ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRICES

by

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1. Introduction. In our previous paper [11], we have proved the unbiasedness of the modified likelihood ratio (= modified LR) test (i) for the equality of covariance matrix to a given matrix and (ii) for the equality of two covariance matrices. We have also shown the unbiasedness of the LR test (iii) for sphericity and (iv) for the equality of mean vector and covariance matrix to a given vector and matrix.

In this paper asymptotic expansions of the distributions of the test criteria for (i) and (iv) both under hypothesis and alternatives are derived, by inverting the characteristic function directly. The asymptotic expansion of the power function of the LR test for sphericity (iii), is obtained by using the differential operator due to Welch [12], and also the limiting non-null distribution of the LR test for the equality of k covariance matrices is derived in a similar way. This method has been shown to be useful in other problems in multivariate analysis by Ito [3], Siotani [8], Okamoto [5], and etc.

All the limiting non-null distributions of these test criteria are shown to be normal distributions, whereas the limiting distributions under hypothesis are χ^2-distributions as in Anderson [1]. This shows the difference between the LR test criteria for covariance matrices and the criterion for the linear hypothesis as in Sugiura and Fujikoshi [10].

2. Expansion of the distribution of the criterion for $\Sigma = \Sigma_0$.

Let $p \times 1$ vectors $X_1, \ldots, X_N$ be a random sample from a p-variate normal distribution with unknown mean vector μ and covariance matrix Σ (non-singular). The LR criterion for testing the hypothesis $H_1 : \Sigma = \Sigma_0$ against the alternatives $K_1 : \Sigma \neq \Sigma_0$, for some given positive definite matrix Σ_0, is given by

(2.1) $$\lambda = (e/N)^{Np/2}|S\Sigma_0^{-1}|^{N/2} \text{ etr } - (1/2)\Sigma_0^{-1}S,$$

where etr means exp tr and $S = \Sigma_{\alpha=1}^{N} (X_\alpha-\bar{X})(X_\alpha-\bar{X})'$, $\bar{X} = (1/N)\Sigma_{\alpha=1}^{N}X_\alpha$. This LR test is not unbiased. However, if we modify this criterion by reducing the sample size N to the degrees of freedom $n = N-1$, it has some desirable property, that is, the unbiasedness is shown by Sugiura and Nagao [11] and the monotonicity of the power function with respect to p characteristic roots of $\Sigma\Sigma_0^{-1}$ is established by Nagao [4]. So we shall consider the asymptotic expansion of the modified LR statistic λ^* instead of λ.

(2.2) $$\lambda^* = (e/n)^{np/2}|S\Sigma_0^{-1}|^{n/2} \text{ etr } - (1/2)\Sigma_0^{-1}S.$$

The limiting distribution of the statistic $-2 \log \lambda$ under the hypothesis is the χ^2-distribution with $p(p+1)/2$ degrees of freedom, which can be seen in Anderson [1, p. 267]. The hth moment of the statistic λ^* under alternative K_1 is given by

(2.3) $$E[\lambda^{*h}|K_1] = (\frac{2e}{n})^{nhp/2} \frac{\Gamma_p(n(1+h)/2)}{\Gamma_p(n/2)} \frac{|\Sigma\Sigma_0^{-1}|^{nh/2}}{|I+h\Sigma\Sigma_0^{-1}|^{n(1+h)/2}},$$

where $\Gamma_p(x) = \pi^{p(p-1)/4} \Pi_{\alpha=1}^{p} \Gamma(x - (\alpha-1)/2)$. Hence the characteristic function of $-2 \log \lambda^*$ under the hypothesis H_1 is expressed as

(2.4) $$C_{H_1}(t) = (\frac{n}{2e})^{itpn} \frac{\Gamma_p(n(1-2it)/2)}{\Gamma_p(n/2)} (1-2it)^{-np(1-2it)/2}.$$

We shall use the following asymptotic formula for the gamma function as in Anderson [1, p. 204].

(2.5) $$\log \Gamma(x+h) = \log(2\pi)^{\frac{1}{2}} + (x+h-\tfrac{1}{2}) \log x - x - \sum_{r=1}^{k} \frac{(-1)^r B_{r+1}(h)}{r(r+1)\, x^r} + O(x^{-k-1}).$$

This holds for large value of x with fixed h. The Bernoulli polynomial $B_r(h)$

of degree r is given by $\tau e^{h\tau}/(e^{\tau}-1) = \Sigma_{r=0}^{\infty} (\tau^r/r!)B_r(h)$. Some of these are listed below;

$$(2.6) \qquad B_1(h) = h - \tfrac{1}{2} \qquad B_3(h) = h^3 - \tfrac{3}{2}h^2 + \tfrac{1}{2}h$$
$$B_2(h) = h^2 - h + \tfrac{1}{6} \qquad B_4(h) = h^4 - 2h^3 + h^2 - \tfrac{1}{30}.$$

Applying the formula (2.5) to each gamma function in (2.4), we get

$$(2.7) \quad \log C_{H_1}(t) = -\frac{p(p+1)}{4}\log(1-2it) - \sum_{r=1}^{k} \frac{(-2)^r B_{r+1}}{r(r+1)n^r}\{(1-2it)^{-r}-1\}+O(n^{-k-1}),$$

where $B_{r+1} = \Sigma_{\alpha=1}^{p} B_{r+1}((1-\alpha)/2)$. This formula implies the asymptotic expansion of the characteristic function for $-2\log\lambda^*$.

$$(2.8) \quad C_{H_1}(t) = (1-2it)^{-p(p+1)/4}[1+B_2n^{-1}\{(1-2it)^{-1}-1\} + (1/6)n^{-2}\{(3B_2^2-4B_3)$$
$$\cdot(1-2it)^{-2}-6B_2^2(1-2it)^{-1}+(3B_2^2+4B_3)\}+(1/6)n^{-3}\{(4B_4-4B_2B_3+B_2^3)$$
$$\cdot(1-2it)^{-3}+B_2(4B_3-3B_2^2)(1-2it)^{-2}+B_2(4B_3+3B_2^2)(1-2it)^{-1}-$$
$$(4B_4+4B_2B_3+B_2^3)\}] + O(n^{-4}),$$

where

$$(2.9) \quad \begin{aligned} B_2 &= p(2p^2+3p-1)/24 \\ B_3 &= -p(p-1)(p+1)(p+2)/32 \\ B_4 &= p(6p^4+15p^3+10p^2-30p+3)/480. \end{aligned}$$

Inverting this characteristic function, using the well-known fact that $(1-2it)^{-f/2}$ is the characteristic function of the χ^2-distribution with f degrees of freedom, we have the following theorem.

