BOUNDS ON THE CHROMATIC AND ACHROMATIC NUMBERS OF COMPLIMENTARY GRAPHS

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ABSTRACT

In the present note, exact upper bounds on the sums of chromatic and achromatic numbers of complimentary graphs are determined which prove in particular a conjecture by Hedetniemi (1966) and imply a bound due to Nordhaus and Gaddum (1956).

1. INTRODUCTION

The graphs considered below are assumed to be non-null, finite, undirected which are to have no loops and no multiple edges.

A graph G consists of a set V(G) of vertices together with a set E(G) of unordered pairs [u,v] of distinct vertices u, v $\in$ V(G). The elements of E(G) are called edges of the graph G. If [u,v] is an edge of G, the vertices u and v are said to be adjacent. The order of a graph G is the number of its vertices. For any S $\subseteq$ V(G), the subgraph G' of G, induced by S, is defined as follows: V(G') = S and for any u, v $\in$ S, [u,v] $\in$ E(G') if and only if [u,v] $\in$ E(G). Two graphs G and $\overline{G}$ are called complimentary if they have the same set of vertices and any two vertices are adjacent in one of G or $\overline{G}$ but not in both, i.e., V(G) = V($\overline{G}$) and [u,v] $\in$ E(G) if and only if [u,v] $\notin$ E($\overline{G}$).

Consider a graph G and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ represent k distinct colors. Any function f which assigns to each vertex v of G a unique color $f(v) \in \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is called coloring or more specifically a k-coloring of G. If $f(v) = \alpha$, we say that v is colored α or that v is an α-vectex. For any S $\subseteq$ V(G), f(S) denotes the set of colors $\{f(v)/v \in S\}$. Any k-coloring f of G induces a decomposition of V(G),

$$V(G) = V_1 \cup V_2 \cup \ldots \cup V_k,\ V_i \cap V_j = \phi,\ i \neq j, \tag{1.1}$$

where V_i is precisely the set of all α_i-vectices. Conversely, any decomposition (1.1) of V(G) induces a k-coloring f of G such that $f(v) = \alpha_i$ whenever $v \in V_i$, $1 \leq i \leq k$. Thus, there is a natural 1-1 correspondence between k-colorings of G and decompositions of V(G) into k mutually disjoint sets.

In the present note, we shall consider k-colorings f of a graph G with (1.1) as the induced decomposition of V(G) which satisfy one or both of the following two conditions:

(R) For any two vertices u and v of G, $[u,v] \in E(G)$ implies $f(u) \neq f(v)$ or, equivalently, for each i, $1 \leq i \leq k$, u, v $\in V_i$ implies u and v are not adjacent.

(C) For any two colors α_i and α_j, $1 \leq i < j \leq k$, there is an edge $[u,v] \in E(G)$ with $f(u) = \alpha_i$ and $f(v) = \alpha_j$ or, equivalently, for each i and j, $1 \leq i < j \leq k$, there exist vertices $u \in V_i$, $v \in V_j$ such that u and v are adjacent.

A k-coloring f of G is called regular, pseudo-complete or complete according as it satisfies the condition (R), (C) or both (R) and (C), respectively. The chromatic number of G, denoted by $\chi(G)$, is the minimum number k for which a regular k-coloring of G exists. The pseudo-achromatic number of G, denoted by $\psi_s(G)$, is defined to be the maximum number k for which a pseudo-complete k-coloring of G exists. Finally, the achromatic number of G, denoted by $\psi(G)$, is the maximum number k for which a complete k-coloring of G exists. From the definitions, it is obvious that for any graph G, $\psi(G) \leq \psi_s(G)$. Also, if $\chi(G) = k$ and f is any regular k-coloring of G, then it is easily seen that f must also be pseudo-complete so that clearly $\chi(G) \leq \psi(G)$. Hence, for any graph G, we always have

$$\chi(G) \leq \psi(G) \leq \psi_s(G). \tag{1.2}$$

Further, there exist graphs G for which the strict inequalities $\chi(G) < \psi(G)$ and/or $\psi(G) < \psi_s(G)$ may hold. For instance, if G is the graph of Figure 1, then, it is easily verified that $\chi(G) = 2$, $\psi(G) = 3$ and $\psi_s(G) = 4$.

