ON THE MOMENTS OF MARKOV RENEWAL PROCESSES

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SUMMARY

Recently Kshirsagar and Gupta [5] obtained expressions for the asymptotic values of the first two moments of a Markov renewal process. The method they employed involved formal inversion of matrices of Laplace-Stieltjes transforms. Their method also required the imposition of a non-singularity condition. In this paper we derive the asymptotic values using known renewal theoretic results. This method of approach utilizes the fundamental matrix of the imbedded ergodic Markov chain. Although our results differ in form from those obtained by Kshirsagar and Gupta [5] we show that they reduce to their results under the added non-singularity condition. As a by-product of the derivation we find explicit expressions for the moments of the first passage time distributions in the associated semi-Markov process, generalising the results of Kemeny and Snell [4] obtained for Markov chains.

I. INTRODUCTION

1.1 Outline

This paper is primarily concerned with the derivation of matrix expressions for the asymptotic values of the first two moments of a Markov renewal process. This first section deals with the definitions, notation, and known results of semi-Markov processes, Markov renewal processes and general renewal processes. In section 2 we derive matrix expressions for the moments of the first passage time distributions of a semi-Markov process. These results are utilized in section 3 in determining the matrix expressions for the first two moments of a Markov renewal process. In the final section the results obtained recently by Kshirsagar and Gupta [5] are compared with those obtained in this paper.

1.2 Semi-Markov processes and Markov renewal processes

Consider a stochastic process which moves from one to another of a finite number of states $A_1, A_2, \ldots, A_m$ with successive states forming a Markov chain, whose transition matrix is given by $P = [p_{ij}]$. Furthermore, the process stays in a given state a random length of time, "the wait", the distribution function $\Omega_{ij}(\cdot)$ of which depends on the initial state A_i as well as the one to be visited next, A_j. Let us write Z_t for the state occupied at time t, then $\{Z_t; t \geq 0\}$ is called a semi-Markov process. Associated with this process is a Markov renewal process which records at each time t the number of times the Z_t process has visited each of the possible states up to time t.

This descriptive definition can be formalised and for a more detailed and extensive treatment the reader is referred to the papers by W. L. Smith ([11], [12]) and R. Pyke ([7], [8]).

Let us define, a (possibly) defective probability distribution, $F_{ij}(\cdot)$ by

$$F_{ij}(t) = p_{ij}\ \Omega_{ij}(t)\ , \qquad i,j = 1,2,\ldots,m.$$

Let $\underline{F}(t) = [F_{ij}(t)]$.

Then $\underline{F}(\cdot)$ is a matrix valued function on $(-\infty,\infty)$ with the following properties

(i) $F_{ij}(t) = 0$ for $t<0$

(ii) $\sum_{j=1}^{m} F_{ij}(+\infty) = 1$ for $1\leq i\leq m$

Property (i) differs from that given by Pyke [7]. In this paper we permit "instantaneous" transitions from state to state but we do make the additional restriction that the "wait" random variables are not all zero with probability one.

1.3 General renewal processes

Let $\{X_n\}$, $n = 0,1,2\ldots$ be an infinite sequence of independent, non negative random variables which are not all zero with probability one. We assume that X_0 has a distribution function $K(\cdot)$ and that each X_n, for $n\geq 1$, has a distribution function $F(\cdot)$ which is not necessarily identical with $K(\cdot)$. $\{X_n\}$, $n = 0,1,2,\ldots$ is called a general renewal process.

Let $S_{-1} = 0$; $S_k = X_0+X_1+\ldots+X_k$, $(k = 0,1,\ldots)$, and for all $t\geq 0$, define the random variable N_t as the greatest integer k such that $S_{k-1}\leq t$. Thus N_t is the number of renewals that will have occurred up to and including time t in such a general renewal process.

Let $\nu_r = EX_0^r$ $(r = 1,2,\ldots)$, and $\mu_r = EX_1^r$ $(r = 1,2,\ldots)$, provided these moments exist.

V. K. Murthy, [6], generalising the results of W. L. Smith, [13], showed that, under the assumption that $F(\cdot)$ belongs to a class of distribution functions for which, for some finite k, its kth convolution has an absolutely continuous component; the following results hold.

(i) If $\mu_2<\infty$ and $\nu_1<\infty$, then as $t\to+\infty$

$$EN_t = \frac{t}{\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1}\right) + o(1). \tag{1.1}$$

(ii) If $\mu_3<\infty$ and $\nu_2<\infty$, then as $t\to+\infty$

$$\text{var } N_t = \left(\frac{\mu_2-\mu_1^2}{\mu_1^3}\right) t + \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} - \frac{\nu_1\mu_2}{\mu_1^3} + \frac{(\nu_2-\nu_1^2)}{\mu_1^2} + \frac{\nu_1}{\mu_1}\right) + o(1) \tag{1.2}$$

Murthy actually finds expressions for the first eight cumulants of N_t for a general renewal process but there are some errors in his calculations.

Instead of using equation (1.2) we shall find it more convenient in the latter parts of this paper to consider the second factorial moment. From Murthy [6], we have the following additional result.

(iii) If $\mu_3<\infty$ and $\nu_2<\infty$, then as $t\to+\infty$

$$EN_t(N_t+1) = \frac{t^2}{\mu_1^2} + 2\left(\frac{\mu_2}{\mu_1^3} - \frac{\nu_1}{\mu_1^2}\right) t + \left(\frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{2\nu_1\mu_2}{\mu_1^3} + \frac{\nu_2}{\mu_1^2}\right) + o(1). \tag{1.3}$$

Note that when $K(x) = F(x)$ we have $\nu_i = \mu_i$ and the results above reduce to those for an ordinary renewal process, namely,

$$EN_t = \frac{t}{\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + o(1). \tag{1.4}$$

$$\text{var } N_t = \left(\frac{\mu_2 - \mu_1^2}{\mu_1^3}\right) t \quad \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2}\right) + o(1). \tag{1.5}$$

1.4 Markov renewal processes

Let J_n denote the state of the system after the nth transition, (J_0 being the initial state of the system). Thus J_n is one of 1,2,...,m. Let X_n denote the time spent in J_{n-1} before transition into J_n. We define $X_0 = 0$.

The two dimensional stochastic process $\{(J_n, X_n),\ n \geq 0\}$ defined on a complete probability space such that

$$P\{J_n=k,\ X_n \leq t \mid J_0, J_1, X_1, J_2, X_2, \ldots, J_{n-1}, X_{n-1}\} = F_{J_{n-1},k}(t)$$

for all $t \varepsilon (-\infty, \infty)$ and $1 \leq k \leq m$, is called a semi-Markov sequence.

The process $\{J_n,\ n \geq 0\}$ is a Markov chain with transition matrix $P = [p_{ij}] = [F_{ij}(+\infty)]$.

Let $S_n = \sum_{i=0}^{n} X_i$. We define integer valued stochastic processes $\{N_t,\ t \geq 0\}$ where $N_t = \sup\{n \geq 0 \mid S_n \leq t\}$ and $\{N_i(t),\ t \geq 0\}$ where $N_i(t)$ is the number of times $J_n = i$ for $0 < n \leq N_t$.

The vector stochastic process $\underline{N}(t) = \{N_1(t),\ N_2(t),\ \ldots,\ N_m(t)\}$ is called a Markov renewal process and the process $\{Z_t,\ t \geq 0\}$ where $Z_t = J_{N_t}$ is called a semi-Markov process.

Let $M_{ij}(t) = E\{N_j(t) \mid Z_0 = i\}$ for $t \geq 0$ and zero elsewhere, and let $\underline{M}(t) = [M_{ij}(t)]$. The functions $M_{ij}(t)$ are called the renewal functions and $\underline{M}(t)$ the renewal function of the process (Pyke [8]); i.e., $M_{ij}(t)$ is the expected number of visits of the process $\{Z_t,\ t \geq 0\}$ to the state A_j up to time t given the process started in state A_i.