<u>Theorem 2.1</u>. Under the hypothesis $H_1: \Sigma = \Sigma_0$, the distribution of the modified LR criterion $-2\log\lambda^*$ defined by (2.2) can be expanded asymtotically as

$$(2.10)\quad P(-2\log\lambda^* \le z) = P(\chi_f^2 \le z) + B_2 n^{-1}\{P(\chi_{f+2}^2 \le z) - P(\chi_f^2 \le z)\}$$

$$+ (1/6)n^{-2}\{(3B_2^2-4B_3)P(\chi_{f+4}^2 \le z) - 6B_2^2 P(\chi_{f+2}^2 \le z) + (3B_2^2+4B_3)P(\chi_f^2 \le z)\}$$

$$+ (1/6)n^{-3}\{(4B_4-4B_2B_3+B_2^3)P(\chi_{f+6}^2 \le z) + B_2(4B_3-3B_2^2)P(\chi_{f+4}^2 \le z)$$

$$+B_2(4B_3+3B_2^2)P(\chi_{f+2}^2 \le z) - (4B_4+4B_2B_3+B_2^3)P(\chi_f^2 \le z)\} + O(n^{-4}),$$

where χ_f^2 means the χ^2-variate with f degrees of freedom and $f = p(p+1)/2$; the constant B_r is given by (2.9).

Now we shall consider the asymptotic expansion of the power function of this criterion $-2\log\lambda^*$. The characteristic function of $-2n^{-\frac{1}{2}}\log\lambda^*$ under alternative K_1 can be written from formula (2.3) for the moment of λ^* as

$$(2.11)\qquad c_{K_1}(t) = \left(\frac{n}{2e}\right)^{itp\sqrt{n}} \frac{\Gamma_p((n/2)-\sqrt{n}it)}{\Gamma_p(n/2)} \frac{|\Sigma\Sigma_0^{-1}|^{-\sqrt{n}it}}{|I-2itn^{-\frac{1}{2}}\Sigma\Sigma_0^{-1}|^{(n/2)-\sqrt{n}it}}.$$

Applying the asymptotic formula (2.5) for the gamma function to each term of $\Gamma_p((n/2)-\sqrt{n}it)/\Gamma_p(n/2)$, we have

$$(2.12)\qquad \log\frac{\Gamma_p((n/2)-\sqrt{n}it)}{\Gamma_p(n/2)} = \sqrt{n}itp\log\frac{2}{n} - pt^2 + p\sum_{r=1}^{2k}\frac{(2it)^r}{n^{r/2}}\left\{\frac{2(it)^2}{(r+1)(r+2)}\right.$$

$$\left. + \frac{p+1}{4r}\right\} - \sum_{r=1}^{k}\frac{(-2)^r B_{r+1}}{r(r+1)n^r}\{(1-\frac{2it}{\sqrt{n}})^{-r}-1\} + O(n^{-k-\frac{1}{2}}).$$

Since the asymptotic formula $-\log|I-Z/n| = \Sigma_{r=1}^{k} n^{-r}\mathrm{tr}\,(Z^r)/r + O(n^{-k-1})$ holds for any positive definite matrix Z, we have

$$(2.13)\quad -((n/2)-\sqrt{n}it)\log|I-2itn^{-\frac{1}{2}}\Sigma\Sigma_0^{-1}| = \sqrt{n}it\,\mathrm{tr}\,\Sigma\Sigma_0^{-1} - t^2\{\mathrm{tr}(\Sigma\Sigma_0^{-1})^2$$

$$-2\,\mathrm{tr}\Sigma\Sigma_0^{-1})\} + \sum_{r=1}^{2k}\frac{2^{r+1}(it)^{r+2}}{n^{r/2}}\left\{\frac{\mathrm{tr}(\Sigma\Sigma_0^{-1})^{r+2}}{r+2} - \frac{\mathrm{tr}(\Sigma\Sigma_0^{-1})^{r+1}}{r+1}\right\} + O(n^{-k-\frac{1}{2}}).$$

Substituting these two expressions to (2.11), we can see that

$$(2.14)\quad \log C_{K_1}(t) = -\sqrt{n}it\,\{\log|\Sigma\Sigma_0^{-1}| + \mathrm{tr}(I-\Sigma\Sigma_0^{-1})\} - t^2\mathrm{tr}(I-\Sigma\Sigma_0^{-1})^2$$
$$+\sum_{r=1}^{2k}\frac{(2it)^r}{n^{r/2}}\left[\frac{p(p+1)}{4r}+\frac{2p(it)^2}{(r+1)(r+2)}+2(it)^2\left\{\frac{\mathrm{tr}(\Sigma\Sigma_0^{-1})^{r+2}}{r+2}-\frac{\mathrm{tr}(\Sigma\Sigma_0^{-1})^{r+1}}{r+1}\right\}\right]$$
$$-\sum_{r=1}^{k}\frac{(-2)^r B_{r+1}}{r(r+1)n^r}\left\{\left(1-\frac{2it}{\sqrt{n}}\right)^{-r}-1\right\} + O(n^{-k-\frac{1}{2}}),$$

which implies that the statistic $\lambda^{**} = -2n^{-\frac{1}{2}}\log\lambda^{*} - \sqrt{n}\{\mathrm{tr}(\Sigma\Sigma_0^{-1}-I) - \log|\Sigma\Sigma_0^{-1}|\}$ converges in law to the normal distribution with mean zero and variance $\tau^2 = 2\mathrm{tr}(I-\Sigma\Sigma_0^{-1})^2$ and further it enables us to expand the characteristic function of λ^{**}/τ up to any order asymptotically. We shall write it up to order n^{-1}.

$$(2.15)\qquad C_{\lambda^{**}/\tau}(t) = e^{-t^2/2}\,\{1+n^{-\frac{1}{2}}A_1 + n^{-1}A_2\},$$

where the coefficients A_1 and A_2 of each term are given by

$$A_1 = it\tau^{-1}p(p+1)/2 + (2/3)\tau^{-3}(it)^3(p+2\mathrm{tr}_3 - 3\mathrm{tr}_2)$$

$$A_2 = (1/8)\tau^{-2}(it)^2p(p+1)(p^2+p+4)+(1/3)\tau^{-4}(it)^4\{p(p^2+p+2)-3p(p+1)\mathrm{tr}_2$$
$$+2(p^2+p-4)\mathrm{tr}_3+6\mathrm{tr}_4\} + (2/9)\tau^{-6}(it)^6(p+2\mathrm{tr}_3-3\mathrm{tr}_2)^2,$$

with the abbreviated notation $\mathrm{tr}_j = \mathrm{tr}(\Sigma\Sigma_0^{-1})^j$. By inverting this characteristic function, we can get the following theorem.