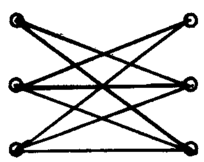

Figure 1.

From the above observations, it is evident that the concept of pseudo-achromatic number of a graph as defined here is a proper generalization of the concept of achromatic number of a graph, introduced by Hedetniemi [1].

Let G and $\overline{G}$ be complimentary graphs defined on a set of p vertices. Then it is known [2] that

$$(1.3) \qquad 2\sqrt{p} \leq \chi(G) + \chi(\overline{G}) \leq p + 1.$$

In the present note, we determine the following upper bounds:

$$(1.4) \qquad \chi(G) + \psi_s(\overline{G}) \leq p + 1,$$

$$(1.5) \qquad \psi_s(G) + \psi_s(\overline{G}) \leq \{\tfrac{4}{3} p\}.$$

The bounds (1.4) and (1.5) are shown to be exact. As a corollary to (1.4), we obtain the bound

$$(1.6) \qquad \chi(G) + \psi(\overline{G}) \leq p + 1,$$

conjectured by Hedetriemi [1] and also the upper bound in (1.3) due to Nordhaus and Gaddum [2]. From (1.5), we obtain the (exact) upper bound

$$(1.7) \qquad \psi(G) + \psi(\overline{G}) \leq \{\tfrac{4}{3} p\}$$

which answers a question by Hedetniemi [1] in the negative.

2. BOUNDS

In the following, $|A|$ denotes, as usual, the number of elements in the set A; $[x]$ denotes the integral part of the number x and $\{x\}$ is the smallest integer greater than or equal to x.

We first prove the following

Theorem 2.1. If G and $\overline{G}$ are complimentary graphs of order p, then

$$(2.1) \qquad \chi(G) + \psi_s(\overline{G}) \leq p + 1,$$

$$(2.2) \qquad \chi(G) + \psi(\overline{G}) \leq p + 1,$$

$$(2.3) \qquad \chi(G) + \chi(\overline{G}) \leq p + 1.$$

For any $p \geq 1$, the bounds (2.1), (2.2) and (2.3) are attainable.

<u>Proof:</u> We shall first prove (2.1). The bounds (2.2) and (2.3) then follow immediately from (2.1) and (1.2).

Let G and $\overline{G}$ be complimentary graphs of order p, and let $\psi_s(\overline{G}) = k$. Obviously, $1 \leq k \leq p$. If $k = 1$, then since clearly $\chi(G) \leq p$, we have the inequality (2.1). We may therefore assume that $k > 1$. Now, consider any pseudo-complete k-coloring of $\overline{G}$ and let

(2.4) $$V(G) = V(\overline{G}) = V_1 \cup V_2 \cup \ldots \cup V_k, \; V_i \cap V_j = \phi, \quad i \neq j,$$

be the induced decomposition of V(G). Let G_r denote the subgraph of G induced by the set of vertices $V(G_r) = V_1 \cup V_2 \cup \ldots \cup V_r$ and let $|V(G_r)| = p_r$, $r = 1, 2, \ldots, k$. We assert that $\chi(G_r) \leq p_r - r + 1$.