We shall also be interested in $V_{ij}(t) = \mathrm{var}\{N_j(t) \mid Z_0 = i\}$ for $t \geq 0$ and zero elsewhere. Let $\underline{V}(t) = [V_{ij}(t)]$.

Let $\underline{G}(t) = [G_{ij}(t)]$ where $G_{ij}(t) = P\{N_j(t) > 0 \mid Z_0 = i\}$ for $t > 0$ and zero elsewhere. Thus $G_{ij}(t)$ is the distribution function of the time that the process $\{Z_t,\ t \geq 0\}$ visits state A_j for the first time, starting in state A_i.

We shall consider only those processes for which $G_{ij}(+\infty) = 1$ for all $i,j = 1,2,\ldots m$. This ensures that "return to state A_j" is a recurrent event and hence gives rise to a renewal process. Equivalently, we assume that the imbedded Markov chain, $\{J_n, n\geq 0\}$, is irreducible (i.e., each state in the chain can be reached from every other state). This means that we restrict attention to ergodic chains. We shall make the additional assumption of aperiodicity. However, all the results presented in this paper hold for periodic chains with, in some cases, minor modifications.

The connection between the $G_{ij}(t)$ and the $F_{ij}(t)$ is given by the following lemma.

Lemma 1.1

$$G_{ij}(t) = F_{ij}(t) + \sum_{\substack{k=1 \\ k\neq j}}^{m} \int_{0-}^{t} F_{ik}(t-u)\,dG_{kj}(u) \tag{1.6}$$

Proof: Pyke [8], p. 1245.

II. MOMENTS OF THE FIRST PASSAGE TIME DISTRIBUTIONS FOR A SEMI-MARKOV PROCESS

2.1 General results

Let $m_{ij}^{(r)} = \int_{0-}^{\infty} x^r \, dG_{ij}(x)$,

$\mu_{ij}^{(r)} = \int_{0-}^{\infty} x^r \, dF_{ij}(x)$, $r = 0,1,2,\ldots,$ whenever these moments exist.

Note that $m_{ij}^{(0)} = 1$, and that $\mu_{ij}^{(0)} = p_{ij}$

We shall write $m_{ij} = m_{ij}^{(1)}$ and $\mu_{ij} = \mu_{ij}^{(1)}$.

It is sometimes convenient to write $\mu_i^{(r)} = \sum_{j=1}^{m} \mu_{ij}^{(r)}$. Note that $\mu_i^{(0)} = 1$ and let $\mu_i^{(1)} = \mu_i$.

The following lemma connects the $m_{ij}^{(r)}$ (the rth moment of the first passage time from state A_i to state A_j) with the $\mu_{ij}^{(r)}$ (p_{ij} times the rth moment of the distribution of the wait in state A_i before transition to state A_j at the next step).

<u>Lemma 2.1:</u>

$$m_{ij}^{(r)} = \sum_{k\neq j} p_{ik} m_{kj}^{(r)} + \sum_{s=1}^{r-1} \binom{r}{s} \{ \sum_{k\neq j} \mu_{ik}^{(r-s)} m_{kj}^{(s)} \} + \mu_i^{(r)} \tag{2.1}$$

provided the moments exist.

<u>Proof</u>:

From Lemma 1.1 we obtain

$$\int_0^{\infty} x^r dG_{ij}(x) = \int_0^{\infty} x^r dF_{ij}(x) + \sum_{k\neq j} \int_0^{\infty} x^r \int_0^x d_u G_{kj}(u) d_x F_{ij}(x-u)$$

Changing the order of integration

$$m_{ij}^{(r)} = \mu_{ij}^{(r)} + \sum_{k\neq j} \int_0^{\infty} \left(\int_u^{\infty} x^r d_x F_{ik}(x-u) \right) d_u G_{kj}(u)$$

$$= \mu_{ij}^{(r)} + \sum_{k\neq j} \int_0^{\infty} \left(\int_0^{\infty} (u+t)^r dF_{ik}(t) \right) dG_{kj}(u)$$

$$= \mu_{ij}^{(r)} + \sum_{k\neq j} \int_0^{\infty} \left(\int_0^{\infty} \sum_{s=0}^{r} \binom{r}{s} u^s t^{r-s} dF_{ik}(t) \right) dG_{kj}(u)$$

$$= \mu_{ij}^{(r)} + \sum_{s=0}^{r} \binom{r}{s} \{ \sum_{k \neq j} \mu_{ik}^{(r-s)} m_{kj}^{(s)} \}$$

and the result follows upon simplification.

Cor. 2.1.1:

(a) $$m_{ij} = \sum_{k \neq j} p_{ik} m_{kj} + \mu_i \tag{2.2}$$

(b) $$m_{ij}^{(2)} = \sum_{k \neq j} p_{ik} m_{kj}^{(2)} + 2 \sum_{k \neq j} \mu_{ij} m_{kj} + \mu_i^{(2)} \tag{2.3}$$

Let us write $\underline{m}_a^{(r)'} = (m_{1a}^{(r)}, \ldots, m_{ma}^{(r)})$ for $a = 1,\ldots,m$.

In particular let $\underline{m}_a' = \underline{m}_a^{(1)'}$.

Let $P^{(r)} = [\mu_{ij}^{(r)}]$ for $r = 0,1,\ldots,$ with the convention that $P = P^{(0)} = [p_{ij}]$.

Furthermore, define $P_a^{(r)} = [(1-\delta_{aj})\mu_{ij}^{(r)}]$, $r = 1,2,\ldots;$

$$P_a = [(1-\delta_{aj})p_{ij}].$$

Thus, P_a is the transition matrix P with the a^{th} column replaced by zeros.

Also let $\underline{\mu}^{(r)'} = (\mu_1^{(r)}, \ldots, \mu_m^{(r)})$.

Theorem 2.2

$$(I-P_j)\underline{m}_j^{(r)} = \sum_{s=1}^{r-1} \binom{r}{s} P_j^{(r-s)} \underline{m}_j^{(s)} + \underline{\mu}^{(r)} \tag{2.4}$$

Proof: This result follows directly from equation (2.1) using the notation established above.

Cor. 2.2.1:

(a) $$(I-P_j)\underline{m}_j = \underline{\mu}^{(1)} \tag{2.5}$$

(b) $$(I-P_j)\underline{m}_j^{(2)} = 2P_j^{(1)} \underline{m}_j + \underline{\mu}^{(2)} \tag{2.6}$$

Let us define $M^{(r)} = [m_{ij}^{(r)}]$; $M = M^{(1)}$;

$$\Lambda^{(r)} = \text{diag}\ (\mu_1^{(r)}, \ldots, \mu_m^{(r)}).$$

If A is a matrix $[a_{ij}]$ we denote by ${}_dA$, the diagonal matrix $[\delta_{ij} a_{jj}]$. We shall also use the convention that I is the identity matrix, J is the matrix whose elements are all unity, and 0 is the null matrix; (all square matrices of order m).

Note that $\Lambda^{(r)}J = P^{(r)}J$.

Theorem 2.3:

$$M^{(r)} = P[M^{(r)} - {}_dM^{(r)}] + \sum_{s=1}^{r-1}\binom{r}{s}P^{(r-s)}[M^{(s)} - {}_dM^{(s)}] + P^{(r)}J. \quad (2.7)$$

Proof: This result follows directly from equation (2.1).