<u>Theorem 2.2</u>. Under the alternative $K_1:\Sigma \neq \Sigma_0$, the distribution of the modified LR criterion $-2\log\lambda^{*}$ defined by (2.2) can be expanded asymptotically as

$$(2.16)\quad P((1/\sqrt{n}\tau)\,[-2\log\lambda^{*}-n\{tr(\Sigma\Sigma_0^{-1}-I)-\log|\Sigma\Sigma_0^{-1}|\}]\le z)$$

$$= \Phi(z)-\frac{1}{6\sqrt{n}}[3\tau^{-1}\Phi'(z)p(p+1)+4\tau^{-3}\Phi'''(z)(p+2tr_3-3tr_2)]$$

$$+\frac{1}{72n}[9\tau^{-2}\Phi''(z)p(p+1)(p^2+p+4)+24\tau^{-4}\Phi^{(4)}(z)\{p(p^2+p+2)-3p(p+1)tr_2$$

$$+2(p^2+p-4)tr_3+6tr_4\}+16\tau^{-6}\Phi^{(6)}(z)(p+2tr_3-3tr_2)^2]+O(n^{-3/2}),$$

where $tr_j = tr(\Sigma\Sigma_0^{-1})^j$ and $\tau^2 = 2tr(I-\Sigma\Sigma_0^{-1})^2$; $\Phi^{(r)}(z)$ means the rth derivative of the standard normal distribution function $\Phi(z)$.

It may be interesting to note that the asymptotic mean and variance of the statistic $-2n^{-\frac{1}{2}}\log\lambda^{*}$ vanish, if we put $\Sigma = \Sigma_0$ in the above theorem. This shows some singularity of the limiting distribution at the hypothesis. The statistic $-2n^{-\frac{1}{2}}\log\lambda^{*}$ converges to zero under the hypothesis, in fact, $-2\log\lambda^{*}$ converges in law to the χ^2-distribution.

3. Expansion of the distribution of the criterion for $\Sigma = \Sigma_0$ and $\mu = \mu_0$. Let a p-variate random sample of size N from the normal distribution with mean vector μ and covariance matrix Σ be denoted by $X_1, X_2, \ldots, X_N$. The LR statistic for testing the hypothesis H_2: $\Sigma = \Sigma_0$ and $\mu = \mu_0$ against alternatives K_2: $\Sigma \ne \Sigma_0$ or $\mu \ne \mu_0$, where Σ_0 and μ_0 are a given positive definite matrix and a given vector, is expressed as

$$(3.1)\quad \lambda = \left(\frac{e}{N}\right)^{Np/2}|S\Sigma_0^{-1}|^{N/2}\,\mathrm{etr}\,-\tfrac{1}{2}\Sigma_0^{-1}\{S+N(\overline{X}-\mu_0)(\overline{X}-\mu_0)'\},$$

where $S = \Sigma_{\alpha=1}^{N}(X_\alpha-\overline{X})(X_\alpha-\overline{X})'$ and $\overline{X} = (1/N)\Sigma_{\alpha=1}^{N}X_\alpha$. For this problem, it has been shown by Sugiura and Nagao [11] that the LR test without modification is unbiased. So we shall consider the asymptotic expansion of the distribution of the LR statistic λ. The hth moment of λ under alternative K_2 is expressed as

$$(3.2)\quad E[\lambda^h|K_2] = \left(\frac{2e}{N}\right)^{Nhp/2} \frac{\Gamma_p((n+Nh)/2)}{\Gamma_p(n/2)} \frac{|\Sigma\Sigma_0^{-1}|^{Nh/2}}{|I+h\Sigma\Sigma_0^{-1}|^{N(1+h)/2}}$$

$$\times \operatorname{etr}[-\frac{Nh}{2} \Sigma_0^{-1}(\mu-\mu_0)(\mu-\mu_0)'\{I-h\Sigma_0^{-1}(\Sigma^{-1}+h\Sigma_0^{-1})^{-1}\}],$$

where $n = N-1$. The characteristic function of $-2 \log \lambda$ under the hypothesis H_2 is given from (3.2) as

$$(3.3)\quad C_{H_2}(t) = \left(\frac{N}{2e}\right)^{Npit} \frac{\Gamma_p((n/2)-Nit)}{\Gamma_p(n/2)} (1-2it)^{Np(1-2it)/2}.$$

Applying the asymptotic formula (2.5) for the gamma function to each term of $\Gamma_p((n/2)-Nit)/\Gamma_p(n/2)$, we have

$$(3.4)\quad C_{H_2}(t) = (1-2it)^{-(p/2)-p(p+1)/4} \exp[-\sum_{r=1}^{k} \frac{(-2)^r B'_{r+1}}{r(r+1)N^r}\{(1-2it)^{-r}-1\}]$$

$$+ O(N^{-k-1}),$$

where $B_{r+1}' = \Sigma_{\alpha=1}^{p} B_{r+1}(-\alpha/2)$. Noting that this expression is of the same form as (2.7), we can immediately obtain the following theorem.

<u>Theorem 3.1</u>. Under the hypothesis H_2: $\Sigma = \Sigma_0$ and $\mu = \mu_0$, the distribution of the LR criterion $-2 \log \lambda$ given in (3.1) can be expanded asymptotically as

$$(3.5)\quad P(-2 \log \lambda \le z) = P(\chi_f^2 \le z) + B_2'N^{-1}\{P(\chi_{f+2}^2 \le z)$$

$$-P(\chi_f^2 \le z)\} + (1/6)N^{-2}\{(3B'^2 - 4B_3')\, P(\chi_{f+4}^2 \le z) - 6B_2'^2\, P(\chi_{f+2}^2 \le z)$$

$$+(3B_2'^2+4B_3')\, P(\chi_f^2 \le z)\} + (1/6)N^{-3}\, \{(4B_4'-4B_2'B_3'+B_2'^3)\, P(\chi_{f+6}^2 \le z)$$

$$+B_2'(4B_3' - 3B_2'^2)\, P(\chi_{f+4}^2 \le z) + B_2'(4B_3' + 3B_2'^2)\, P(\chi_{f+2}^2 \le z)$$

$$- (4B_4' + 4B_2'B_3' + B_2'^3)\, P(\chi_f^2 \le z)\} + O(n^{-4}),$$

where $f = p + p(p+1)/2$ and the constant B'_r is given by $B'_2 = p(2p^2 + 9p + 11)/24$, $B'_3 = -p(p+1)(p+2)(p+3)/32$ and $B'_4 = p(6p^4+45p^3+110p^2+90p+3)/480$.

