APPROXIMATION OF A MONOTONE FUNCTION

BY BERNSTEIN POLYNOMIALS

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For a real-valued function f on [0,1] the Bernstein polynomial of order n is defined as $B_n^f(x) = \Sigma_{i=0}^n \, f(i/n)\binom{n}{i} x^i (1-x)^{n-i}$. For f Lebesgue-integrable on (0,1) we consider the polynomials $P_n^f(x) = \frac{d}{dx} B_{n+1}^F(x)$, where $F(x) = \int_0^x f(y)dy$. Explicitly (see Lorentz [2], Chapter II),

$$(1) \quad P_n^f(x) = \sum_{i=0}^{n} (n+1) \int_{i/(n+1)}^{(i+1)/(n+1)} f(y)dy \binom{n}{i} x^i (1-x)^{n-i} .$$

In this note the following result is proved.

<u>Theorem</u>. There is a numerical constant C such that if f is monotone and Lebesgue integrable on (0,1), then

$$(2) \int_0^1 | P_n^f(x) - f(x) | dx \leq Cn^{-\frac{1}{2}} \int_0^1 | f(x) - a | x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} dx$$

for n = 1,2,..., where a is an arbitrary real number.

<u>Remark 1</u>. The theorem implies that if F is convex (or concave) and absolutely continuous on [0,1], and $F'(x)\, x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ is integrable, then $\mathrm{var}_{(0,1)}(B_n^F - F) = O(n^{-\frac{1}{2}})$. (I am indebted to Professor G.G. Lorentz for this observation.)

<u>Remark 2</u>. The theorem shows that if f is monotone and

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$$\int_0^1 |f(x)|\, x^{-\frac{1}{2}}\, (1-x)^{-\frac{1}{2}} dx < \infty, \tag{3}$$

then

$$\int_0^1 |P_n^f(x) - f(x)|\, dx = O(n^{-\frac{1}{2}}). \tag{4}$$

By Lemma 1 below, (3) implies square integrability of the monotone function f. The converse does not hold. The author does not know whether square integrability is sufficient for (4). The following two propositions show that a slightly stronger condition than square integrability implies (3), and that a slightly weaker condition is not enough for (4) to hold.

<u>Proposition 1</u>. If f is monotone and

$$\int_0^1 f^2(x)\, \{\log(1 + |f(x)|)\}^{2+\delta}\, dx < \infty \tag{5}$$

for some $\delta > 0$, then (3) holds.

<u>Proposition 2</u>. For every $\delta > 0$ there is a monotone function f such that

$$\int_0^1 f^2(x)\, \{\log(1 + |f(x)|)\}^{-1-\delta}\, dx < \infty \tag{6}$$

and (4) does not hold.

The proof of the theorem is preceded by three lemmas.

<u>Lemma 1</u>. If f in monotone, then

$$\left\{\int_0^1 f^2(x) dx\right\}^{\frac{1}{2}} \leq \int_0^1 |f(x)|\, x^{-\frac{1}{2}}\, (1-x)^{-\frac{1}{2}}\, dx. \tag{7}$$

We may assume that f in nondecreasing. Then there is a number b, $0 \leq b \leq 1$, such that $f(x) \leq 0$ for $x < b$ and $f(x) \geq 0$ for $x > b$. If

$b < x < y < 1$, then $f(x)(1-x)^{-\frac{1}{2}} \le f(y)(1-y)^{-\frac{1}{2}}$, hence

$$f(x)(1-x)^{\frac{1}{2}} \le \int_x^1 f(y)(1-y)^{-\frac{1}{2}}dy \le \int_b^1 f(y)y^{-\frac{1}{2}}(1-y)^{-\frac{1}{2}}dy.$$ Therefore

$$\int_b^1 f^2(x)dx = \int_b^1 f(x)(1-x)^{-\frac{1}{2}}f(x)(1-x)^{\frac{1}{2}}dx \le \left\{\int_b^1 |f(x)|x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx\right\}^2 .$$

An analogous inequality holds with $\int_b^1$ replaced by $\int_0^b$. The two inequalities imply (7).

Lemma 2. There is a numerical constant C_1 such that if f is nondecreasing,

$$(8) \quad \sum_{i=1}^{n-1} \left\{f\left(\frac{i+1}{n+1}\right) - f\left(\frac{i}{n+1}\right)\right\}\left(\frac{i}{n}\right)^{\frac{1}{2}}\left(1-\frac{i}{n}\right)^{\frac{1}{2}} \le C_1\int_0^1 |f(x)|x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx.$$

Denote the sum on the left of (8) by S_n and let $h(x) = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$. Then

$S_n = \Sigma_{i=1}^n f(i/(n+1)\{h((i-1)/n) - h(i/n)\}$. It is easy to show (using $|h'(x)| \le \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}$) that $|h((i-1)/n)-h(i/n)| \le n^{-1}\left(\frac{i}{n+1}\right)^{-\frac{1}{2}}\left(1-\frac{i}{n+1}\right)^{-\frac{1}{2}}$, $1 \le i \le n$.