Cor. 2.3.1:

(a) $M = P[M - {}_dM] + P^{(1)}J.$ (2.8)

(b) $M^{(2)} = P[M^{(2)} - {}_dM^{(2)}] + 2P^{(1)}[M - {}_dM] + P^{(2)}J.$ (2.9)

2.2 Solving for the moments using the theory of determinants.

In this section we outline a method for obtaining the $m_{ij}^{(r)}$ using equation (2.4).

We earlier made the assumption that $P = [p_{ij}]$ is the transition matrix of an ergodic Markov chain. Under this assumption we can produce a solution vector $\underline{u}' = (u_1, \ldots, u_m)$, such that $\underline{u}' = P\underline{u}'$, in terms of the subdeterminants of I-P. Let D_i denote the determinant formed by striking out the i^{th} row and i^{th} column of I-P.

Theorem 2.4:

A Markov chain, with transition matrix P, is ergodic if and only if $D_i > 0$ for $i = 1,2,\ldots,m$.

If $\underline{u}' = (D_1, \ldots, D_m)$ then $\underline{u}'P = \underline{u}'$.

Proof: This theorem is due to Mihoc and can be found in Frechet [2]. It is also stated in Barlow and Proschan [1], p. 129.

Let $\underline{\xi}' = (1,1,\ldots 1)$, and note that $\underline{\xi}\underline{u}'$ is a matrix.

Theorem 2.5: If P is an ergodic transition matrix with period d, and $\underline{u}' = (u_1, \ldots, u_m)$ where $u_i = \frac{D_i}{\sum_{j=1}^{m} D_j}$

then (a) $\lim_{n\to\infty} P^{nd} = \underline{\xi}\underline{u}' = L$

(b) $PL = LP = L$

(c) $\underline{u}'$ is the unique probability vector satisfying $\underline{u}'P = \underline{u}'$.

Proof: Barlow and Proschan, [1], p. 129.

Theorem 2.6:

If P is an ergodic transition matrix, then

$$\text{(a)}\quad \underline{m}_j = (I-P_j)^{-1}\underline{\mu}^{(1)} \tag{2.10}$$

$$\text{(b)}\quad \underline{m}_j^{(2)} = (I-P_j)^{-1}[2P_j^{(1)}\underline{m}_j + \underline{\mu}^{(2)}]. \tag{2.11}$$

Proof: For $(I-P_j)^{-1}$ to exist we require $\det(I-P_j) \neq 0$. But evaluating this determinant by considering the cofactors of the j^{th} column, it is easily seen that $\det(I-P_j) = D_j > 0$ (by ergodicity); and hence the results follow from equations (2.5) and (2.6).

Let us examine equation (2.10) in a little more detail so as to find explicit expressions for the m_{ij} in terms of the cofactors of I-P and $I-P_j$.

Firstly, consider an arbitrary matrix $A = [a_{ij}]$. Let A_{ij} be the cofactor of the $(i,j)^{th}$ element of A (i.e., striking out the i^{th} row and the j^{th} column of A).

Let $A^+ = \text{adj } A = [A_{ji}]$;

then $A^+A = AA^+ = I \det A$.

From this above relation we have for each $i,j = 1,2,\ldots,m$

$$\sum_{k=1}^{m} a_{kj}A_{ki} = \sum_{k=1}^{m} a_{ik}A_{jk} = \delta_{ij} \det A.$$

Now, let B_{ij} be the cofactor of the $(i,j)^{th}$ element of I-P, and let B_{ij}^r be the cofactor of the $(i,j)^{th}$ element of $I-P_r$.

Lemma 2.7:

$$\det\,(I-P_j) = B_{jj} = D_j > 0.$$

Proof:

$$\det\,(I-P_j) = \sum_{k=1}^{m} [I-P_j]_{kj}(\text{cofactor of } [I-P_j]_{kj})$$

$$= \sum_{k=1}^{m} [\delta_{kj} - (1-\delta_{jj})p_{kj}]B^j_{kj}$$

$$= \sum_{k=1}^{m} \delta_{kj}B^j_{kj}$$

$$= B^j_{jj} = B_{jj}\ .$$

Theorem 2.8:

$$m_{ij} = \frac{1}{B_{jj}} \sum_{k=1}^{m} B^j_{ki}\mu^{(1)}_k\ . \qquad (2.12)$$

Proof: From (2.10) and the definition of $(I-P_j)^{-1}$,

$$\underline{m}_j = \frac{1}{B_{jj}}\,\text{adj}(I-P_j)\underline{\mu}^{(1)}\ .$$

The result follows by considering the i^{th} row of $\underline{m}_j$ and noting that $[\text{adj}(I-P_j)]_{ik} = B^j_{ki}$.

It is possible to carry this approach further and express the B^j_{ki} in terms of the second order minors of I-P, but, since the method presented in the next section, 2.3, gives us the solution in a compact form, we shall not proceed any further with this above method.

Let $B = [B_{ji}] = \text{adj}(I-P)$. Since $\det(I-P) = 0$ we can conclude that

$$B(I-P) = (I-P)B = 0$$

i.e. $$B = BP = PB\ .$$

Comparison of this result with statement (b) of Theorem 2.5, implies that $B = k_0L$, where k_0 is a constant determined so that the column sums of L are unity.

Let $L = [\ell_{ij}]$, then, since $L = \underline{\xi}\underline{u}'$, $\ell_{ij} = u_j$. If we make the additional assumption of aperiodicity then the mean recurrence time, ℓ_j, of state A_j in the ergodic Markov chain with transition matrix P, is finite for all j and $u_j = \frac{1}{\ell_j} > 0$.

Theorem 2.9:

$$m_{jj} = \ell_j \sum_{k=1}^{m} \frac{\mu_k^{(1)}}{\ell_k}$$

Proof: Since $B = k_0 L$ and $\sum_j \frac{1}{\ell_j} = 1$, we have that $B_{ij} = \frac{k_0}{\ell_i}$. Hence $\sum_i B_{ij} = k_0$, and $\frac{B_{kj}}{B_{jj}} = \frac{\ell_j}{\ell_k}$.

Now, from Theorem 2.8,

$$m_{jj} = \frac{1}{B_{jj}} \sum_{k=1}^{m} B_{kj}^{j} \mu_k^{(1)}$$

and the result follows, noting that $B_{kj}^{j} = B_{kj}$.

2.3 Solving for the moments using generalised inverses.

In this section we obtain explicit expressions for M and $M^{(2)}$ using equations (2.8) and (2.9).

Throughout this section we make the assumption that P is the transition matrix of an aperiodic, ergodic Markov chain and hence the remarks made in Section 2.2 concerning the matrix L still hold.

Define $\lambda_r = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{1}{\ell_i} \mu_{ij}^{(r)} = \sum_{i=1}^{m} \frac{1}{\ell_i} \mu_i^{(r)}$ (2.13)

The constant λ_1 has been termed the "asymptotic mean increment" by Keilson and Wishart, [3].

Lemma 2.10:

(a) $LP^{(r)}L = \lambda_r L$ (2.14)

(b) $LP^{(r)}J = \lambda_r J$ (2.15)

The result presented in Theorem 2.9 can be obtained by an alternative, more direct, proof as follows.

Theorem 2.11:

(a) $m_{jj} = \lambda_1 \ell_j$ (2.16)

(b) $m_{jj}^{(2)} = [\lambda_2 + 2 \sum_{k \neq j} \sum_{s=1}^{m} \frac{1}{\ell_s} \mu_{sk}^{(1)} m_{kj}] \ell_j$ (2.17)

Proof: Premultiply equation (2.8) by L to obtain

$$LM = LM - L_d M + LP^{(1)} J$$

Using equation (2.15), this reduces to

$$L_d M = \lambda_1 J \tag{2.18}$$

Taking the $(i,j)^{th}$ element of equation (2.18)

$$\sum_k \ell_{ik} \delta_{kj} m_{jj} = \lambda_1$$

i.e. $\frac{1}{\ell_j} m_{jj} = \lambda_1$, and hence equation (2.16) follows.

