A NOTE ON THE P-ARY REPRESENTATION OF INTEGERS

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S. Ikeda and I. Maeda

Dept. of Statistics, Nihon Univ. and Dept. of Economics, Dokkyo College

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This article gives a new form of the p-ary representation of any non-negative integer for prime p, which is a generalization of Ikeda's formula for $p = 2$ [1].

For any non-negative integer N, the usual formula of the binary representation is given by

$$N = \sum_{k=o}^{s} C_k . 2^k$$

where $C_k = \left[\frac{N}{2^k}\right] - 2\left[\frac{N}{2^{k+1}}\right]$, $k=o, 1, \ldots, s$; $s = \max\{h;\ 2^h \leqq N\}$.

On the other hand, Ikeda [1] showed that N is expanded in the form

$$N = \sum_{k=o}^{s} \binom{N}{2^k}_2 .\ 2^k$$

where, for any positive integer R, $R_2 = o$ if R is even, and $= 1$ if R is odd. This implies, by the uniqueness of the binary representation, that

(1) $$\frac{N}{2^k} = \binom{N}{2^k} \quad (\text{mod. } 2).$$

The purpose of this article is to prove the following identity.

$$\left[\frac{N}{p^k}\right] - p\left[\frac{N}{p^{k+1}}\right] = \binom{N}{p^k}_p \ , \tag{2}$$

or, equivalently,

$$\left[\frac{N}{p^k}\right] = \binom{N}{p^k} \qquad (\text{mod. } p) \ , \tag{3}$$

where N and k are non-negative integers, p is any given prime, [ ] is the ordinary Gauss symbol, and $\binom{N}{n}_p$ denotes the remainder when $\binom{N}{n}$ is divided by p. Here, we use the convention that the number of n-combinations out of N, $\binom{N}{n}$, is equal to zero if $n > N$.

PROOF OF (2)

(I) In the case $0 \leqq N < p^k$, it is evident that

$$\left[\frac{N}{p^k}\right] - p\left[\frac{N}{p^{k+1}}\right] = 0 = \binom{N}{p^k}_p$$

for any non-negative integer k.

(II) In the case $p^k \leqq N < p^{k+1}$, N is expressed uniquely in the following form

$$N = jp^k + i,$$

for some integers j and i; $1 \leqq j \leqq p-1$ and $0 \leqq i \leqq p^k - 1$.

Then, for the left hand side of (2) it is easy to see that

$$\left[\frac{jp^k + i}{p^k}\right] - p\left[\frac{jp^k + i}{p^{k+1}}\right] = j$$

for any non-negative k. Thus, putting

$$I^k_{j,i} = \binom{N}{p^k}_p = \binom{jp^k + i}{p^k}_p$$

we have to show that the relation

$$I^k_{j,i} = j \tag{4}$$

holds true for any non-negative integer k.

In the first place, we shall prove the relation (4) when i=o, that is,

$$I^{k}_{j,o} = j, \text{ or equivalently, } \binom{jp^k}{p^k} \equiv j \qquad (\text{mod. } p).$$

Since

$$\binom{jp^k}{p^k} = j\binom{jp^k - 1}{p^k - 1},$$

it suffices to prove that the relation

(5) $$\binom{jp^k - 1}{p^k - 1} \equiv 1 \qquad (\text{mod. } p)$$

holds true for any non-negative k. When k = o, this relation is trivial.

Let us consider the expansion

(6) $$\binom{jp^k - 1}{p^k - 1} = \prod_{h=1}^{p^k-1} \frac{jp^k - h}{h}$$

for any positive k.

In the above expansion, h is expressed in general as

$$h = p^{m(h)} \cdot \delta(h)$$

where $\delta(h)$ is prime to p and m(h) is an integer such that $0 \leqq m(h) \leqq k-1$.

If h is prime to p, then m(h) = 0 and hence h = $\delta(h)$.

Thus, it follows from (6) that

(7) $$\binom{jp^k - 1}{p^k - 1} = \frac{\prod_{h=1}^{p^k-1} (jp^{k-m(h)} - \delta(h))}{\prod_{h=1}^{p^k-1} \delta(h)}$$

Here, we have

$$\prod_{h=1}^{p^k-1} (jp^{k-m(h)} - \delta(h)) = p \cdot Q(p) + (-1)^{p^k-1} \prod_{h=1}^{p^k-1} \delta(h),$$

where Q(p) is a polynomial in p.

Thus, we obtain

$$\binom{jp^k - 1}{p^k - 1} = \frac{p\cdot Q(p)}{\prod_{h=1}^{p^k-1} \delta(h)} + (-1)^{p^k-1}.$$

Recalling that p^k-1 is even when $p \geqq 3$, we thus have,

$$\binom{jp^k - 1}{p^k - 1} = \frac{p\cdot Q(p)}{\prod_{h=1}^{p^k-1} \delta(h)} + 1.$$

Since $\prod_{h=1}^{p^k-1} \delta(h)$ is prime to p, it divides Q(p), and hence we have (5) for $p \geqq 3$.

When p = 2, it holds that

$$\binom{j\cdot 2^k - 1}{2^k - 1} = 2\cdot \frac{Q(2)}{\prod_{h=1}^{2^k-1} \delta(h)} - 1 \equiv 1 \qquad (\text{mod. } 2).$$

Thus, the relation (5) is true for any prime p.

In the second place, we shall prove the relations

(8) $$I^k_{j,1} = I^k_{j,i-1}$$

for $i=1,2,\ldots,p^k-1$ and $j=1,2,\ldots,p-1$, $(k \geqq 1)$.

Putting, as before, $h = p^{m(h)}\cdot\delta(h)$ for $h=1,2,\ldots,p^k-1$, we have

$$\binom{jp^k + i}{p^k} - \binom{jp^k + i - 1}{p^k} = \prod_{h=1}^{p^k-1} \frac{jp^k + i - h}{h}$$

$$= p^{k-m(i)} \cdot \frac{j \prod_{h=1, h\neq i}^{p^k-1} (jp^{k-m(h)} + p^{m(i)-m(h)}\cdot \delta(i) - \delta(h))}{\prod_{h=1}^{p^k-1} \delta(h)}.$$

Since $\prod_{h=1}^{p^k-1} \delta(h)$ is prime to p and $0 \leq m(i) \leq k-1$, the second factor of the last expression of the above equalities should be a positive integer.

Hence

$$\binom{jp^k + i}{p^k} - \binom{jp^k + i - 1}{p^k} = 0 \qquad \pmod{p},$$

which proves (8).

This completes the proof of (4) in the case (II).

(III) In the case $p^{k+1} \leq N$, we may put for some q, j and i,

$$N = qp^{k+1} + jp^k + i,$$

where q is a positive integer, $j = o, 1, \ldots, p-1$ and $i = o, 1, \ldots, p^k - 1$.

Let us put

$$J^k_{j,i} = \binom{qp^{k+1} + jp^k + i}{p^k}_p .$$

Then, by a similar argument in the case (II), we have

$$J^k_{j,o} = J^k_{j,1} = \ldots = J^k_{j,p^k-1}, \qquad j = o, 1, \ldots, p-1$$

for any k positive and

$$J^k_{j,o} = j, \qquad j = 0, 1, \ldots, p-1$$

for any non-negative k. This implies that (2) is true in the case (III).

Hence, in all the cases, we proved the relations (2) for any prime number p and non-negative integer k.

REFFERENCE

[1] S. Ikeka, "A note on the Hamming weight of binary vectors", to appear in the Journal of the College of Industrial Technology, Nihon University, Tokyo, Vol.2, No.1 (1968).