ON A FLUCTUATION THEOREM FOR PROCESSES

WITH INDEPENDENT INCREMENTS II

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On A Fluctuation Theorem For Processes With Independent Increments II[1]

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1. Introduction and summary. Let $\xi(t)$, $t\geq 0$, $\xi(0) = 0$ be a separable stochastic process with stationary independent increments whose sample functions are continuous on the right. Write $\bar{\xi}(t) = \sup_{0\leq s\leq t} \xi(s)$ and $T_t = \min[u: 0\leq u\leq t;\ \xi(u) = \bar{\xi}(t)]$. The object of this paper is to establish the following theorem:

Theorem. The limiting distribution

(1) $$\lim_{t\to\infty} \Pr(t^{-1}T_t < x) = F(x)$$

exists if and only if

(2) $$\lim_{t\to\infty} t^{-1}\int_0^t \Pr\big(\xi(u)>0\big)du = \alpha,\ 0\leq\alpha\leq 1,$$

and then $F(x)$ is related to α by

(3) $$F(x) = F_\alpha(x) = \frac{\sin\pi\alpha}{\pi}\int_0^x v^{-(1-\alpha)}(1-v)^{-\alpha}dv,\ 0<\alpha<1,\ 0\leq x\leq 1,$$

$$F_0(x) = 0 \text{ if } x<0,\ 1 \text{ if } x\geq 0$$

$$F_1(x) = 0 \text{ if } x<1,\ 1 \text{ if } x\geq 1.$$

This theorem is the exact counterpart to a theorem of Spitzer ([4], Theorem 7.1) 7.1) for sums of independent and identically distributed random variables. It contains as a special case the well-known arc-sine limit theorem for the Brownian motion process. An earlier version of the theorem was obtained by Heyde [3] under the additional condition $\int_0^1 t^{-1}\Pr\big(\xi(t)>0\big)dt<\infty$, the violation of which leads to $\Pr\big(\bar{\xi}(t) = 0\big) = 0$, $t>0$. It should be remarked that T_t has the same distribution as $N_t = \mu\{u: 0\leq u\leq t; \xi(u)>0\}$, μ denoting ordinary Lebesgue measure. This follows from the well-known corresponding result of Sparre-Andersen for sums of independent and identically distributed random variables by a straightforward limiting argument.

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2. Proof of the theorem. We introduce a sequence $X_{m,i}$, $i = 1,2,3,\ldots$ of independent and identically distributed random variables defined for integer valued $m \geq 1$ by

$$X_{m,i} = \xi(2^{-m}i) - \xi(2^{-m}(i-1)),$$

and write $S_{m,0} = 0$, $S_{m,n} = \sum_{i=1}^{n} X_{m,i} = \xi(2^{-m}n)$, $n \geq 1$. Write also,

$$T_{m,n} = \min[k : 0 \leq k \leq n,\ S_{m,k} = \max_{0 \leq j \leq n} S_{m,j}].$$

Then, from well-known results of Sparre-Andersen,

$$\Pr(T_{m,n} = k) = \Pr(T_{m,k} = k)\Pr(T_{m,n-k} = 0)\ ,\ 0 \leq k \leq n,$$

and for $0 \leq t < 1$,

$$\sum_{k=}^{\infty} \Pr\big(\max_{0 \leq j \leq k} S_{m,j} = 0\big)t^k = \sum_{k=0}^{\infty} \Pr(T_{m,k} = 0)t^k = \exp\{\sum_{1}^{\infty}\frac{t^k}{k}\Pr(S_{m,k} \leq 0)\}$$

$$\sum_{k=0}^{\infty} \Pr\{T_{m,k} = k)t^k = \exp\{\sum_{1}^{\infty}\frac{t^k}{k}\Pr(S_{m,k} > 0)\},$$

(see for example Spitzer [4]), and we readily find upon taking generating functions that for $0 \leq v < 1$, $u > 0$,

(4) $$\sum_{n=0}^{\infty} E\big(e^{-uT_{m,n}}\big)v^n = (1-v)^{-1}\exp\{-\sum_{n=1}^{\infty}\frac{v^n}{n}\big(1-e^{-nu}\big)\Pr(S_{m,n} > 0)\}.$$

Now, put $v = e^{-2^{-m}z}$, $u = 2^{-m}z$ in (4), rewrite it in the form

$$\big(1-e^{-2^{-m}z}\big)\sum_{n=0}^{\infty} E\big(e^{-2^{-m}sT_{m,n}}\big)e^{-2^{-m}nz}$$

$$= \exp\{-\sum_{n=1}^{\infty}\frac{1}{n}e^{-2^{-m}zn}\big(1-e^{-2^{-m}sn}\big)\Pr\big(\xi(2^{-m}n) > 0\big)\},$$

and let $m \to \infty$. It follows from a straightforward argument in the spirit of Baxter and Donsker [1] (see Lemma 2 and the first part of the proof of Theorem 1) that

(5) $$z\int_0^{\infty} e^{-zt}E\big(e^{-sT_t}\big)dt = \exp\{-\int_0^{\infty} t^{-1}\Pr\big(\xi(t) > 0\big)e^{-zt}\big(1-e^{-st}\big)dt\}.$$

This result provides the basis for the proof of the theorem.