To prove the assertion, it is evidently sufficient to show that G_r possesses a $(p_r - r + 1)$-coloring f_r which is regular. Now, for $r = 1$, we may define f_1 by assining p_1 distinct colors to the vertices of G_1 so that any two vertices are colored differently. Let us assume as induction hypothesis that we have already defined a (fixed) regular $(p_r - r + 1)$-coloring f_r of G_r for some r, $1 \leq r < k$. Now consider the graph G_{r+1}. Since the decomposition (2.4) is induced by a pseudo-complete coloring of $\overline{G}$, by definition it is clear that there exist vertices $v_1 \in V_1$, $v_2 \in V_2$, $\ldots$, $v_r \in V_r$ such that for some vertices $u_i \in V_{r+1}$ $(1 \leq i \leq r)$ which need not all be distinct, $[v_i, u_i] \in E(\overline{G})$ so that clearly $[v_i, u_i] \notin E(G_{r+1})$. Now, we define a coloring f_{r+1} of G_{r+1} by using the coloring f_r of G_r. If each of the colors $f_r(v_1)$, $f_r(v_2)$, $\ldots$, $f_r(v_r)$ is also assigned by f_r to some vertex in $V(G_r) - \{v_1, v_2, \ldots, v_r\}$, then evidently the number of colors actually used by f_r does not exceed $|V(G_r) - \{v_1, v_2, \ldots, v_r\}| = p_r - r$. In this case, we define f_{r+1} as follows: assign $|V_{r+1}| = p_{r+1} - p_r$ new colors to the vertices in V_{r+1} and let $f_{r+1}(v) = f_r(v)$ for the rest of the vertices $v \in V(G_{r+1})$. And, if there is a color α among $f_r(v_1)$, $f_r(v_2)$, ..., $f_r(v_r)$ which is not assigned by f_r to any vertex in $V(G_r) - \{v_1, v_2, \ldots, v_r\}$ so that all the α-vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ $(1 \le t \le r)$, say, are among v_1, v_2, ..., v_r, then, we define f_{r+1} as follows: assign $p_{r+1} - p_r$ new colors to the vertices in V_{r+1}; delete the color α by putting $f_{r+1}(v_{i_j}) = f_{r+1}(u_{i_j})$, $1 \le j \le t$, where $u_{i_j} \in V_{r+1}$ is such that $[v_{i_j}, u_{i_j}] \notin E(G_{r+1})$; for all other vertices $v \in V(G_{r+1})$, let $f_{r+1}(v) = f_r(v)$. In either case, clearly, f_{r+1} is a $(p_{r+1} - \overline{r+1} + 1)$-coloring of G_{r+1}, and it is easily verified that f_{r+1} is regular. The assertion is now proved by finite induction. In particular, since $G_k = G$, we have $\chi(G) \le p-k+1$ from which we obtain (2.1) immediately.

To see that the bounds (2.1)-(2.3) are attainable, it is sufficient to let G be a complete graph of order p, i.e., G has p vertices each pair of which is adjacent in G so that no two vertices are adjacent in $\overline{G}$. Then, clearly $\chi(G) = p$, $\chi(\overline{G}) = \psi(\overline{G}) = \psi_s(\overline{G}) = 1$ and the equalities in (2.1)-(2.3) hold. This completes the proof of the theorem.

We shall now prove the following

<u>Theorem 2.2</u>: If G and $\overline{G}$ are complimentary graphs of order p, then

(2.5) $$\psi_s(G) + \psi_s(\overline{G}) \le \{\tfrac{4}{3} p\},$$

(2.6) $$\psi(G) + \psi_s(\overline{G}) \le \{\tfrac{4}{3} p\},$$

(2.7) $$\psi(G) + \psi(\overline{G}) \le \{\tfrac{4}{3} p\}.$$

For any $p \ge 1$, the bounds (2.5), (2.6) and (2.7) are attainable.

<u>Proof</u>: We shall first prove (2.5). The bounds (2.6) and (2.7) then follow immediately from (2.5) and (1.2).