The limiting distribution of $-2 \log \lambda$ is stated in Anderson [1, p.268]. Now we shall consider the asymptotic expansion of the distribution of $-2 \log \lambda$ under the alternative K_2. The characteristic function of $-2N^{-\frac{1}{2}} \log \lambda$ under K_2 can be obtained from the moment of λ in (3.2) as $C_{K_2}(t) = C^{(1)}_{K_2}(t)\, C^{(2)}_{K_2}(t)$, where

$$(3.6)\quad C^{(1)}_{K_2}(t) = \left(\frac{N}{2e}\right)^{\sqrt{N}pit} \frac{\Gamma_p((n/2)-\sqrt{N}it)}{\Gamma_p(n/2)} \frac{|\Sigma\Sigma_0^{-1}|^{-\sqrt{N}it}}{|I-2itN^{-\frac{1}{2}}\Sigma\Sigma_0^{-1}|^{(N/2)-\sqrt{N}it}}$$

$$C^{(2)}_{K_2}(t) = \operatorname{etr}\sqrt{N}it\, \Sigma_0^{-1}(\mu-\mu_0)(\mu-\mu_0)'\{I + 2itN^{-\frac{1}{2}}\, \Sigma_0^{-1}(\Sigma^{-1}-2itN^{-\frac{1}{2}}\Sigma_0^{-1})^{-1}\}.$$

The first factor $C^{(1)}_{K_2}(t)$ has a similar expression to the characteristic function $C_{K_1}(t)$ in (2.11). The same computation gives us the following expression

$$(3.7)\quad \log C^{(1)}_{K_2}(t) = \sqrt{N}it\{\operatorname{tr}(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}|\} - t^2\operatorname{tr}(\Sigma\Sigma_0^{-1} - I)^2$$
$$+ \sum_{r=1}^{2k} \frac{(2it)^r}{N^{r/2}} \left\{\frac{p(p+3)}{4r} + \frac{2p(it)^2}{(r+1)(r+2)} + 2(it)^2\left(\frac{\operatorname{tr}(\Sigma\Sigma_0^{-1})^{r+2}}{r+2} - \frac{\operatorname{tr}(\Sigma\Sigma_0^{-1})^{r+1}}{r+1}\right)\right\}$$
$$- \sum_{r=1}^{k} \frac{(-2)^r B'_{r+1}}{r(r+1)N^r} \left\{\left(1 - \frac{2it}{\sqrt{N}}\right)^{-r} - 1\right\} + O(N^{-k-\frac{1}{2}}).$$

Applying the formula $(I - N^{-\frac{1}{2}} Z)^{-1} = \Sigma_{r=0}^{2k+1} N^{-r/2} Z^r + O(N^{-k-1})$ to the second factor $C^{(2)}_{K_2}(t)$, we have

$$(3.8)\quad \log C^{(2)}_{K_2}(t) = \sqrt{N}it\,(\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0) + \sum_{r=0}^{2k} \frac{2^{r+1}(it)^{r+2}}{N^{r/2}} (\mu-\mu_0)'$$
$$\cdot\, \Sigma_0^{-1}(\Sigma\Sigma_0^{-1})^{r+1}(\mu-\mu_0) + O(N^{-k-\frac{1}{2}}).$$

Hence we have an asymptotic formula for the characteristic function log $C_{K_2}(t)$ by adding the expression (3.7) and (3.8). This shows that the statistic $\lambda^* = -2N^{-\frac{1}{2}} \log \lambda - \sqrt{N}\,\{\mathrm{tr}(\Sigma\Sigma_0^{-1} - I) - \log |\Sigma\Sigma_0^{-1}| + (\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)\}$ is distributed asymptotically according to the normal distribution with mean zero and variance $\tau^2 = 2\{\mathrm{tr}(\Sigma\Sigma_0^{-1} - I)^2 + 2(\mu-\mu_0)'\ \Sigma_0^{-1}\Sigma\Sigma_0^{-1}(\mu-\mu_0)\}$ and further the characteristic function of λ^*/τ can be expanded asymptotically as

$$(3.9)\qquad e^{-t^2/2}\,\{1 + N^{-\frac{1}{2}}A_1 + N^{-1}A_2 + O(N^{-2})\},$$

where the coefficients A_1 and A_2 are given by

$$A_1 = (1/6)\,\{3\tau^{-1}\,it\,p(p+3) + 4\tau^{-3}\,(it)^3\,(p+6d_2+2tr_3 - 3tr_2)\}$$

$$A_2 = (1/72)[9\tau^{-2}(it)^2p(p+3)(p^2+3p+4)+24\tau^{-4}(it)^4\{p(p+1)(p+2) + 6p(p+3)d_2+24d_3+6tr_4+2(p+4)(p-1)tr_3-3p(p+3)tr_2\} +16\tau^{-6}(it)^6(p+6d_2+2tr_3-3tr_2)^2],$$

with the abbreviated notations $d_r = (\mu-\mu_0)'\Sigma_0^{-1}(\Sigma\Sigma_0^{-1})^r(\mu-\mu_0)$ and $tr_j = \mathrm{tr}(\Sigma\Sigma_0^{-1})^j$. Inverting this characteristic function we immediately obtain the following theorem.

<u>Theorem 3.2</u>. Under the alternative K_2: $\Sigma \neq \Sigma_0$ or $\mu \neq \mu_0$, the distribution of the LR criterion $-2 \log \lambda$ given in (3.1) can be expanded asymptotically as

$$(3.10)\qquad P((1/\sqrt{N}\tau)[-2\log\lambda - N\{\mathrm{tr}(\Sigma\Sigma_0^{-1}-I) - \log|\Sigma\Sigma_0^{-1}|+d_0\}] \leq z)$$
$$= \Phi(z) - \frac{1}{6\sqrt{N}}\,[4\tau^{-3}\Phi'''(z)(p+6d_2+2tr_3-3tr_2) + 3\tau^{-1}\Phi'(z)\,p(p+3)]$$
$$+ \frac{1}{72N}\,[16\tau^{-6}\Phi^{(6)}(z)(p+6d_2+2tr_3-3tr_2)^2+24\tau^{-4}\Phi^{(4)}(z)\{p(p+1)(p+2) + 6p(p+3)d_2+24d_3+6tr_4+2(p+4)(p-1)tr_3-3p(p+3)tr_2\} +9\tau^{-2}\Phi''(z)p(p+3)(p^2+3p+4)] + O(N^{-3/2}),$$

where $d_r = (\mu-\mu_0)'\Sigma_0^{-1}(\Sigma\Sigma_0^{-1})^r(\mu-\mu_0)$, $tr_j = tr(\Sigma\Sigma_0^{-1})^j$ and $\tau^2=2\{tr(\Sigma\Sigma_0^{-1}-I)^2+2d_1\}$; $\Phi^{(r)}(z)$ means the rth derivative of the standard normal distribution function $\Phi(z)$.