Similarly, for equation (2.17), from equation (2.9)

$$L_d M^{(2)} = 2LP^{(1)}[M - {}_dM] + \lambda_2 J \tag{2.19}$$

Taking the $(i,j)^{th}$ element of equation (2.19)

$$\sum_k \frac{1}{\ell_k} \delta_{kj} m_{jj}^{(2)} = 2 \sum_k \left(\sum_s \frac{1}{\ell_s} \mu_{sk}^{(1)} \right) \left(m_{kj} - \delta_{kj} m_{jj} \right) + \lambda_2$$

and equation (2.17) follows.

From Theorem 2.11 we can conclude that the diagonal elements of M are easily obtained. Furthermore, once M is found ${}_dM^{(2)}$ will be easily determined.

Let us put ${}_dM = D$, where D is a diagonal matrix with diagonal elements $d_{jj} = \lambda_1 \ell_j$. Thus equation (2.8) can be rewritten as

$$(I-P)M = P^{(1)}J - PD .$$

Note that $\det(I-P) = 0$, and hence $(I-P)$ is a singular matrix. To solve this equation for M by matrix methods we resort to the use of generalised inverses.

We denote a generalised inverse of a matrix A by A^-. (A^- is, in general, not unique).

C. R. Rao ([9], [10]) discusses the properties of generalised inverses extensively. The following lemma contains a selection of Rao's results, sufficient for our needs.

Lemma 2.12:

(a) A^- exists, if and only if, $AA^-A = A$.

(b) A necessary and sufficient condition for the equation $AXB = C$ to have a solution is

$$AA^-CB^-B = C$$

in which case the general solution is

$$X = A^-CB^- + W - A^-AWBB^-$$

where W is arbitrary.

Proof: (a) Rao [9], p. 24

(b) Rao [10], p. 269

Cor. 2.12.1: A necessary and sufficient condition for the equation $AX = C$ to have a solution is $AA^-C = C$ in which case the general solution is

$$X = A^-C + (I - A^-A)W$$

where W is arbitrary.

Proof: Put $B = I$ (in which case $B^- = I$) in Lemma 2.12(b).

To apply this lemma, to obtain the solution of equation (2.8), we need to find a generalised inverse of $I-P$.

Theorem 2.13: If P is an ergodic transition matrix, then the matrix $Z = [I-(P-L)]^{-1}$ exists, and

(a) $PZ = ZP$

(b) $Z\underline{\xi} = \underline{\xi}$

(c) $\underline{u}'Z = \underline{u}'$

(d) $(I-P)Z = I-L$

Proof: Kemeny and Snell, [4], p. 100.

Cor. 2.13.1:

(a) $ZJ = J$

(b) $LZ = ZL = L$

Proof: Result (a) follows directly from (b) of Theorem 2.13, while result (a) follows from (b) and (c) of the same theorem.

Kemeny and Snell refer to this matrix Z as the "fundamental matrix" for the Markov chain determined by P. However, it appears that its full importance has not hitherto been realised. We show that it is a generalised inverse of I-P, with the additional property that it has full rank.

Theorem 2.14: $Z = [I-(P-L]^{-1}$ is a generalised inverse of I-P.

Proof: Let $(I-P)^{-} = Z$, then

$$\begin{aligned} (I-P)(I-P)^{-}(I-P) &= (I-P)Z(I-P) \\ &= (I-L)(I-P) && \text{(Theorem 2.13(d))} \\ &= I-P && \text{(Theorem 2.5(b))} \end{aligned}$$

Thus Z satisfies the criterion to be a generalised inverse of I-P, as given by Lemma 2.12 (a).

We shall make repeated use of the following easily proved results.

Lemma 2.15: Let Λ be any diagonal matrix and let X be an arbitrary square matrix, then,

(a) $_d(X\Lambda) = (_dX)\Lambda$

(b) $_d(XJ) = \frac{1}{\lambda_1} {}_d(XL)D$

(c) $_d(J\Lambda) = \Lambda$

(d) $J_dL = L$

Theorem 2.16: If the imbedded Markov chain of the semi-Markov process is ergodic then

$$M = [\frac{1}{\lambda_1}\{ZP^{(1)}L - J_d(ZP^{(1)}L)\} + I - Z + J_dZ]D \tag{2.20}$$

where D is the diagonal matrix with diagonal elements $d_{jj} = \lambda_1 \ell_j$.

Proof: Using Cor. 2.12.1, the general solution of

$$(I-P)M = P^{(1)}J - PD$$

is given by

$$M = ZP^{(1)}J - ZPD + [I - Z(I-P)]W \tag{2.21}$$

where W is an arbitrary matrix, provided the consistency restriction $(I-P)Z(P^{(1)}J - PD) = P^{(1)}J - PD$ is satisfied.

Now $(I-P)Z(P^{(1)}J - PD) = (I-L)(P^{(1)}J - PD)$ (Theorem 2.13(d))

$= P^{(1)}J - PD - \lambda_1 J + L_dM$ (Lemma 2.10(a))

$= P^{(1)}J - PD$ (by (2.18))

and thus the required consistency condition is satisfied.

Furthermore, using the results of Theorem 2.13, it is easily seen that $I - Z(I-P) = L$ and hence equation (2.21) becomes

$$M = ZP^{(1)}J - ZPD + LW$$

Let $W = [w_{ij}]$, then $[LW]_{ij} = \sum_k \frac{1}{\ell_k} w_{kj} = b_{jj}$, say; i.e., an expression independent of i.

Let $B_1 = \text{diag}(b_{11},\ldots,b_{mm})$, then

$$M = ZP^{(1)}J - ZPD + JB_1 \tag{2.22}$$

Thus, instead of having m^2 arbitrary elements of W there are only m, the diagonal elements of B_1. These can be found using the results for ${}_dM$, i.e., D.

From (2.22)

$$ {}_dM = d(ZP^{(1)}J) - {}_d(ZPD) - {}_d(JB_1) $$

Using the results of Lemma 2.15 we have

$$ D = \frac{1}{\lambda_1}{}_d(ZP^{(1)}L)D - {}_d(ZP)D + B_1 $$

From this equation we have an expression for B_1.

Substituting for B_1 in equation (2.22) we obtain

$$ M = ZP^{(1)}J - ZPD + J[I + {}_d(ZP) - \frac{1}{\lambda_1}\ {}_d(ZP^{(1)}L)]D $$

Using Lemma 2.15(d) and equation (2.18) this above equation can be written as

$$ M = [\frac{1}{\lambda_1}\{ZP^{(1)}L - J_d(ZP^{(1)}L)\} - ZP + J_d(ZP) + J]D $$

Using Theorem 2.13(d) in conjunction with Lemma 2.15(d)

$$ \begin{aligned} - ZP + J_d(ZP) + J &= - L - Z + I + J_dL + J_dZ - J + J \\ &= I - Z + J_dZ; \end{aligned} $$

and equation (2.20) follows.

Theorem 2.16 presents a new result. It generalises the results obtained by Kemeny and Snell, [4], (p.79) for the mean first passage matrix of a Markov chain. The technique used by Kemeny and Snell is different from the one presented here since they do not at any stage utilize the generalised inverse method of solving linear equations. The following corollary shows that if all the $\Omega_{ij}(\cdot)$ degenerate at one for all i,j; i.e., the semi-Markov process degenerates to a Markov chain; then (2.20) gives the results obtained by Kemeny and Snell.

