THE PETTIS-STIELTJES (STOCHASTIC) INTEGRAL

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1. INTRODUCTION

Various definitions for integrals of functions with values in an arbitrary Banach space $\mathcal{X}$ (vector-valued integrals) have been given in at least three topologies: The weak and strong topologies on $\mathcal{X}$ and pointwise in the scalar field topology. See, for example, [1], [4] and [5]. In each case conditions must be determined so that the integral exists as a well-defined vector in $\mathcal{X}$.

The purpose of this paper is to define and exhibit some of the properties of a Stieltjes integral for vector-valued measures (functions) in the weak topology on $\mathcal{X}$. As the definition is motivated by B. J. Pettis' definition of a Lebesgue integral in $\mathcal{X}$, the integral will be called the Pettis-Stieltjes integral.

In Section 2, some properties of vector measures used in the analysis are mentioned. The basic definition of the Pettis-Stieltjes integral and a listing of some of the commonly indicated integral properties comprise Section 3. The main result of Section 4 is a representation for the Pettis-Stieltjes stochastic integral in the form of a generalized integration by parts formula. The latter entails the use of a duality formula and an unsymmetric Fubini theorem. In Section 5, a comparison with other stochastic integrals is noted as well as another condition for existence via the definition of a modified Stieltjes integral in the strong topology on $\mathcal{X}$.

Examples are given in Section 6 and there is an appendix cataloging properties of the scalar valued modified Stieltjes integral. For the interval T in the real line, $\mathcal{R}$, take $-\infty \leq a = \inf_{t\in T} t < \sup_{t\in T} t = b \leq \infty$.

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2. VECTOR MEASURES

This section contains some preliminary data on vector-valued measures, definitions of their variations and comparisons.

Consider the measure space $(T, \mathcal{Q})$, where T is an interval in $\mathcal{R}$, the reals, and $\mathcal{Q}$ is a sigma field of subsets of T. Let μ be a set function (measure) on $\mathcal{Q}$ to the Banach space $\mathcal{X}$ over $\mathcal{R}$.

DEFINITION 2.1. $\mu: \mathcal{Q} \to \mathcal{X}$ is strongly (weakly) countably additive if for $\{A_n\}$ disjoint in $\mathcal{Q}$

$$\mu(\bigcup_n A_n) = \sum_n \mu A_n$$

where convergence is in the norm (weak) topology.

Three definitions for a function $x: T \to \mathcal{X}$ to be of bounded variation are given in Hille-Phillips, [4, p.60], and two are shown to be equivalent. Below, variation functions for the measure μ are introduced and equivalence is extended to yet another definition from Dunford-Schwartz, [1, p.320]. The supremum is taken over all finite partitions of $A \in \mathcal{Q}$ unless otherwise noted.

$\mathcal{X}^*$, $\mathcal{X}^{**}$ denote the first and second dual spaces of $\mathcal{X}$.

DEFINITION 2.2.

i) The weak variation of μ on $A \in \mathcal{Q}$ with respect to $x^* \in \mathcal{X}^*$ is

$$W_A(\mu, x^*) \equiv \sup \sum_{k=1}^{n} |x^*(\mu A_k)| .$$

ii) The (total) weak variation of μ on $A \in \mathcal{Q}$ is

$$W_A(\mu) \equiv \sup\{ W_A(\mu, x^*) : \|x^*\| \leq 1 \}.$$

iii) The semi-variation of μ on $A \in \mathcal{Q}$ is

$$\|\mu\|(A) \equiv \sup \left\| \sum_{k=1}^{n} \alpha_k \, \mu A_k \right\|$$

where the supremum is also taken over all $|\alpha_k| \leq 1$.

iv) The variation of μ on $A \in \mathcal{Q}$ is

$$V_A(\mu) \equiv \sup \left\| \sum_{k=1}^{n} \mu A_k \right\| .$$

v) The strong variation of μ on $A \in \mathcal{Q}$ is

$$S_A(\mu) \equiv \sup \sum_{k=1}^{n} \| \mu A_k \| .$$

<u>PROPOSITION</u> <u>2.3</u>.

i) $$V_A(\mu) \leq \|\mu\|(A) \leq W_A(\mu) \leq S_A(\mu)$$

ii) $$\|\mu\|(A) = \beta_1 V_A(\mu)$$

$$W_A(\mu) = \beta_2 V_A(\mu) , \quad \beta_1 \leq \beta_2 , \quad \beta_j \in [1, 2].$$

See, also, Section 6.