4. Expansion of the distribution of the criterion for sphericity. The LR statistic for testing the sphericity hypothesis $H_3:\Sigma = \sigma^2 I$ against the alternatives $K_3: \Sigma \neq \sigma^2 I$, where σ^2 is unspecified, from a p-variate random sample of size N with normal distribution having common covariance matrix Σ, is

$$\lambda = |S|^{N/2}\left(\frac{1}{p}trS\right)^{-Np/2}, \tag{4.1}$$

where $S = \Sigma_{\alpha=1}^{N}(X_\alpha-\bar{X})(X_\alpha-\bar{X})'$. Unbiasedness of this test criterion was established by Gleser [2] and Sugiura and Nagao [11], whereas we already know that in order to get unbiasedness in the k-sample situation, the statistic

$$\lambda^* = |S|^{n/2}\left(\frac{1}{p}trS\right)^{-np/2}, \tag{4.2}$$

is preferable instead of λ by Sugiura and Nagao [11]. Under the hypothesis H_3, the hth moment of λ^* is expressed as

$$E[\lambda^{*h}|H_3] = p^{hp}\,\frac{\Gamma(pn/2)\Gamma_p((n/2)+h)}{\Gamma((pn/2)+h)\Gamma_p(n/2)}\ , \tag{4.3}$$

which implies the asymptotic expansion of the null distribution of this criterion as in Anderson [1, p.263].

$$P(-2\rho\log\lambda^* \leq z) = P(\chi_f^2 \leq z) + (1/288p^2m^2)(p+2)(p-1)(p-2) \times(2p^3+6p^2+3p+2)\ \{P(\chi_{f+4}^2 \leq z) - P(\chi_f^2 \leq z)\} + O(m^{-3}), \tag{4.4}$$

where $f = \{p(p+1)/2\}-1$, $m = \rho n$ and $\rho = 1 - (2p^2+p+2)/6pn$. The correction factor ρ is so chosen that the term of order m^{-1} in the expansion vanishes.

It may be remarked that in the previous two sections we cannot use the correction factor ρ as in this case. The reason is that the hth moment of the test statistic has two gamma functions containing h in (4.3), but only one gamma function containing h in (2.3) and (3.2).

We shall now consider the asymptotic expansion of the non-null distribution of the LR criterion $-2\rho\log\lambda^*$. Since the hth moment of λ^* under alternatives cannot be written explicitly, some different technique is necessary. It is provided by using differential operators. Noting that S has the Wishart distribution $W_p(n,\Sigma)$ under K_3, we can write the characteristic function of $-2\rho m^{-\frac{1}{2}}\log\lambda^*$ as

$$(4.5)\qquad C_{K_3}(t) = \frac{1}{\Gamma_p(n/2)2^{np/2}} \int \frac{|S|^{(n-p-1)/2\ -\sqrt{m}it}\ \mathrm{etr}\ -\frac{1}{2}\Sigma^{-1}S}{|\Sigma|^{n/2}\ (\mathrm{tr}\ S/p)^{-\sqrt{m}itp}}\,dS.$$

By the transformation $S\rightarrow HSH'$ for some orthogonal matrix H of order p, we may assume $\Sigma = \Gamma = \mathrm{diag}\ (\lambda_1,\ldots,\lambda_p)$ where λ_j are characteristic roots of Σ. Put $U = (1/m)S$, then the statistic U converges in probability to Γ as m tends to infinity. We can express the characteristic function as

$$(4.6)\qquad C_{K_3}(t) = \frac{m^{np/2}\ p^{-\sqrt{m}itp}}{\Gamma_p(n/2)2^{np/2}|\Gamma|^{n/2}} \int \frac{|U|^{(n-p-1)/2-\sqrt{m}it}}{(\mathrm{tr}\ U)^{-\sqrt{m}itp}}\ \mathrm{etr}(-\tfrac{m}{2}\Gamma^{-1}U)dU.$$

Transform the variable U to D and R by $U=D^{\frac{1}{2}}RD^{\frac{1}{2}}$ such that the matrix D is diagonal and diagonal element is given by that of U. Then $|\partial U/\partial(D,R)| = |D|^{(p-1)/2}$ and $\mathrm{tr}\Gamma^{-1}U = \mathrm{tr}\Gamma^{-1}D$, so we have

$$(4.7)\quad c_{K_3}(t) = \frac{m^{np/2}\, p^{-\sqrt{m}itp}}{\Gamma_p(n/2)2^{np/2}|\Gamma|^{n/2}} \int |R|^{(n-p-1)/2-\sqrt{m}it} dR$$

$$\times \int (\mathrm{tr}D)^{\sqrt{m}itp} |D|^{(n/2)-\sqrt{m}it-1}\ \mathrm{etr}\,(-\tfrac{m}{2}\Gamma^{-1}D)dD$$

$$= \frac{m^{np/2} p^{-\sqrt{m}itp}\ \Gamma_p((n/2)-\sqrt{m}it)}{\Gamma_p(n/2)2^{np/2}|\Gamma|^{n/2}\Gamma((n/2)-\sqrt{m}it)^p} \int (\mathrm{tr}D)^{\sqrt{m}itp} |D|^{(n/2)-\sqrt{m}it-1}$$

$$\mathrm{etr}(-\tfrac{m}{2}\Gamma^{-1}D)dD.$$

Put $f(\Lambda) = (\mathrm{tr}\Lambda)^{\sqrt{m}itp}$ and $\partial = \mathrm{diag}\ (\partial/\partial\lambda_1, \ldots, \partial/\partial\lambda_p)$, where $\Lambda = \mathrm{diag}\ (\lambda_1, \ldots, \lambda_p)$. Noting that the diagonal matrix D converges in probability to Γ as m tends to infinity, we shall expand the function f(D) in the expression (4.7) about $D = \Gamma$, that is, $f(D) = \mathrm{etr}[(D-\Gamma)\partial]f(\Lambda)|_{\Lambda=\Gamma}$. This gives by taking the integration regarding the operator ∂ as constant as in Okamoto [5] etc.,

$$(4.8)\quad \int |D|^{(n/2)-\sqrt{m}it-1}(\mathrm{etr} - \tfrac{m}{2}\Gamma^{-1}D)\ f(D)dD$$

$$= (2/m)^{np/2 -\sqrt{m}itp}\ \Gamma((n/2) -\sqrt{m}it)^p \mathrm{etr}\{-\Gamma\partial - (\tfrac{n}{2} -\sqrt{m}it)\log|\Gamma^{-1} - \tfrac{2}{m}\partial|\} f(\Lambda)|_{\Lambda=\Gamma}.$$

Hence we have the following expression of the characteristic function for $-2\rho\log\lambda^*$.