Hence

$$(9) \quad S_n \le n^{-1}\Sigma_{i=1}^n \left|f\left(\frac{i}{n+1}\right)\right|\left(\frac{i}{n+1}\right)^{-\frac{1}{2}}\left(1-\frac{i}{n+1}\right)^{-\frac{1}{2}}.$$

Due to the mononicity of f, (8) is readily obtained from (9).

Lemma 3. For $0 < x < 1$,

$$(10) \quad \sum_{j=i}^{n}\binom{n}{j} x^j(1-x)^{n-j} \le \left(\frac{i}{n}\right)^{-i}\left(1-\frac{i}{n}\right)^{-n+i} x^i(1-x)^{n-i}, \quad nx \le i \le n-1,$$

$$(11) \quad \sum_{j=0}^{i}\binom{n}{j} x^j(1-x)^{n-j} \le \left(\frac{i}{n}\right)^{-i}\left(1-\frac{i}{n}\right)^{-n+i} x^i(1-x)^{n-i}, \quad 1 \le i \le nx.$$

Inequalities (10) and (11) follow, for instance, from Theorem 1 in [1], where further references are given.

Proof of the Theorem. It is sufficient to prove (2) for f nondecreasing and a = 0 (since the left side does not change if f is replaced by f + const). Let $f_i = f(i/(n+1))$, $1 \leq i \leq n$. Since f is nondecreasing,

$$P_n^f(x) \leq \sum_{i=0}^{n-1} f_{i+1} \binom{n}{i} x^i (1-x)^{n-i} + (n+1) \int_{n/(n+1)}^{1} f(y)dy\, x^n$$

$$= f_1 + \sum_{i=1}^{n-1} (f_{i+1} - f_i) \sum_{j=i}^{n} \binom{n}{j} x^j (1-x)^{n-j} - f_n x^n + (n+1) \int_{n/(n+1)}^{1} f(y)dy\, x^n .$$

If we majorize the sum $\Sigma_{j=i}^{n}$ by 1 for $i \leq [nx]$ and apply (10) to the terms with $i \geq [nx] + 1$, we obtain

$$\text{(12)} \quad P_n^f(x) - f_{[nx]+1} \leq \sum_{i=[nx]+1}^{n-1} (f_{i+1} - f_i) \left(\frac{i}{n}\right)^{-i} \left(1 - \frac{i}{n}\right)^{-n+i} x^i (1-x)^{n-i} - f_n x^n + (n+1) \int_{n/(n+1)}^{1} f(y)dy\, x^n .$$

In a similar way we find that

$$P_n^f(x) \geq f_n - \sum_{i=1}^{n-1} (f_{i+1} - f_i) \sum_{j=0}^{i} \binom{n}{j} x^j (1-x)^{n-j} - f_1 (1-x)^n + (n+1) \int_{0}^{1/(n+1)} f(y)dy (1-x)^n$$

and, applying (11) to the terms with $i \leq [nx]$, that

$$\text{(13)} \quad P_n^f(x) - f_{[nx]+1} \geq - \sum_{i=1}^{[nx]} (f_{i+1} - f_i) \left(\frac{i}{n}\right)^{-i} \left(1 - \frac{i}{n}\right)^{-n+i} x^i (1-x)^{n-i} - f_1 (1-x)^n + (n+1) \int_{0}^{1/(n+1)} f(y)dy\, (1-x)^n .$$

Note that the sum of the last two terms in (12) is nonnegative, so that the right side of (12) is nonnegative. Similarly, the right side of (13) is nonpositive. Hence it follows from (12) and (13) that

$$|P_n^f(x)-f_{[nx]+1}| \le \sum_{i=1}^{n-1} (f_{i+1}-f_i)(\tfrac{i}{n})^{-i}(1-\tfrac{i}{n})^{-n+i}x^i(1-x)^{n-i} + f_1(1-x)^n$$

(14)

$$-(n+1)\int_0^{1/(n+1)} f(y)dy(1-x)^n - f_n x^n + (n+1)\int_{n/(n+1)}^{1} f(y)dy\ x^n$$

By Sterling's formula,

$$(15)\ \int_0^1 x^i(1-x)^{n-i}dx = \frac{i!(n-i)!}{(n+1)!} \le C_2 n^{-\frac{1}{2}}(\tfrac{i}{n})^{i+\frac{1}{2}}(1-\tfrac{i}{n})^{n-i+\frac{1}{2}},\ 1 \le i \le n-1,$$

where C_2 is a numerical constant. Integrating both sides of (14), applying (15), and noting that $f_1 \le f_n$, we obtain

$$\int_0^1 |P_n^f(x) - f_{[nx]+1}|dx \le C_2 n^{-\frac{1}{2}}\sum_{i=1}^{n-1} (f_{i+1}-f_i)(\tfrac{i}{n})^{\frac{1}{2}}(1-\tfrac{i}{n})^{\frac{1}{2}}$$