<u>Proof</u>: Let $W_A(\mu, \alpha) \equiv \sup \sum_{k=1}^{n} |\alpha_k \, x^*(\mu A_k)|$ over all $|\alpha_k| \leq 1$ and

$\|x^*\| \leq 1$, then $$W_A(\mu, \alpha) = W_A(\mu)$$

and adapting the argument in [4], get

$$\|\mu\|(A) \leq W_A(\mu, \alpha)$$

as well as $W_A(\mu) \le 2V_A(\mu)$. The remainder of the inequalities are evident.

COROLLARY 2.4. The following are equivalent.

i) μ has finite (total) weak variation (is of weak bounded variation) on A .

ii) μ has finite semi-variation on A .

iii) μ has finite variation on A .

Moreover, the finiteness of any of the variation functions $V_T(\mu)$, $||\mu||(T)$ or $W_T(\mu)$ is equivalent to μ having either type of countable additivity when $\mathcal{Q}$ is the Borel field, $\underset{\sim}{B}(T)$.

PROPOSITION 2.5. The statements below are equivalent for $\mu: \underset{\sim}{B}(T) \to \mathcal{X}$:

i) μ is strongly countably additive on $\underset{\sim}{B}(T)$.

ii) μ is weakly countably additive on $\underset{\sim}{B}(T)$.

iii) μ is of weak bounded variation on T , for μ finite.

Statements (i) and (ii) are equivalent for arbitrary $\mathcal{Q}$.

Proof: (i) <=> (ii) in [4].

Let $m(\cdot) = x^*[\mu(\cdot)]$, $x^* \in \mathcal{X}^*$.

(ii) => (iii): the scalar case result is known (Hahn decomposition) for $m(\cdot)$.

(iii) => (ii): this follows from the correspondence between Lebesque-Stieltjes measures and non-decreasing, bounded functions on T . Let $f(\cdot) = m((-\infty, \cdot])$, then f is the difference of two such functions.

Note. There is no confusion about μ being of finite weak variation or finite (total) weak variation since they are, also, equivalent:

(iii) => (i) => $||\mu||(T) < \infty$, [DS, p.320], => $W_T(\mu) < \infty$.

Let BV(T) be the space of functions of bounded variation on T under the supremum norm

$$\|f(\cdot)\|_u \equiv \sup_{t\in T}|f(t)| .$$

In Section 4, we will be interested in the case where the vector measure μ is induced by a function $x: T \to \mathcal{X}$. x defines μ on the field of half-open intervals by

$$\mu(\ (a,\ t]\ ) \ = \ x(t) - x(a)$$

and when x is of weak bounded variation on T , i.e., the scalar function

$$x^*[x(\cdot)] \ = \ g^*(\cdot) \in BV(T)$$

for all $x^* \in \mathcal{X}^*$, then $\Delta g^*(\cdot) = x^*[\mu]$ is countably additive on the field and extends uniquely to $dg^*(\cdot)$ countably additive on $\underset{\sim}{\mathcal{B}}(T)$ by the correspondence mentioned in the above proof. So the vector measure $\mu = dx$ is defined by the values $x^*(dx) = dg^*$.

When dealing with the measure induced by a function $x: T \to \mathcal{X}$, we will always use $\underset{\sim}{\mathcal{Q}} = \underset{\sim}{\mathcal{B}}(T)$ and take the usual partitions $\{t_{kn}\}$, $k = 0, \ldots, n$; $n = 1, 2, \ldots$. All of the above remarks are valid; in particular, for $V_A(x)$, use partitions $\{(s_{kn}, t_{kn})\}$, for all finite collections of non-overlapping intervals. For example,

$$W_A(x,\ x^*) \ = \ \sup \sum_{k=1}^{n} |\Delta x^*[x(t_{kn})]| .$$

When A is an interval with endpoints c and d , write W_{cd} or V_{cd} . If $x: T \to \mathcal{X} = \mathcal{R}$, all of the above reduce to the ordinary variation of the function x .