$$(4.9)\quad c_{K_3}(t) = (\tfrac{m}{2p})^{\sqrt{m}itp} \frac{\Gamma_p((n/2)-\sqrt{m}it)}{\Gamma_p(n/2)|\Gamma|^{\sqrt{m}it}}\ \mathrm{etr}\{-\Gamma\partial-(\tfrac{n}{2} -\sqrt{m}it)\log|I - \tfrac{2}{m}\Gamma\partial|\}$$

$$\cdot f(\Lambda)|_{\Lambda=\Gamma}.$$

The first factor can be expanded by using the asmptotic formula (2.5) for the gamma function with respect to m instead of n as

$$(4.10)\quad \log \Gamma_p((n/2)-\sqrt{m}it)/\Gamma_p(n/2) = -\sqrt{m}itp \log (m/2) - pt^2$$
$$+ (1/6\sqrt{m})\{(p^2+2p-2)it + 4p(it)^3\} + (1/6m)\{(p^2+2p-2)(it)^2+4p(it)^4\} + O(m^{-3/2}),$$

which implies

$$(4.11)\quad \Gamma_p((n/2)-\sqrt{m}it)/\Gamma_p(n/2)=(2/m)^{\sqrt{m}itp}e^{-pt^2}[1 + (1/6\sqrt{m})\{(p^2+2p-2)it$$
$$+ 4p(it)^3\} + (1/72m)\{(p^2+2p-2)(p^2+2p+10)(it)^2 + 8p(p^2+2p+4)(it)^4$$
$$+ 16p^2(it)^6\} + O(m^{-3/2})].$$

The problem is how to evaluate the exponential part of $C_{K_3}(t)$ in (4.9). Since $\partial f(\mathbf{A}) = \sqrt{m}itp\ (\mathrm{tr}\mathbf{A})f(\mathbf{A})$, we must regard the order of ∂ as $\sqrt{m}$. Applying the formula $\log|I-(2/m)\Gamma\partial| = -\Sigma_{r=1}^{2k} (2/m)^r \mathrm{tr}(\Gamma\partial)^r/r + O(m^{-k-\frac{1}{2}})$ to (4.9) we can write

$$(4.12)\qquad \mathrm{etr}\{ - \Gamma\partial - ((n/2)- \sqrt{m}it) \log |I - (2/m)\Gamma\partial|\} f(\mathbf{A})$$
$$= \exp [A_0(\partial) + \frac{1}{\sqrt{m}} A_1(\partial) + \frac{1}{m} A_2(\partial) + O (\frac{1}{m\sqrt{m}})]f(\mathbf{A}),$$

where

$$A_0(\partial) = \frac{1}{m} \mathrm{tr}(\Gamma\partial)^2 - \frac{2it}{\sqrt{m}} \mathrm{tr}\ (\Gamma\partial)$$
$$A_1(\partial) = \frac{4}{3m\sqrt{m}} \mathrm{tr}(\Gamma\partial)^3- \frac{2it}{m} \mathrm{tr}(\Gamma\partial)^2+ \frac{2p^2+p+2}{6p\sqrt{m}}\ \mathrm{tr}\ \Gamma\partial$$
$$A_2(\partial) = \frac{2}{m^2} \mathrm{tr}(\Gamma\partial)^4- \frac{8it}{3m\sqrt{m}} \mathrm{tr}(\Gamma\partial)^3+ \frac{2p^2+p+2}{6mp} \mathrm{tr}\ (\Gamma\partial)^2.$$

Note that the result of applying each term $A_0(\partial)$, $A_1(\partial)$ and $A_2(\partial)$ to $f(\mathbf{A})$ is $O(1).f(\mathbf{A})$. We can expand the expression (4.12) as

$$(4.13)\quad \{ 1 + \frac{A_1(\partial)}{\sqrt{m}} + \frac{1}{m} (A_2(\partial) +\tfrac{1}{2} A_1(\partial)^2) + O(\frac{1}{m\sqrt{m}})\}\{\exp A_0(\partial)\}f(\mathbf{A})\Big|_{\mathbf{A}=\Gamma}\ .$$

We now evaluate the first part - $[\exp A_0(\partial)]f(\mathbf{A})$.

$$(4.14)\qquad \exp\{\frac{1}{m}\operatorname{tr}(\Gamma\partial)^2 - \frac{2it}{\sqrt{m}}\operatorname{tr}\Gamma\partial\}f(\Lambda)$$

$$= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{1}{m}\operatorname{tr}(\Gamma\partial)^2 - \frac{2it}{\sqrt{m}}\operatorname{tr}\Gamma\partial\right)^r f(\Lambda)$$

$$= \left\{\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{k+\ell=r}\binom{r}{k}\frac{(-2it)^{\ell}}{m^{k+(\ell/2)}}\sum_{(k),(\ell)}\frac{k!}{k_1!\dots k_p!}\,\frac{\ell!}{\ell_1!\dots\ell_p!}\prod_{\alpha=1}^{p}\lambda_\alpha^{2k_\alpha+\ell_\alpha}\partial_\alpha^{2k_\alpha+\ell_\alpha}\right\}$$

$$.f(\Lambda),$$

where $\Sigma_{(k),(\ell)}$ means the sum of all possible combinations of non-negative integers $k_1, \dots, k_p$ and $\ell_1, \dots, \ell_p$ such that $k_1+\dots+k_p=k$ and $\ell_1+\dots+\ell_p=\ell$. Since the equality $\prod_{\alpha=1}^{p}\partial_\alpha^{2k_\alpha+\ell_\alpha} f(\Lambda) = (\sqrt{m}itp)_{2k+\ell}(\operatorname{tr}\Lambda)^{\sqrt{m}itp-2k-\ell}$ holds where $(a)_k = a(a-1)\dots(a-k+1)$, we can simplify the above expression as

$$(4.15)\qquad \sum_{r=0}^{\infty}\frac{1}{r!}\sum_{k+\ell=r}\binom{r}{k}\frac{(-2it)^{\ell}(\sqrt{m}itp)_{r+k}}{m^{k+(\ell/2)}}\left\{\frac{\operatorname{tr}\Gamma^2}{(\operatorname{tr}\Lambda)^2}\right\}^k\left(\frac{\operatorname{tr}\Gamma}{\operatorname{tr}\Lambda}\right)^{\ell}(\operatorname{tr}\Lambda)^{\sqrt{m}itp}\ .$$