Let G and $\bar{G}$ be complimentary graphs of order p, and let $\psi_s(G) = k$. Obviously, $1 \leq k \leq p$. If $k = 1$, then since clearly $\psi_s(\bar{G}) \leq p$, we have the inequality (2.5). We may therefore assume that $k > 1$. Now, consider any pseudo-complete k-coloring of G and let

(2.8) $$V(G) = V(\bar{G}) = V_1 \cup V_2 \cup \ldots \cup V_k, \quad V_i \cap V_j = \phi, \ i \neq j,$$

be the induced decomposition of $V(\bar{G})$. Clearly, $V_i \neq \phi$ for $i = 1, 2, \ldots, k$. Let the number of sets among $V_1, V_2, \ldots, V_k$ which consist of exactly one vertex each be r where $r \geq 0$. To be definite, we may assume that $V_i = \{v_i\}$, $i = 1, 2, \ldots, r$ (if $r > 0$). Then, the remaining sets $V_{r+1}, \ldots, V_k$ consist of at least two vertices each. Since the sets V_i are mutually disjoint, we have evidently

(2.9) $$k \leq r + [\tfrac{p-r}{2}].$$

Now, let $\psi_s(\bar{G}) = k'$ and f be any pseudo-complete k'-coloring of $\bar{G}$. Since the decomposition (2.8) is induced by a pseudo-complete coloring of G, it is observed that each pair of vertices in $\{v_1, v_2, \ldots, v_r\}$ must be adjacent in G so that no two vertices in $\{v_1, v_2, \ldots, v_r\}$ are adjacent in $\bar{G}$. Therefore, since f is a pseudo-complete coloring of $\bar{G}$, it is easily seen that there can be at most one color in $\{f(v_1), f(v_2), \ldots, f(v_r)\}$ which is not assigned by f to any vertex in $V(\bar{G}) - \{v_1, v_2, \ldots, v_r\}$. Hence, we have evidently, $k' \leq |V(\bar{G}) - \{v_1, v_2, \ldots, v_r\}| + 1$, or

(2.10) $$k' \leq p - r + 1.$$

We shall next prove the following inequality.

(2.11) $$k' \leq p - [\tfrac{k}{2}].$$

To this end, consider the sets of colors $f(V_1), f(V_2), \ldots, f(V_k)$. If for each index i, $1 \leq i \leq k$, we have either $|f(V_i)| < |V_i|$ or $f(V_i) \cap f(V_j) \neq \phi$ for some j, $j \neq i$, then it is easily seen (by induction on k) that

$k' = |\bigcup_{i=1}^{k} f(V_i)| \leq |\bigcup_{i=1}^{k} V_i| - [\frac{k}{2}] = p - [\frac{k}{2}]$ and hence, a fortiori, we have the inequality (2.11). We may therefore assume that $f(V_1)$, say, is such that $|f(V_1)| = |V_1| = p_1$ and $f(V_1) \cap f(V_j) = \phi$ for $2 \leq j \leq k$. Let, if possible, there be another set $f(V_2)$, say, such that $|f(V_2)| = |V_2|$ and $f(V_2) \cap f(V_j) = \phi$ for $j \neq 2$. Now, since there exist vertices $v_1 \in V_1$, $v_2 \in V_2$ such that $[v_1, v_2] \in E(G)$ or $[v_1, v_2] \notin E(\overline{G})$, it is seen that there can be no edge $[v,u] \in E(\overline{G})$ with $f(v) = f(v_1)$ and $f(u) = f(v_2)$. This, however, contradicts the fact that f is a pseudo-complete coloring of $\overline{G}$. Hence, we must have for every index i, $2 \leq i \leq k$, either $|f(V_i)| < |V_i|$ or $f(V_i) \cap f(V_j) \neq \phi$ for some j, $j \neq i$. Hence, as above, we have $k' - p_1 = |\bigcup_{i=2}^{k} f(V_i)| \leq |\bigcup_{i=2}^{k} V_i| - [\frac{k-1}{2}] = p - p_1 - [\frac{k}{2}]$ whence we obtain (2.11) immediately.