Cor. 2.16.1: If the semi-Markov process degenerates to Markov chain, then the mean first passage matrix, M_1, for this ergodic Markov chain is given by

$$M_1 = [I - Z + J_d Z]D_1 \tag{2.23}$$

where D_1 is a diagonal matrix with diagonal elements $d_{jj} = \ell_j$.

Proof: For a Markov chain, $\mu_{ij} = p_{ij}$, and hence $P^{(1)} = P$.

Also since $\mu_i = 1$ it is easily seen that $\lambda_1 = 1$. Therefore

$$\begin{aligned}\frac{1}{\lambda_1}[ZP^{(1)}L - J_d(ZP^{(1)}L)] &= ZPL - J_d(ZPL) \\ &= ZL - J_d(ZL) \\ &= L - J_dL \\ &= 0,\end{aligned}$$

and thus equation (2.20) becomes (2.23).

Theorem 2.17: If the imbedded Markov chain of the semi-Markov process is ergodic, then

$${}_dM^{(2)} = \frac{1}{\lambda_1}[2\{\frac{1}{\lambda_1}\ {}_d(LP^{(1)}ZP^{(1)}L) - {}_d(ZP^{(1)}L) - {}_d(LP^{(1)}Z) + \lambda_1\ {}_dZ\}D + \lambda_2 I]D \tag{2.24}$$

Proof: From equation (2.19), substituting for $M - {}_dM$,

$$L_dM^{(2)} = 2LP^{(1)}[\frac{1}{\lambda_1}ZP^{(1)}L - \frac{1}{\lambda_1}J_d(ZP^{(1)}L) - Z + J_dZ]\,D + \lambda_2 J$$

Since, from equation (2.18), $L = \lambda_2 J({}_dM)^{-1}$, we have

$$\lambda_1 J_dM^{(2)} = [2LP^{(1)}\{\frac{1}{\lambda_1}ZP^{(1)}L - \frac{1}{\lambda_1}J_d(ZP^{(1)}L) - Z + J_dZ\}D + \lambda_2 J]D$$

Taking into consideration only the diagonal elements we have, with the aid of Lemma 2.15, that

$$\begin{aligned}\lambda_1\ {}_dM^{(2)} = [2\{\frac{1}{\lambda_1}\ {}_d(LP^{(1)}ZP^{(1)}L) - \frac{1}{\lambda_1}\ {}_d(LP^{(1)}J)_d(ZP^{(1)}L) - {}_d(LP^{(1)}Z) + \\ + {}_d(LP^{(1)}J)_dZ\}D + \lambda_2 I]D\end{aligned}$$

From Lemma 2.10, $LP^{(1)}J = \lambda_1 J$, and hence ${}_d(LP^{(1)}J) = \lambda_1 I$. Substitution of this result gives equation (2.24).

Cor. 2.17.1: If the semi-Markov process degenerates to a Markov chain, and if $M_1^{(2)}$ is the second moment first passage matrix for this ergodic chain,then

$${}_dM_1^{(2)} = [2\,({}_dZ)D_1 - I]D_1 \qquad (2.25)$$

Proof: For a Markov chain $P^{(2)} = P$, $P^{(1)} = P$, $\lambda_1 = 1$, $\lambda_2 = 1$

Also

$$ZP^{(1)}L = ZPL = ZL = L$$

$$LP^{(1)}Z = LPZ = LZ = L$$

$$LP^{(1)}ZP^{(1)}L = LPL = L^2 = L$$

Thus equation (2.24) becomes

$${}_dM_1^{(2)} = [2\{{}_dL - {}_dL - {}_dL + {}_dZ\}D_1 + I]D_1$$

$$= [I - 2({}_dL)D_1 + 2({}_dZ)D_1]\,D_1$$

which becomes equation (2.25) upon observing $({}_dL)D_1 = I$

This result for ergodic Markov chains was found by Kemeny and Snell, [4], p. 83.

Theorem 2.18:

$$M^{(2)} = [I - Z + J_dZ]{}_dM^{(2)} + 2[J_d(ZP^{(1)}) - ZP^{(1)}]D$$
$$+ 2[ZP^{(1)}M - J_d(ZP^{(1)}M] + ZP^{(2)}J - J_d(ZP^{(2)}J) \qquad (2.26)$$

Proof: We parallel the method used in Theorem 2.16. Starting with equation (2.9) the required solution is given by

$$M^{(2)} = 2ZP^{(1)}[M - {}_dM] + ZP^{(2)}J - ZP\,{}_dM^{(2)} + JB_2 \qquad (2.27)$$

where B_2 is an arbitrary diagonal matrix which can easily be found since ${}_dM^{(2)}$ has already been determined.

From equation (2.27),

$${}_dM^{(2)} = 2_d(ZP^{(1)}M) - 2_d(ZP^{(1)})D + {}_d(ZP^{(2)}J) - {}_d(ZP)\,{}_dM^{(2)} + B_2$$

B_2 is now determined by this above equation and hence substitution for B_2 in equation (2.27) gives, upon simplification,

$$M^{(2)} = [J - ZP + J_d(ZP)]\,{}_dM^{(2)} + 2[J_d(ZP^{(1)}) - ZP^{(1)}]D$$
$$+ 2[ZP^{(1)}M - J_d(ZP^{(1)}M)] + ZP^{(2)}J - J_d(ZP^{(2)}J)$$

and the result follows as in the proof of Theorem 2.16.

Further simplification is possible in the case of a Markov chain and it can be shown that equation (2.26) leads to the corresponding result found by Kemeny and Snell, [4], p. 83.

III. MOMENTS OF MARKOV RENEWAL PROCESSES

3.1 The asymptotic value of the first moment

In this section we are interested in obtaining the asymptotic value of the matrix $\underline{M}(t) = [M_{ij}(t)]$, where the $M_{ij}(t)$ are the renewal functions of the Markov renewal process.

Since we have assumed that the imbedded Markov chain is irreducible we have that "return to state A_j, given the process started in A_i" is a recurrent event. We shall make the additional assumption that this Markov chain is aperiodic.

Under these assumptions we are able to make use of the results of general renewal processes. In particular, from equation (1.1) with $\mu_1 = m_{jj}$, $\mu_2 = m_{jj}^{(2)}$, and $\nu_1 = m_{ij}$, we obtain

$$M_{ij}(t) = \frac{t}{m_{jj}} + \left(\frac{m_{jj}^{(2)}}{2m_{jj}^2} - \frac{m_{ij}}{m_{jj}}\right) + o(1) \tag{3.1}$$

Kshirsagar and Gupta,[5], in their paper on finding the asymptotic value of $\underline{M}(t)$ do not use this approach but instead use the Laplace-Stieltjes transforms of the first passage time distributions $G_{ij}(t)$'s and apply the standard Tauberian arguments. They found it necessary to consider many special cases depending on the form of the matrices $P^{(1)}$ and P. We shall show later that the results obtained in this paper agree with those obtained by Kshirsagar and Gupta.

