calculating the polynomial of the graph H as given in (3.1).

Lemma 3.2. For any two vertices x, y in H, $d(x,y) \leq 3$.

Proof: If in (3.2) we set $A_{ij} = 0$, $A^2_{ij} = 0$, then $A^3_{ij} = 36$, but this implies that $d(i,j) \leq 3$ for all vertices i,j in H.

Lemma 3.3. Consider the matrix

$$B = \tfrac{1}{4}[A^2-(n-2)A-3(n-3)I].$$

Let $n_2(i)$ denote the number of vertices j, such that $d(i,j) = 2$, and $n_3(i)$ denote the number of vertices k, such that $d(i,k) = 3$. If $n_2(i) = \frac{3}{2}(n-3)(n-4)$, for all vertices i in H, then

i) B is a (0,1) matrix

ii) $\Delta(x,y) = n-2$, for all vertices x,y in H, such that $d(x,y) = 1$

iii) $\Delta(x,y) = 4$, for all vertices x,y in H, such that $d(x,y) = 2$.

Proof: Since H is regular and 3(n-3) is the dominant eigenvalue of A, it follows H is regular of degree $n_1 = 3(n-3)$.

Divide the set of vertices of H, with respect to a particular vertex i into four subsets S_0, S_1, S_2, S_3 as follows:

S_0: i

S_1: $j_1, j_2, \ldots, j_t, j_{n_1}$, such that $d(i,j_t) = 1$, $t = 1, 2, \ldots, n_1$

S_2: $k_1, k_2, \ldots, k_s, k_{n_2(i)}$, such that $d(i,k_s) = 2$, $s = 1,2,\ldots,n_2(i)$

S_3: $\ell_1, \ell_2, \ldots, \ell_r, \ell_{n_3(i)}$, such that $d(i,\ell_r) = 3$, $r = 1,2,\ldots,n_3(i)$

Thus the vertices in S_t are the tth associates of the vertex i. The following relations can be deducted easily from (3.2) by noting that $AJ = JA$.

(3.3) $$A^3_{ii} = \sum_{t=1}^{n_1} A^2_{ij_t} = 3(n-2)(n-3).$$

(3.4) $$A^4_{ii} = \sum_{t=1}^{n_1} A^3_{ij_t} = 3(n-3)(n^2+7n-37).$$

Also, since $A^tJ = \{3(n-3)\}^tJ$, we get

(3.5) $$\sum_{j=1}^{v} A^2_{ij} = (A^2J)_{ii} = 9(n-3)^2.$$

(3.6) $$A^2_{ii} = \sum_{t=1}^{n_1} A_{ij_t} = 3(n-3).$$

Also

(3.7) $$\sum_{r=1}^{n_3(i)} A^2_{i\ell_r} = 0.$$

Hence it follows from (3.3), (3.5), (3.6), (3.7) that

$$(3.8) \qquad \sum_{s=1}^{n_2(i)} A^2_{ik_s} = \sum_{j=1}^{v} A^2_{ij} - \sum_{t=1}^{n_1} A^2_{ij_t} - \sum_{r=1}^{n_3(i)} A^2_{i\ell_r} - A^2_{ii}$$

$$= 6(n-3)(n-4).$$

Consider

$$(3.9) \qquad X_i = b^2_{ii} + \sum_{t=1}^{n_1(i)} b^2_{ij_t} + \sum_{s=1}^{n_2(i)} (b_{ik_s}-1)^2 + \sum_{r=1}^{n_3(i)} b^2_{i\ell_r}$$

$$= \sum_{j=1}^{v} b^2_{ij} - 2\sum_{s=1}^{n_2(i)} b_{ik_s} + n_2(i).$$

We first show that

$$X_i = n_2(i) - \tfrac{3}{2}(n-3)(n-4).$$

Since

(3.10) $\quad B = \frac{1}{4}[A^2-(n-2)A-3(n-3)I]$, we get

$$(3.11) \qquad B^2_{ii} = \tfrac{1}{16}[A^4_{ii}-2(n-2)A^3_{ii}+(n^2-10n+22)A^2_{ii}+6(n^2-5n+6)A_{ii}+9(n-3)^2I_{ii}]$$

Substituting values from (3.3), (3.4), (3.6), in (3.11) we get

$$B^2_{ii} = \tfrac{3}{2}(n-3)(n-4).$$

But

$$\sum_{j=1}^{v} b^2_{ij} = B^2_{ii}.$$

Hence

$$(3.12) \qquad \sum_{j=1}^{v} b^2_{ij} = \tfrac{3}{2}(n-3)(n-4).$$

Also from (3.10)

$$\sum_{s=1}^{n_2(i)} b_{ik_s} = \frac{1}{4} \sum_{s=1}^{n_2(i)} A^2_{ik_s}.$$

It follows from (3.8) that

(3.13)
$$\sum_{s=1}^{n_2(i)} b_{ik_s} = \frac{3}{2}(n-3)(n-4).$$

Substituting values from (3.12), (3.13) in (3.9) we get

$$X_i = n_2(i) - \frac{3}{2}(n-3)(n-4).$$

Now if $n_2(i) = \frac{3}{2}(n-3)(n-4)$ for all i in H, then $X_i = 0$, for all i in H. Then, it follows from (3.9) that B is a (0,1) matrix which proves i).

To prove ii), we note that if $A_{ij_t} = 1$, then from (3.10), (3.3) and (3.6) it follows

$$\sum_{t=1}^{n_1} b_{ij_t} = 0.$$

But since $b_{ij} = 0$ or 1, this implies $b_{ij_t} = 0$, and hence from (3.10) it follows that $A^2_{ij} = n-2$.

To prove iii) we note that if $A_{ij} = 0$, $A^2_{ij} \neq 0$ then $b_{ij} \neq 0$ and hence $A^2_{ij} = 4$.

4. <u>Theorem</u>. If H is a graph satisfying the following properties:

P_1) The number of vertices is $\binom{n}{3}$.

P_2) H is connected and regular.

P_3) $n_2(x) = \frac{3}{2}(n-3)(n-4)$ for all x in H.

P_4) The distinct eigenvalues of A(H) are -3, 2n-9, n-7, 3(n-3).

then, for $n > 16$, H is tetrahedral.

Proof: From lemmas (3.1) - (3.3) and the hypothesis, H clearly satisfies the following conditions:

a_1) The number of vertices in $\binom{n}{3}$

a_2) H is connected and regular of degree $3(n-3)$

a_3) $\Delta(x,y) = n-2$, for $d(x,y) = 1$

a_4) $\Delta(x,y) = 4$, for $d(x,y) = 2$.

Hence if $n > 16$, H is tetrahecral [1].

Note: It is conjectured that the property P_3) of the theorem is implied by the other properties P_1), P_2), P_4).

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