3. THE PETTIS-STIELTJES INTEGRAL

The basic definition is motivated by the definition of the Pettis integral, see Hille-Phillips [4, p.77], and is made possible by the following proposition, which is similar to the ordinary case. Let f be measurable with respect to $(T, \mathcal{Q})$ and we say that f is weakly integrable with respect to μ if $f(\cdot) \in L_1(T, \mathcal{Q}, x^*[\mu(\cdot)])$ for all $x^* \in \mathcal{X}$, where μ is (weakly) countably additive on $\mathcal{Q}$.

<u>PROPOSITION</u> <u>3.1</u>. Let μ be countably additive on $\mathcal{Q}$ and f weakly integrable with respect to μ . Then there exists $x^{**} \in \mathcal{X}^{**}$ such that

$$x^{**}(x^*) = \int_T f(t)\, x^*[\mu(dt)]$$

for all $x^* \in \mathcal{X}^*$.

The scalar-valued integral on the right side is the ordinary Lebesgue integral.

<u>Proof</u>. Let $I(x^*) = \int_T f(t)\, x^*[\mu(dt)]$ for $x^* \in \mathcal{X}^*$. I is obviously linear and

$$W_{at}(\mu, x^*) \leq \beta \|x^*\| V_{at}(\mu)$$

for some $\beta \in [1, 2]$ by Proposition 2.3. So

$$|I(x^*)| \leq \int_T |f(t)|\, dW_{at}(\mu, x^*) \leq \|x^*\|\, \beta \int_T |f(t)|\, dV_{at}(\mu) .$$

Therefore I is bounded and, hence, in $\mathcal{X}^{**}$.

With the above as a justification, x^{**} may be set equal to the symbol

$$\int_T f(t)\mu(dt) \in \mathcal{X}^{**} .$$

More precisely, for each f, there is an $x_f^{**} \in \mathcal{X}^{**}$ which may or may not be in the image of $\mathcal{X}$ under the natural mapping

$$i: \mathcal{X} \to i(\mathcal{X}) \subset \mathcal{X}^{**} .$$

If this correspondence obtains, we may define the Pettis-Stieltjes integral.

DEFINITION 3.2. The scalar-valued function f on $(T, \mathcal{Q})$ is Pettis-Stieltjes (PS-) integrable with respect to $\mu: \mathcal{Q} \to \mathcal{X}$, a Banach space, if for all $A \in \mathcal{Q}$, there exists $x_A \in \mathcal{X}$ such that

$$x^*(x_A) = \int_A f(t)\ x^*[\mu(dt)]$$

for all $x^* \in \mathcal{X}^*$. By definition

$$x_A = \text{PS-}\int_A f(t)\ \mu(dt) \in \mathcal{X}$$

and write $f \in PS(\mu)$.

For notational purposes, we will often write

$$\langle f, \mu\rangle_A \equiv \text{PS-}\int_A f(t)\ \mu(dt) .$$

Let $\mathcal{X}$ be reflexive. Then if μ is weakly countably additive, $f \in PS(\mu)$ if and only if f is weakly integrable with respect to μ; hence, always for $\mathcal{X} = L_p$ over an arbitrary measure space, $1 < p < \infty$.

COROLLARIES 3.3.

i) The PS-integral is uniquely defined.

ii) μ must be weakly countably additive for the definition.

iii) $f_1, f_2 \in PS(\mu) \Rightarrow \alpha_1 f_1 + \alpha_2 f_2 \in PS(\mu)$, $\alpha_j \in \mathcal{R}$,

$f \in PS(\mu_1), PS(\mu_2) \Rightarrow f \in PS(\beta_1\mu_1 + \beta_2\mu_2)$, $\beta_j \in \mathcal{R}$,

and $\langle f, \mu\rangle_A$ is a bilinear function for fixed $A \in \mathcal{Q}$.

iv) If $\mathcal{X} = \mathcal{R}$, then $\langle f, \mu\rangle$ reduces to the ordinary Lebesgue integral.

Proof. Use the definition and the fact that $x^*(x) = x^*(y)$ for all $x^* \in \mathcal{X}^*$ implies $x = y$.

PROPOSITION 3.4. Let $f \in PS(\mu)$ and $\nu(A) = \langle f, \mu\rangle_A$, then $\nu: \mathcal{Q} \to \mathcal{X}$ is (strongly) countably additive.