Using the same matrix notation as earlier introduced, equation (3.1) can be written as

$$\underline{M}(t) = tJ({}_dM)^{-1} + \tfrac{1}{2}\, J[({}_dM)^{-1}]^2\, {}_dM^{(2)} - M({}_dM)^{-1} + o(1) \tag{3.2}$$

The following lemmas collect together some useful results

Lemma 3.1:

(a) $({}_dM)^{-1} = \frac{1}{\lambda_1} {}_dL$

(b) $J({}_dM)^{-1} = \frac{1}{\lambda_1} L$

Proof: Result (a) follows direct from equation (2.16) while result (b) follows from equation (2.18)

Lemma 3.2: If X is an arbitrary square matrix, then

(a) $J_d(LX) = LX, \quad L_d(LX) = LX_dL$

(b) $J_d(XL) = JX_dL, \quad L_d(XL) = JX({}_dL)^2$

Theorem 3.3:

$$\underline{M}(t) = \frac{t}{\lambda_1} L + \frac{\lambda_2}{2\lambda_1^2} L + \frac{1}{\lambda_1^2} LP^{(1)}ZP^{(1)}L - \frac{1}{\lambda_1} ZP^{(1)}L - \frac{1}{\lambda_1} LP^{(1)}Z + Z - I + o(1) \tag{3.3}$$

Proof: Using the result of Lemma 3.1(b), equation (3.2) becomes

$$\underline{M}(t) = \frac{t}{\lambda_1} L + \frac{1}{2\lambda_1} L\, {}_dM^{(2)} ({}_dM)^{-1} - M({}_dM)^{-1} + o(1)$$

Substituting for $L_dM^{(2)}$ from equation (2.19) we obtain

$$\underline{M}(t) = \frac{t}{\lambda_1} L + \frac{1}{2\lambda_1} \{2LP^{(1)}M({}_dM)^{-1} - 2LP^{(1)} + \frac{\lambda_2}{\lambda_1} L\} - M({}_dM)^{-1} + o(1) \tag{3.4}$$

Theorem 2.16 implies that

$$M({}_dM)^{-1} = \frac{1}{\lambda_1} ZP^{(1)}L - \frac{1}{\lambda_1} J_d(ZP^{(1)}L) + I - Z + J_dZ \tag{3.5}$$

Substitution of (3.5) into (3.4) gives

$$\begin{aligned}\underline{M}(t) = \frac{t}{\lambda_1} L &+ \frac{1}{\lambda_1^2} LP^{(1)}ZP^{(1)}L - \frac{1}{\lambda_1^2} LP^{(1)}J_d(ZP^{(1)}L) \\ &+ \frac{1}{\lambda_1} LP^{(1)} - \frac{1}{\lambda_1} LP^{(1)}Z + \frac{1}{\lambda_1} LP^{(1)}J_dZ \\ &- \frac{1}{\lambda_1} LP^{(1)} + \frac{\lambda_2}{2\lambda_1^2} L - \frac{1}{\lambda_1} ZP^{(1)}L \\ &+ \frac{1}{\lambda_1} J_d(ZP^{(1)}L) - I + Z - J_dZ + o(1)\,.\end{aligned}$$

Application of Lemma 2.10, namely that $LP^{(1)}J = \lambda_1 J$, and simplification yields the required result.

Computationwize, the calculation of $\underline{M}(t)$ using equation (3.3) appears rather long. However, this equation can be simplified by factorisation to yield

$$\underline{M}(t) = \frac{t}{\lambda_1} L + \frac{\lambda_2}{2\lambda_1^2} L + [I - \frac{1}{\lambda_1} LP^{(1)}]Z[I - \frac{1}{\lambda_1} P^{(1)}L] + I + o(1).$$

3.2 The asymptotic value of the second moment

Rather than consider the second moment directly we shall focus attention on the

$$W_{ij}(t) = E\{N_j(t)(N_j(t) + 1)|Z_0 = i\}.$$

Should one be interested in $V_{ij}(t) = \text{var}\{N_j(t)|Z_0 = i\}$ or the second moment $E\{N_j^2(t)|Z_0 = i\}$, the results of this section together with those of 3.1 will suffice.

To determine $W_{ij}(t)$ we apply equation (1.3) with $\mu_r = m_{jj}^{(r)}$ and $\nu_r = m_{ij}^{(r)}$ to obtain

$$W_{ij}(t) = \left[\frac{t^2}{m_{jj}^2} + 2\frac{m_{jj}^{(2)}}{m_{jj}^3} - \frac{m_{ij}}{m_{jj}^2}\right] t$$

$$+ \left[\frac{3m_{jj}^{(2)2}}{2m_{jj}^4} - \frac{2m_{jj}^{(3)}}{3m_{jj}^3} - \frac{2m_{ij}m_{jj}^{(2)}}{m_{jj}^3} + \frac{m_{ij}^{(2)}}{m_{jj}^2}\right] + o(1) \qquad (3.6)$$

Theoretically, to find an explicit expression for $\underline{W}(t) = [W_{ij}(t)]$ all we need is ${}_dM^{(r)}$ $(r = 1,2,3)$ and $M^{(r)}$ $(r = 1,2)$.

In fact,

$$\underline{W}(t) = t^2 J\{({}_dM)^{-1}\}^2 + 2t[J\,{}_dM^{(2)}\{({}_dM)^{-1}\}^3 - M\{({}_dM)^{-1}\}^2]$$

$$+ \tfrac{3}{2}J({}_dM^{(2)})^2\{({}_dM)^{-1}\}^4 - \tfrac{2}{3}J\,{}_dM^{(3)}\{({}_dM)^{-1}\}^3 - 2M\,{}_dM^{(2)}\{({}_dM)^{-1}\}^3$$

$$+ M^{(2)}\{({}_dM)^{-1}\}^2 + o(1) \qquad (3.7)$$

i.e., $\underline{W}(t) = A_1 t^2 + A_2 t + A_3 + o(1)$.

We shall determine explicit expressions for A_1 and A_2 and indicate how to find a similar expression for A_3.

Theorem 3.4:

$$\underline{W}(t) = A_1 t^2 + A_2 t + A_3 + o(1)$$

where

$$A_1 = \frac{1}{\lambda_1^2} L_d L \tag{3.8}$$

$$\begin{aligned} A_2 = {} & \frac{2\lambda_2}{\lambda_1^3} L_d L + \frac{4}{\lambda_1^3} LP^{(1)} ZP^{(1)} L_d L \\ & - \frac{2}{\lambda_1^2} ZP^{(1)} L_d L - \frac{2}{\lambda_1^2} L_d (ZP^{(1)} L) - \frac{4}{\lambda_1^2} LP^{(1)} Z_d L \\ & + \frac{2}{\lambda_1} Z_d L + \frac{2}{\lambda_1} L_d Z - \frac{2}{\lambda_1}\, {}_d L \end{aligned} \tag{3.9}$$

Proof: From equation (3.7)

$$A_1 = J\{({}_dM)^{-1}\}^2 = \frac{1}{\lambda_1} L({}_dM)^{-1} \qquad \text{(by Lemma 3.1(b))}$$

$$= \frac{1}{\lambda_1^2} L_d L \qquad \text{(by Lemma 3.1(a))}$$

Similarly, from equation (3.7)

$$A_2 = 2J_d M^{(2)}\{({}_dM)^{-1}\}^3 - 2M\{({}_dM)^{-1}\}^2$$

$$= [\frac{2}{\lambda_1} L_d M^{(2)}({}_dM)^{-1} - 2M({}_dM)^{-1}]({}_dM)^{-1} \qquad \text{(by Lemma 3.1(b))}$$

$$= [\frac{2}{\lambda_1}\{2LP^{(1)} M({}_dM)^{-1} - 2LP^{(1)} + \frac{\lambda_2}{\lambda_1} L\} - 2M({}_dM)^{-1}]\frac{1}{\lambda_1}\, {}_d L$$

(by equation (2.16) and Lemma 3.1)

Using equation (3.5) for $M({}_dM)^{-1}$,

$$A_2 = [\frac{4}{\lambda_1^2} LP^{(1)}ZP^{(1)}L - \frac{4}{\lambda_1^2} LP^{(1)}J_d(ZP^{(1)}L) + \frac{4}{\lambda_1} LP^{(1)}$$

$$- \frac{4}{\lambda_1} LP^{(1)}Z + \frac{4}{\lambda_1} LP^{(1)}J_dZ - \frac{4}{\lambda_1} LP^{(1)} + \frac{2\lambda_2}{\lambda_1^2} L$$

$$- \frac{2}{\lambda_1} ZP^{(1)}L + \frac{2}{\lambda_1} J_d(ZP^{(1)}L) - 2I + 2Z - 2J_dZ] \frac{1}{\lambda_1} {}_dL$$

This expression can be simplified by using Lemma 2.10 and Lemma 2.15(d) to give the required expression for A_2.