We can easily see that $(\sqrt{m}itp)_{r+k}$ can be expanded asymptotically with respect to m as

$$(4.16)\qquad (\sqrt{m}itp)_{r+k} = (\sqrt{m}itp)^{r+k} - (\sqrt{m}itp)^{r+k-1}(1/2)\{(k)_2+2rk+(r)_2\}$$

$$+ (\sqrt{m}itp)^{r+k-2}(1/24)\{3(k)_4+4(3r+2)(k)_3+6r(3r+1)(k)_2+12r^2(r-1)k$$

$$+ r(r-1)(3r-1)(r-2)\} + O(m^{(r+k-3)/2}),$$

which makes it possible to simplify the summation in (4.15) giving

$$(4.17)\qquad \{\exp A_0(\partial)\}f(\Lambda) = \{\exp B_0(\Lambda)\}\{1 + \frac{B_1(\Lambda)}{\sqrt{m}} + \frac{B_2(\Lambda)}{m}\}\ f(\Lambda),$$

$$B_0(\Lambda) = -pt^2\left(p\frac{\operatorname{tr}\Gamma^2}{(\operatorname{tr}\Lambda)^2} - 2\frac{\operatorname{tr}\Gamma}{\operatorname{tr}\Lambda}\right)$$

$$B_1(\Lambda) = -2(it)^3p\left(p\frac{\operatorname{tr}\Gamma^2}{(\operatorname{tr}\Lambda)^2} - \frac{\operatorname{tr}\Gamma}{\operatorname{tr}\Lambda}\right)^2 - itp\frac{\operatorname{tr}\Gamma^2}{(\operatorname{tr}\Lambda)^2}$$

$$B_2(\Lambda) = \frac{1}{6}\Big[12p^2(it)^6\Big(p\,\frac{tr\Gamma^2}{(tr\Lambda)^2} - \frac{tr\Gamma}{tr\Lambda}\Big)^4 + 4p(it)^4\Big(p\,\frac{tr\Gamma^2}{(tr\Lambda)^2} - \frac{tr\Gamma}{tr\Lambda}\Big)^2$$

$$\times\Big(13p\,\frac{tr\Gamma^2}{(tr\Lambda)^2} - 4\,\frac{tr\Gamma}{tr\Lambda}\Big) + 3(it)^2\Big\{11p^2\,\frac{(tr\Gamma^2)^2}{(tr\Lambda)^4} - 8p\,\frac{tr\Gamma^2 tr\Gamma}{(tr\Lambda)^3}\Big\}\Big].$$

Applying the operator $[1 + m^{-\frac{1}{2}} A_1(\partial) + m^{-1}(A_2(\partial) + A_1(\partial)^2/2)]$ to this expression we can see that the exponential part of (4.9) can be written asymptotically as

$$(4.18)\quad \{1 + \frac{1}{\sqrt{m}}(B_1(\Lambda)+A_1(\partial)) + \frac{1}{m}(B_2(\Lambda)+A_1(\partial)B_1(\Lambda)+A_2(\partial)+A_1(\partial)^2/2)$$

$$+ O(m^{-3/2})\}.\{\exp B_0(\Lambda)\}\, f(\Lambda)\,\big|_{\Lambda=\Gamma}.$$

Noting that the formulas

$$m^{-\ell/2}tr(\Gamma\partial)^{\ell}\, e^{B_0(\Lambda)} f(\Lambda) = (itp)^{\ell}(tr\Lambda)^{\sqrt{m}itp-\ell}\, e^{B_0(\Lambda)}\, tr\Gamma^{\ell}$$

$$\cdot\Big[1 + \frac{1}{\sqrt{m}itp}\Big\{-\frac{\ell(\ell-1)}{2} + 2\ell pt^2\Big(p\,\frac{tr\Gamma^2}{(tr\Lambda)^2} - \frac{tr\Gamma}{tr\Lambda}\Big)\Big\} + O(\tfrac{1}{m})\Big]$$

$$m^{-(k+\ell)}tr(\Gamma\partial)^k tr(\Gamma\partial)^{\ell}\, e^{B_0(\Lambda)} f(\Lambda) = (itp)^{k+\ell} tr\Gamma^k tr\Gamma^{\ell}\, e^{B_0(\Lambda)}(tr\Lambda)^{\sqrt{m}itp-k-\ell}$$

$$\{1 + O(m^{-\frac{1}{2}})\},$$

hold for any non-negative integers k and ℓ, we can compute each term in (4.8), obtaining

$$(4.19)\qquad \exp\{-\Gamma\partial - (\tfrac{n}{2} - \sqrt{m}it)\log|I - \tfrac{2}{m}\Gamma\partial|\}\, f(\Lambda)\,\big|_{\Lambda=\Gamma}$$

$$= e^{B_0(\Gamma)} f(\Gamma)\,\{1 + \frac{1}{\sqrt{m}}C_1(\Gamma) + \frac{1}{m}C_2(\Gamma) + O(\frac{1}{m\sqrt{m}})\}$$

$$C_1(\Gamma) = (2/3)(it)^3 p\{-3(t_2-1)^2 + 2t_3 - 3t_2\} + (1/6)it(2p^2+p+2-6t_2)$$

$$C_2(\Gamma)=(2/9)p^2(it)^6\{3(t_2-1)^2+3t_2-2t_3\}^2+(1/9)p(it)^4\{3(t_2-1)^2(26t_2-2p^2-p-10)$$

$$+6(t_2-1)(15t_2-14t_3) + 18t_4 + 2(2p^2+p-16)t_3-3(2p^2+p-4)t_2\}$$

$$+ (1/72)(it)^2 \{396t_2^2 - 288t_3 - 24(2p^2+p+8)t_2+ (2p^2+p+2)(2p^2+p+26)\}.$$

with the abbreviated notation $t_j = p^{j-1}(\mathrm{tr}\Gamma^j)/(\mathrm{tr}\Gamma)^j$. Combining this result with the equation (4.11), we can finally obtain the asymptotic formula for the characteristic function $C_{K_3}(t)$ of the statistic $-2\rho m^{-\frac{1}{2}}\log\lambda$ given by (4.2).

$$(4.20)\qquad C_{K_3}(t) = \exp\left[\sqrt{m}it\log\{(\tfrac{1}{p}\mathrm{tr}\Gamma)^p/|\Gamma|\} - pt^2(p\frac{\mathrm{tr}\Gamma^2}{(\mathrm{tr}\Gamma)^2} - 1)\right]$$

$$\{1 + \frac{1}{\sqrt{m}}D_1(\Gamma) + \frac{1}{m}D_2(\Gamma) + O(\frac{1}{m\sqrt{m}})\}$$

$$D_1(\Gamma) = (2p/3)(it)^3\{-3(t_2-1)^2+2t_3-3t_2+1\}+(1/2)it(p^2+p-2t_2)$$

$$D_2(\Gamma) = (2/9)p^2(it)^6[\{3(t_2-1)^2+3t_2-2t_3\}^2 + 1 - 6(t_2-1)^2+4t_3-6t_2]$$

$$+(p/3)(it)^4\{(t_2-1)^2(26t_2-3p^2-3p-8) + 2(t_2-1)(15t_2-14t_3)$$

$$+ 6t_4+2(p+3)(p-2)t_3-(3p^2+3p-4)t_2+p^2+p+2\}$$

$$+(1/24)(it)^2\{132t_2^2-96t_3-4(5p^2+4p+14)t_2+3p^4+6p^3+23p^2+16p+8\},$$

which implies that the limiting distribution of the statistic $\lambda^{**} = m^{-\frac{1}{2}}[-2\rho\log\lambda^* - m\log\{\mathrm{tr}(\Gamma)/p)^p/|\Gamma|\}]$ is normal with mean 0 and variance $\tau^2 = 2p\{p(\mathrm{tr}\Gamma^2)/(\mathrm{tr}\Gamma)^2 - 1\}$ as m tends to infinity. This result has already been obtained by Olkin and Siotani [6]. Gleser [2] also gives a similar result, but his result seems to be incorrect. By inverting the characteristic function of λ^{**}/τ, we have the following theorem, from (4.20).