We now establish that the condition (2) is sufficient for the existence of the limit distribution (1) and that the form (3) follows. In order to do this, we show firstly that under (2) and when $0<\lambda<1$,

(6) $$\lim_{z\to 0} \int_0^\infty t^{-1}\Pr(\xi(t)>0)e^{-zt}(1-e^{-\lambda zt})\,dt = \alpha \log(1+\lambda).$$

Write

$$A(t) = t^{-1}\int_0^t \Pr(\xi(u)>0)\,du.$$

Then, integration by parts gives

$$\int_0^\infty t^{-1}\Pr(\xi(t)>0)e^{-zt}(1-e^{-\lambda zt})\,dt = \int_0^\infty A(t)C(z,t)\,dt,$$

where

$$C(z,t) = t^{-1}e^{-zt}[(1+tz) - (1+tz(1+\lambda)e^{-z\lambda t}].$$

We note that $C(z,t)\geq 0$ and $\lim_{z\to 0} C(z,t) = 0$. Thus, for $z>0$,

$$\int_0^\infty A(t)C(z,t)\,dt = \int_0^\infty [A(t)-\alpha]C(z,t)\,dt + \alpha\int_0^\infty C(z,t)\,dt$$
$$= \int_0^\infty [A(t)-\alpha]C(z,t)\,dt + \alpha\log(1+\lambda)$$

upon performing a simple integration. Now, in view of (2) we can, given $\varepsilon>0$ arbitrarily small, choose T so large that $|A(t)-\alpha|<\varepsilon$ for $t\geq T$ and then

$$\left|\int_0^\infty [A(t)-\alpha]C(z,t)\,dt\right| \leq \int_0^T |A(t)-\alpha|C(z,t)\,dt + \varepsilon\log(1+\lambda) \to \varepsilon\log(1+\lambda)$$

as $z\to 0$ since $\lim_{z\to 0} C(z,t) = 0$. The result (6) follows immediately. Then, putting $s = \lambda z$ in (5) where $0<\lambda<1$ and making use of (6),

(7) $$\lim_{z\to 0} z\int_0^\infty e^{-zt}E(e^{-\lambda zT_t})\,dt = (1+\lambda)^{-\alpha}.$$

Now,

$$z\int_0^\infty e^{-zt}E(e^{-\lambda zT_t})\,dt = z\int_0^\infty e^{-zt}\sum_{k=0}^\infty \frac{(-\lambda z)^k ET_t^k}{k!}\,dt$$
$$= \sum_{k=0}^\infty \lambda^k A_k(z),$$

where

$$A_k(z) = -\frac{(-z)^{k+1}}{k!}\int_0^\infty e^{-zt}E(T_t^k)\,dt,$$

so that from (7),

$$\lim_{z\to 0} \sum_{k=0}^\infty \lambda^k A_k(z) = (1+\lambda)^{-\alpha} = \sum_{k=0}^\infty \lambda^k \binom{-\alpha}{k},$$

and consequently,

(8) $$\lim_{z\to 0} A_k(z) = \binom{-\alpha}{k},\ k\geq 0\ .$$

But, ET_t^k is monotone in t so, using Theorem 4, 423 of Feller [2], it follows from (8) that as $t\to\infty$,

$$E\left(t^{-1}T_t\right)^k \to (-1)^k\binom{-\alpha}{k}\ ,\ k\geq 0.$$

Furthermore it is easy to verify that

$$(-1)^k\binom{-\alpha}{k} = \int_0^\infty x^k dF_\alpha(x),$$

where $F_\alpha(x)$ is given by (3) and, since the moment problem in this case has a unique solution, the proof of the sufficiency part of the theorem is complete.

Finally, suppose that $t^{-1}T_t$ has a proper limiting distribution. Then, $t^{-1}ET_t\to\alpha$ for some $0\leq\alpha\leq 1$ as $t\to\infty$. Also, upon differentiating with respect to s in (5) and putting s=0,

$$z\int_0^\infty e^{-zt}E(T_t)dt = \int_0^\infty e^{-zt}Pr(\xi(t)>0)dt.$$

Consequently, noting that ET_t is monotone in t, we have from Theorem 4, 423 of [2] that

$$z\int_0^\infty e^{-zt}Pr(\xi(t)>0)dt\to\alpha$$

as $z\to 0$, and the condition (2) follows from Theorem 2, 421 of [2]. This completes the proof of the theorem.

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