The expression for A_3 involves a considerable amount of computation and is omitted. From equation (3.7) it is evident that an expression for ${}_dM^{(3)}$ is required. Actually it is sufficient to find only $L{}_dM^{(3)}$ and this is easily expressed in terms of M and $M^{(2)}$ by premultiplying equation (2.7) by L with r = 3. i.e.,

$$L{}_dM^{(3)} = 3LP^{(1)}[M^{(2)} - {}_dM^{(2)}] + 3LP^{(1)}[M - {}_dM] + \lambda_3J.$$

Simplification of the expression for A_3 can now be effected making use of Theorems 2.16, 2.17, and 2.18.

IV. COMPARISON OF RESULTS WITH THOSE OBTAINED BY KSHIRSAGAR AND GUPTA

4.1 Summary of earlier results

Kshirsagar and Gupta, [5], showed that

$$\underline{M}(t) = \alpha t H_0 + \alpha H_1 + \alpha(\tfrac{1}{2}\alpha k_2 - a_1)H_0 \quad I + o(1) \tag{4.1}$$

$$\underline{W}(t) = t^2 G_1 + tG_2 + G_3 + o(1). \tag{4.2}$$

where

(i) $\operatorname{adj}(I - P + sP^{(1)}) = H_0 + sH_1 + s^2H_2 + \ldots + s^{m-1}H_{m-1}.$ (4.3)

(ii) Under the assumption that $P^{(1)}$ is non singular

$$\det(I - P + sP^{(1)}) = (\det P^{(1)})\, s\, (\beta_1 + s)(\beta_2 + s) \ldots (\beta_{m-1} + s) \tag{4.4}$$

where $\beta_1, \beta_2, \ldots, \beta_{m-1}, 0$ are latent roots of $P^{(1)-1}(I-P)$. Also

$$\frac{1}{\det(I - P + sP^{(1)})} = \frac{a_{m-1}}{s\det P^{(1)}}\{1 - a_1 s + (a_1^2 - a_2)\, s^2 + o(s)\} \tag{4.5}$$

Thus

$$a_1 = \sum_{1}^{m-1} 1/\beta_i \tag{4.6}$$

$$a_2 = \sum_{i>j} 1/\beta_i\beta_j \tag{4.7}$$

$$a_{m-1} = 1/\beta_1\beta_2 \ldots \beta_{m-1} \tag{4.8}$$

(iii) $$\alpha = a_{m-1}/\det P^{(1)} \tag{4.9}$$

(iv) $H_0 P^{(r)} H_0 = k_r H_0 \qquad (r = 2,3,\ldots)$ (4.10)

(v) $$G_1 = \alpha^2 H_{0d} H_0 \tag{4.11}$$

$$G_2 = 2\alpha^2\{\Delta_1 + (\alpha k_2 - 2a_1)H_{0d}H_0\} - 2\alpha_d H_0 \tag{4.12}$$

$$\begin{aligned} G_3 = {} & 2\alpha^2\{(\tfrac{1}{2}\alpha k_2 - 2a_1)_d H_0 + \Delta_2 + \tfrac{1}{2}\alpha\Delta_3 \\ & + (3a_1^2 - 2a_2 - 3\alpha a_1 k_2 - \tfrac{1}{3}\alpha k_3 + \tfrac{1}{4}\alpha k_2^2)H_{0d}H_0\} \\ & - 2\alpha\{(\tfrac{1}{2}\alpha k_2 - a_1)_d H_0 + {}_d H_1\} \end{aligned} \tag{4.13}$$

and

$$\Delta_1 = H_{1d}H_0 + H_{0d}H_1 \tag{4.14}$$

$$\Delta_2 = H_{2d}H_0 + H_{0d}H_2 + H_{1d}H_1 \tag{4.15}$$

$$\Delta_3 = [H_0P^{(2)}H_1]_dH_0 + H_{0d}[H_0P^{(2)}H_1]$$
$$+ [H_1P^{(2)}H_0]_dH_0 + H_{0d}[H_1P^{(2)}H_0] \tag{4.16}$$

4.2 Comparison of results

We show in this section that, under the same assumption that $P^{(1)}$ be non singular, the results presented in Section 4.1 are the same as those presented in Sections 3.1 and 3.2 of this paper.

Since, for any matrix A,

$$A(\text{adj } A) = (\text{adj } A) = I \det A$$

from equation (4.3) we obtain

$$(I - P + sP^{(1)})(H_0 + sH_1 + s^2H_2 + \ldots + s^{m-1}H_{m-1})$$
$$= (H_0 + sH_1 + s^2H_2 + \ldots + s^{m-1}H_{m-1})(I - P + sP^{(1)})$$
$$= I \det(I - P + sP^{(1)})$$
$$= b_0I + sb_1I + s^2b_2I + \ldots + s^mb_mI \quad ; \text{ say} \tag{4.17}$$

From equations (4.16), equating coefficients of s^r we have

(a) $(r = 0)$ $\quad H_0 - PH_0 = H_0 - H_0P = b_0I \quad (4.18)$

(b) $(r = 1,\ldots, m-1)$ $\quad H_r - PH_r + P^{(1)}H_{r-1} = H_r - H_rP + H_{r-1}P^{(1)} = b_rI \quad (4.19)$

(c) $(r = m)$ $\quad P^{(1)}H_{m-1} = H_{m-1}P^{(1)} = b_mI \quad (4.20)$

From equations (4.17) and (4.18) we note that $b_0 = \det(I-P) = 0$ and hence

$$H_0 = PH_0 = H_0P$$

From this result we can conclude that, for some constant k_0

$$H_0 = k_0L \tag{4.21}$$

From equations (4.19) and (4.20), pre- and post-multiplication of each equation by L gives

$$LP^{(1)}H_{r-1} = H_{r-1}P^{(1)}L = b_r L \qquad (r = 1,..,m) \tag{4.22}$$

In particular when $r = 1$, $k_0 LP^{(1)}L = b_1 L$.

The following theorem gives a recurrence relation that enables us to determine H_r once H_{r-1} is known.

Theorem 4.1: For $r = 1,2,...,m-1$

$$H = \frac{k_0}{b_1} LP^{(1)}ZP^{(1)}H_{r-1} - ZP^{(1)}H_{r-1} - \frac{k_0 b_r}{b_1} LP^{(1)}Z + \frac{k_0 b_{r+1}}{b_1} L + b_r Z \tag{4.23}$$

Proof: From equation (4.19) for any $r = 1,2,...,m-1$ we have

$$(I - P)H_r = b_r I - P^{(1)}H_{r-1} \tag{4.24}$$

$$H_r(I - P) = b_r I - H_{r-1}P^{(1)} \tag{4.25}$$

Let us assume that H_{r-1} is determined. We shall derive equation (4.23) for H_r using equation (4.24) together with Cor. 2.12.1 (Alternately, we could use equation (4.25) with an appropriate form of Lemma 2.12 to obtain the same result).