Theorem 4.1. Under the alternative K_3: $\Sigma \neq \sigma^2 I$, the asymptotic expansion of the power function of the LR criterion $-2\rho\log\lambda^*$ given in (4.2) for sphericity is expressed as

$$(4.21)\qquad P((1/\sqrt{m}\tau)[-2\rho\log\lambda^* - m\log\{(\mathrm{tr}\Sigma/p)^p/|\Sigma|\}] \leq z)$$

$$= \Phi(z) - \frac{1}{\sqrt{m}}[(2p/3\tau^3)\Phi'''(z)\{-3(t_2-1)^2+2t_3-3t_2+1\}+(1/2\tau)\Phi'(z)(p^2+p-2t_2)]$$

$$+\frac{1}{m}[(2p^2/9\tau^6)\Phi^{(6)}(z)\{[3(t_2-1)^2+3t_2-2t_3]^2+1-6(t_2-1)^2+4t_3-6t_2\}$$

$$+(p/3\tau^4)\Phi^{(4)}(z)\{(t_2-1)^2(26t_2-3p^2-3p-8)+2(t_2-1)(15t_2-14t_3)$$

$$+6t_4+2(p+3)(p-2)t_3-(3p^2+3p-4)t_2+p^2+p+2\}$$

$$+(1/24\tau^2)\Phi''(Z)\{132t_2^2-96t_3-4(5p^2+4p+14)t_2+3p^4+6p^3+23p^2+16p+8\}],$$

where $t_j = p^{j-1}(\mathrm{tr}\Sigma^j)/(\mathrm{tr}\Sigma)^j$, $\tau^2=2p(t_2-1)$ and $\Phi^{(r)}(z)$ means the rth derivative of the standard normal distribution function $\Phi(z)$.

5. Limiting non-null distribution of the LR criterion for $\Sigma_1=\ldots=\Sigma_k$. Let $X_{\alpha 1}, \ldots, X_{\alpha N_\alpha}$ be a random sample from p-variate normal distribution with mean vector μ_α and covariance matrix Σ_α for $\alpha = 1, 2, \ldots, k$. The LR statistic for testing the hypothesis H_4: $\Sigma_1 = \Sigma_2 = \ldots = \Sigma_k$ against the alternatives K_4: $\Sigma_i \neq \Sigma_j$ for some i, j $(i \neq j)$, is given by

$$(5.1)\qquad \lambda = \{\Pi_{\alpha=1}^{k} |S_\alpha/N_\alpha|^{N_\alpha/2}\} / |S/N|^{N/2},$$

where $S_\alpha = \Sigma_{j=1}^{N_\alpha}(X_{\alpha j}-\bar{X}_\alpha)(X_{\alpha j}-\bar{X}_\alpha)'$, $\bar{X}_\alpha = \Sigma_{j=1}^{N_\alpha} X_{\alpha j}/N_\alpha$ and $S = \Sigma_{\alpha=1}^{k} S_\alpha$, $N = \Sigma_{\alpha=1}^{k} N_\alpha$. If we modify this criterion by reducing the sample size N_α to the degrees of freedom $n_\alpha = N_\alpha - 1$,

$$(5.2)\qquad \lambda^* = \{\Pi_{\alpha=1}^{k}|S_\alpha/n_\alpha|^{n_\alpha/2}\} / |S/n|^{n/2}$$

with $n = \Sigma_{\alpha=1}^{k} n_\alpha$, we have an unbiased test in the univariate case (Pitman [7]), and in the two sample case for arbitrary p (Sugiura dnd Nagao [11]). The asymptotic expansion of the distribution of this criterion under the hypothesis is stated in Anderson [1, p. 255]. The limiting non-null distribution of the statistic $-2n^{-\frac{1}{2}} \log \lambda^*$ can be obtained from the characteristic function by the same argument as in the previous section. However, so far as the limiting distribution is concerned, it is simpler to use the following lemma, which is a direct extension of Siotani and Hayakawa [9].

Lemma. Let $n_\alpha U_\alpha$ have the Wishart distribution $W_p(n_\alpha, \Sigma_\alpha)$ and $n_\alpha = \rho_\alpha n$ for fixed ρ_α such that $\Sigma_{\alpha=1}^{k} \rho_\alpha = 1$. Suppose a real-valued function $f(U_1,\ldots,U_k)$ is continuously differentiable with respect to each variable. Then the statistic

$$(5.3) \qquad \sqrt{n}\{ f(U_1,\ldots,U_k) - f(\Sigma_1,\ldots,\Sigma_k)\}$$

is distributed asymptotically for large n according to the normal distribution with mean zero and variance $2\Sigma_{\alpha=1}^{k} \rho_\alpha^{-1} \operatorname{tr}\{(\partial^{(\alpha)}f)\Sigma_\alpha\}^2$, where $\partial^{(\alpha)}f$ means the symmetric matrix having $\{(1+\delta_{ab})/2\}\ \partial f/\partial u_{ab}^{(\alpha)}$ as its (a,b) element for $U_\alpha = (U_{ab}^{(\alpha)})$ and Kronecker delta δ_{ab}.

Putting $f(U_1,\ldots,U_k) = \log|\Sigma_{\alpha=1}^{k} \rho_\alpha U_\alpha| - \Sigma_{\alpha=1}^{k} \rho_\alpha \log|U_\alpha|$ in the above lemma and noting that the equality $(\{(1+\delta_{ij})/2\}\partial \log|A|/\partial a_{ij}) = A^{-1}$ holds for any positive definite matrix A, we have the following theorem.

Theorem 5.1. For testing the hypothesis H_4: $\Sigma_1 = \ldots = \Sigma_k$ against all alternatives, the non-null distribution of the modified LR statistic.

$$(5.4) \qquad -\frac{2}{\sqrt{n}} [\log\lambda^* - n \log\{|\tilde{\Sigma}|/\Pi_{\alpha=1}^{k} |\Sigma_\alpha|^{\rho_\alpha}\}]$$

is asymptotically normal with mean zero and variance $2\Sigma_{\alpha=1}^{k}\rho_\alpha \mathrm{tr}(\Sigma_\alpha\tilde{\Sigma}^{-1}-I)^2$,

where $\tilde{\Sigma} = \Sigma_{\alpha=1}^{k}\rho_\alpha\Sigma_\alpha$.

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