Now, we shall derive (2.5) from (2.9), (2.10) and (2.11). If $r \geq [\frac{p}{3}] + 1$, then from (2.9) and (2.10), we obtain $\psi_s(G) + \psi_s(\overline{G}) = k + k' \leq p + [\frac{p-r}{2}] + 1 \leq \{\frac{4}{3} p\}$. (The last inequality is obtained by substituting $r = [\frac{p}{3}] + 1$ and some elementary simplification.) If $r \leq [\frac{p}{3}] + 1$, then from (2.9) and (2.11) we obtain similary $\psi_s(G) + \psi_s(\overline{G}) \leq \{\frac{4}{3} p\}$. This completes the proof of (2.5).

It now remains to show that for any $p \geq 1$, the bounds (2.5)-(2.7) are attainable. For $p \leq 3$, this is obvious. In general, it is evidently sufficient if we show that the bound (2.7) is attainable for all p of the form 3r+1, r=1, 2, ... To this end, we construct below examples of graphs G_r of order 3r+1 successively for r = 1, 2, ... and define (2r+1)-colorings f_r and $\overline{f}_r$ of G_r and $\overline{G}_r$, respectively. For r = 1, we define G_1, f_1, $\overline{f}_1$ as follows: $V(G_1) = \{v_0, v_1, v_2, v_3\}$, $E(G_1) = \{[v_0, v_2], [v_1, v_3], [v_2, v_3]\}$; $f_1(v_0) = f_1(v_1) = \alpha_1$, $f_1(v_2) = \alpha_2$, $f_1(v_3) = \alpha_3$; $\overline{f}_1(v_0) = \alpha_1$, $\overline{f}_1(v_1) = \alpha_2$, $\overline{f}_1(v_2) = \overline{f}_1(v_3) = \alpha_3$. Suppose that we have already defined G_r, f_r, $\overline{f}_r$ for some $r \geq 1$. Now, we define G_{r+1}, f_{r+1}, $\overline{f}_{r+1}$ as follows: $V(G_{r+1}) = V(G_r) \cup \{v_{3r+1}, v_{3r+2}, v_{3r+3}\}$, $E(G_{r+1}) = E(G_r) \cup \{[v_{3r+1}, v_{3i-1}] \mid i = 1,2,\ldots,r\} \cup \{[v_{3r+2}, v_{3i}] \mid i = 0, 1, \ldots, r+1\} \cup \{[v_{3r+3}, v_{3i-1}], [v_{3r+3}, v_{3i}] \mid i = 1, 2, \ldots, r\} \cup \{[v_{3r+3}, v_1]\}$; $f_{r+1}(v_{3r+1}) = f_{r+1}(v_{3r+2}) = \alpha_{2r+2}$, $f_{r+1}(v_{3r+3}) = \alpha_{2r+3}$, $f_{r+1}(v_i) = f_r(v_i)$ for $i = 0, 1, \ldots, 3r$; $\overline{f}_{r+1}(v_{3r+1}) = \alpha_{2r+2}$, $\overline{f}_{r+1}(v_{3r+2}) = \overline{f}_{r+1}(v_{3r+3}) = \alpha_{2r+3}$, $\overline{f}_{r+1}(v_i) = \overline{f}_r(v_i)$ for $i = 0, 1, \ldots, 3r$. It is easily checked that f_r and $\overline{f}_r$ are complete $(2r+1)$-colorings of G_r and $\overline{G}_r$, respectively. Hence, by definition, we have $\psi(G_r) + \psi(\overline{G}_r) \geq 2(2r+1) = \{\frac{4}{3}(3r+1)\}$. But, from (2.7), the equality $\psi(G_r) + \psi(\overline{G}_r) = \{\frac{4}{3}(3r+1)\}$ must hold.

This completes the proof of the theorem.

Remark: The research work presented in this note was motivated by the suggestions made by Hedetniemi [1]. He conjectured that for any complimentary graphs G and $\overline{G}$ of order p, we have: (1) $\psi(G) + \psi(\overline{G}) \leq p+1$, and further, asked the question: (2) $\psi(G) + \psi(\overline{G}) \leq p+2$? Clearly, Theorem 2.1 yields a stronger result than (1) and Theorem 2.2 answers the question (2) in the negative.

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