Since Z is a generalised inverse of $I - P$ we have, upon application of Cor. 2.12.1 that

$$H_r = Z(b_r I - P^{(1)}H_{r-1}) + [I - Z(I-P)]W_r$$

i.e.

$$H_r = b_r Z - ZP^{(1)}H_{r-1} + LW_r \tag{4.26}$$

We eliminate the arbitrary matrix W_r from equation (4.26) by premultiplication by $LP^{(1)}$ to obtain

$$LP^{(1)}H_r \begin{cases} = b_r LP^{(1)}Z - LP^{(1)}ZP^{(1)}H_{r-1} + \dfrac{b_1}{k_0} LW_r & \text{(from equation (4.22) with } r = 1) \\ = b_{r+1} L & \text{(from equation (4.22))} \end{cases}$$

Thus,

$$LW_r = \frac{k_0 b_{r+1}}{b_1} L + \frac{k_0}{b_1} LP^{(1)}ZP^{(1)}H_{r-1} - \frac{k_0 b_r}{b_1} LP^{(1)}Z$$

Substitution in equation (4.26) gives the required result.

Cor. 4.1.1

$$H_1 = \frac{k_0^2}{b_1} LP^{(1)}ZP^{(1)}L - k_0 ZP^{(1)}L - k_0 LP^{(1)}Z + \frac{k_0 b_2}{b_1} L + b_1 Z \qquad (4.27)$$

Proof: Put $r = 1$ in equation (4.23) and $H_0 = k_0 L$.

We have now expressed H_0 and H_1 in terms of our known matrices L, $P^{(1)}$, and Z. We now find relationships between k_0, b_1, b_2, α, k_2, a_1 and the λ_1, λ_2.

(i) From Lemma 2.10 $LP^{(r)}L = \lambda_r L$ $(r = 1,2,..)$, thus since $L = \frac{1}{k_0} H_0$, $H_0 P^{(r)} H_0 = \lambda_r k_0 H_0$. Comparison with equation (4.10) shows that $k_r = \lambda_r k_0$ for $r = 2,3,\ldots$.

(ii) From equations (4.4) and (4.5) we have that

$$\det(I - P + sP^{(1)}) = s(\det P^{(1)})\beta_1\beta_2 \ldots \beta_{m-1}(1 + \frac{s}{\beta_1}) \ldots (1 + \frac{s}{\beta_{m-1}})$$

$$= s(\det P^{(1)}) \frac{1}{a_{m-1}} [1 + s(\sum_1^{m-1} \frac{1}{\beta_i}) + s^2(\sum_{i>j} \frac{1}{\beta_i\beta_j})\ldots]$$

Thus, coefficient of s in $\det(I - P + sP^{(1)}) = \frac{\det P^{(1)}}{a_{m-1}}$

$$= \frac{1}{\alpha} \quad \text{(from equation (4.9))}$$

$$= b_1 \quad \text{(from equation (4.17))}$$

Also, coefficient of s^2 in $\det(I - P + sP^{(1)}) = \frac{\det P^{(1)}}{a_{m-1}} (\sum_1^{m-1} \frac{1}{\beta_i})$

$$= \frac{a_1}{\alpha} \quad \text{(from equation (4.6))}$$

$$= b_2 \quad \text{(from equation (4.17))}$$

Coefficient of s^3 in $\det(I - P + sP^{(1)}) = \frac{\det P^{(1)}}{a_{m-1}} \left(\sum_{i>j} \frac{1}{\beta_i \beta_j} \right)$

$$= \frac{a_2}{\alpha} \quad \text{(from equation (4.7))}$$

$$= b_3 \quad \text{(from equation (4.17))}$$

(iii) From equation (4.22) when $r = 1$ we have $k_0 LP^{(1)}L = b_1 L$, but by Lemma 2.10, $LP^{(1)}L = \lambda_1 L$, thus $k_0 \lambda_1 = b_1$.

From these above results (i), (ii), and (iii) we have that

$$\lambda_1 = \frac{b_1}{k_0} = \frac{1}{\alpha k_0} \; ; \; \lambda_2 = \frac{k_2}{k_0} \tag{4.28}$$

Also $a_1 = \frac{b_2}{b_1}$, $a_2 = \frac{b_3}{b_1}$ (4.29)

Lemma 4.2:

The expression (4.1) for $\underline{M}(t)$ is the same as that given by Theorem 3.3.

Proof: Equation (4.1) states that

$$\underline{M}(t) = \alpha t H_0 + \alpha H_1 + \alpha(\tfrac{1}{2}\alpha k_2 - a_1)H_0 - I + o(1)$$

Coeff. of $t = \alpha k_0 L$

$$= \frac{1}{\lambda_1} L \quad \text{(by equation (4.28))}$$

Constant term $= \alpha H_1 + \alpha k_0 (\frac{1}{2} \frac{\alpha k_0 k_2}{k_0} - a_1)L - I$

$$= \alpha H_1 + \frac{1}{\lambda_1}(\frac{\lambda_2}{2\lambda_1} - a_1)L - I \quad \text{(by equation (4.28))}$$

$$= \frac{\alpha k_0^2}{b_1} LP^{(1)}ZP^{(1)}L - \alpha k_0 ZP^{(1)}L - \alpha k_0 LP^{(1)}Z$$

$$+ \frac{\alpha k_0 b_2}{b_1} L + \alpha b_1 Z + \frac{\lambda_2}{2\lambda_1^2} L - \frac{a_1}{\lambda_1} L - I \quad \text{(from Cor. 4.1.1)}$$

which reduces to the required form using equations (4.28) and (4.29).

Lemma 4.3:

The expression (4.2) for $\underline{W}(t)$ is the same as that given by Theorem 3.4.

Proof: We only verify that $G_1 = A_1$ and $G_2 = A_2$.

$$G_1 = \alpha^2 H_{0d}H_0$$

$$= (\alpha k_0)^2 L_d L$$

$$= \frac{1}{\lambda_1^2} L_d L \quad \text{, as required.}$$

$$G_2 = 2\alpha^2\{H_{1d}H_0 + H_{0d}H_1 + (\alpha k_2 - 2a_1)H_{0d}H_0\} - 2\alpha_d H_0$$

Now from Cor. 4.1.1 and equations (4.28) and (4.29)

$$H_1 = k_0[\frac{1}{\lambda_1} LP^{(1)}ZP^{(1)}L - ZP^{(1)}L - LP^{(1)}Z + a_1 L + \lambda_1 Z]$$

Thus
$$G_2 = 2\alpha^2 k_0^2[\frac{1}{\lambda_1} LP^{(1)}ZP^{(1)}L_d L - ZP^{(1)}L_d L - LP^{(1)}Z_d L + a_1 L_d L + \lambda_1 Z_d L + \frac{1}{\lambda_1} L_d(LP^{(1)}ZP^{(1)}L) - L_d(ZP^{(1)}L) - L_d(LP^{(1)}Z) + a_1 L_d L + \lambda_1 L_d Z + \frac{\lambda_2}{\lambda_1} L_d L - 2a_1 L_d L] - 2\alpha k_{0d} L$$

Carrying out simplification using Lemma 3.2 (namely $L_d(LX) = LX_d L$)

$$G_2 = \frac{2}{\lambda_1^2}[\frac{2}{\lambda_1} LP^{(1)}ZP^{(1)}L_d L - ZP^{(1)}L_d L - 2LP^{(1)}Z_d L - L_d(ZP^{(1)}L) + \lambda_1 Z_d L + \lambda_1 L_d Z + \frac{\lambda_2}{\lambda_1} L_d L] - \frac{2}{\lambda_1}\,{}_d L, \quad \text{as required.}$$

The notation used by Kshirsagar and Gupta can be used to simplify some of our earlier results as shown in the following lemma. Computationally, however, the expressions developed in sections 2 and 3 are easier to handle.

Lemma 4.4:

$${}_d M^{(2)} = [\frac{2}{\lambda_1 k_0}\,{}_d H_1 - \frac{2a_1}{\lambda_1}\,{}_d L + \frac{2\lambda_2}{\lambda_1} I]D$$

Proof: This is easily verified by substitution for ${}_d H_1$ (obtained from equation (4.27)) in equation (2.24).

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