(16)

$$-\int_0^{1/(n+1)} f(y)dy + \int_{n/(n+1)}^{1} f(y)dy.$$

Also,

$$\int_0^1 |f_{[nx]+1} - f(x)|dx \le \int_0^1 \{f(\tfrac{nx+1}{n+1}) - f(\tfrac{nx}{n+1})\}dx$$

(17)

$$= \frac{n+1}{n}\left\{\int_{n/(n+1)}^{1} f(x)dx - \int_0^{1/(n+1)} d(x)dx\right\}.$$

Thus

$$\int_0^1 |P_x^f(x) - f(x)|dx \le C_2 n^{-\frac{1}{2}}\sum_{i=1}^{n-1} (f_{i+1}-f_i)(\tfrac{i}{n})(1 - \tfrac{i}{n})^{\frac{1}{2}}$$

(18)

$$+ 3\left\{\int_{n/(n+1)}^{1} f(x)dx - \int_0^{1/(n+1)} f(x)dx\right\}$$

By Schwarz's inequality, each of the last two integrals is $\leq n^{-\frac{1}{2}}\left\{\int_0^1 f^2(x)dx\right\}^{\frac{1}{2}}$. It now follows from (18) and Lemmas 1 and 2 that inequality (2) holds with $C = C_1C_2 + 6$.

Proof of Proposition 1. For definiteness assume that f is nondecreasing. Let $\delta > 0$,

$$g(u) = (1+u)^2\ \{\log(1+u)\}^{2+\delta}\ .$$

The function g is positive, increasing and convex on $(0,\infty)$, and $K = \int_0^1 g(|f(x)|)\,dx$ is finite under condition (5). Let $b \in [0,1]$ be such that $f(x) \leq 0$ for $x < b$ and $f(x) \geq 0$ for $x > b$. Let $0 < x < b$. Then $|f(x)| = -f(x) \leq x^{-1}\int_0^x |f(y)|dy$. Since g is convex, Jensen's inequality implies

$$g(x^{-1}\int_0^x |f(y)|dy) \leq x^{-1}\int_0^x g(|f(y)|)dy \leq x^{-1}K.$$

Since g is nondecreasing, we obtain $|f(x)| \leq g^{-1}(Kx^{-1})$ and

$$\int_0^b |f(x)|x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx \leq \int_0^b g^{-1}(Kx^{-1})x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx.$$

The analogous inequality with $\int_0^b$ and Kx^{-1} replaced by $\int_b^1$ and $K(1-x)^{-1}$ is obtained in a similar way. Hence we have

$$\int_0^1 |f(x)|x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx \leq 2\int_0^1 g^{-1}(Kx^{-1})x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx.$$

Since g^{-1} is increasing and positive, the integral on the right converges if $\int_0^1 g^{-1}(x^{-1})x^{-\frac{1}{2}}dx$ does, which is equivalent to the convergence of

$\int_1^\infty u\, g(u)^{-3/2}\, g'(u)du$. The latter integral converges since the integrand is bounded by const. $u^{-1}(\log u)^{-1-\frac{1}{2}\delta}$.

Proof of Proposition 2. First note that if f is nondecreasing, the integrals in (1) are nondecreasing as i increases. Hence

$P_n^f(x) \leq (n+1)\int_{n/(n+1)}^1 f(y)dy = c_n$, say, for $0 \leq x \leq 1$. Choose a_n so that $f(a_n-) \leq c_n \leq f(a_n+)$. Then $f(x) \geq P_n^f(x)$ for $a_n < x < 1$. Hence

$$(19)\int_0^1 |P_n^f(x)-f(x)|dx \geq \int_{a_n}^1 \{f(x)-P_n^f(x)\}dx \geq \int_{a_n}^1 f(x)dx-(1-a_n)(n+1)\int_{n/(n+1)}^1 f(x)dx.$$

Now let $f(x) = (1-x)^{-\frac{1}{2}}\{-\log(1-x)\}^r$, $r > 0$. Clearly f is nondecreasing on (0,1). With f so defined, the integral in (6) converges if

$\int_{\frac{1}{2}}^1 (1-x)^{-1}\{-\log(1-x)\}^{2r-1-\delta}dx$ converges, which is true for $2r < \delta$. As $\epsilon \to 0+$,

$$\int_{1-\epsilon}^1 f(x)dx \sim 2\epsilon^{\frac{1}{2}}(-\log\epsilon)^r.$$

Hence $c_n \sim 2n^{\frac{1}{2}}(\log n)^r$ as $n \to \infty$. A straightforward calculation shows that the root a_n of $f(a_n) = c_n$ satisfies $1 - a_n \sim c_n^{-2}(\log c_n^2)^{2r} \sim \frac{1}{4}n^{-1}$. These facts imply that the last lower bound in (19) is asymptotically equal to $\frac{1}{2}n^{-\frac{1}{2}}(\log n)^r$, which completes the proof.

REFERENCES

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