Proof. Let $\{A_n\}$ disjoint in $\mathcal{Q}$. Then for $x^* \in \mathcal{X}^*$

$$x^*[\nu(\cup A_n)] = \langle f, x^*(\mu)\rangle_{\cup A_n} = \Sigma \langle f, x^*(\mu)\rangle_{A_n} = \Sigma\, x^*[\nu(A_n)] .$$

So ν is countably additive by Proposition 2.5.

REMARK. Call $\phi: \mathcal{Q} \to \mathcal{X}$ continuous with respect to $\psi: \mathcal{Q} \to \mathcal{X}$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\phi(A)\| < \varepsilon$ whenever $\|\psi(A)\| < \delta$. Then ν is continuous with respect to μ; in fact, ν is continuous with respect to $x^*(\mu)$ for any $x^* \in \mathcal{X}^*$:

Let $x^* \in \mathcal{X}^*$. If $x^*[\mu A] = 0$ then $\nu A = 0$ and the result follows by [4, p.76].

PROPOSITION 3.5. Let L be a bounded, linear operator on $\mathcal{X}$. If $f \in PS(\mu)$, then $f \in PS(L\mu)$ and for $A \in \mathcal{Q}$

$$L \langle f, \mu\rangle_A = \langle f, L\mu\rangle_A .$$

Proof. Let L^* be the adjoint of L. For $x^* \in \mathcal{X}^*$, there is a unique $y^* \in \mathcal{X}^*$ given by $y^* = L^* x^*$ and $y^*(\mu) = x^*(L\mu)$ is countably additive. Thus there exists

$$\langle f, x^*(L\mu)\rangle = \langle f, y^*(\mu)\rangle = y^*(\langle f, \mu\rangle) = x^*(L\langle f, \mu\rangle) .$$

Hence, $\langle f, L\mu\rangle = L\langle f, \mu\rangle$.

REMARK. For $f \in PS(\mu)$, $\nu = \langle f, \mu\rangle$ has finite variation on $\mathcal{Q}$ but does not necessarily have finite strong variation. See Section 6.

PROPOSITION 3.6. Let $f \in PS(\mu_n)$, $n = 1, 2, \ldots$.

i) If $\mu_n - \mu \to 0$ in weak variation then $f \in PS(\mu)$ and for $A \in \mathcal{Q}$

$$\langle f, \mu\rangle_A = \lim_n \langle f, \mu_n\rangle_A \quad \text{(weak)} .$$

ii) $\mu_n \to \mu$ in weak variation in (i) is not sufficient.

Proof. i) μ is countably additive since

$$W_A(\mu, x^*) \le W_A(\mu_n - \mu, x^*) + W_A(\mu_n, x^*)$$

and $f \in PS(\mu)$.

$$| \langle f, x^*(\mu_m - \mu_n)\rangle_A| \le \langle |f|, W_{at}(\mu_m - \mu_n, x^*)\rangle_A \to 0$$

as $m, n \to \infty$, so there exists a weak limit of $\langle f, \mu_n\rangle$ and

$$x^*(\langle f, \mu_n\rangle) = \langle f, x^*(\mu_n)\rangle \to \langle f, x^*(\mu)\rangle .$$

The left side converges to $x^*(\lim_n \langle f, \mu_n\rangle)$ and the right side is $x^*(\langle f, \mu\rangle)$.

ii) Let $\mu_n \equiv -\mu$. $W(\mu_n, x^*) \to W(\mu, x^*)$ but $W(\mu_n - \mu, x^*) = 2W(\mu, x^*) \not\to 0$ and $\langle f, x^*(\mu_n - \mu)\rangle \not\to 0$.

PROPOSITION 3.7. Let $f_n \in PS(\mu)$, $n = 1, 2, \ldots$, and either of the following hold:

i) $f_n \to f$ in $\|\cdot\|_u$.

ii) $f_n \to f$ pointwise, $|f_n| \leq g$ weakly integrable with respect to μ .

Then for $A \in \mathcal{Q}$

$$\langle f, \mu\rangle_A = \lim_n \langle f_n, \mu\rangle_A \quad \text{(weak)} .$$

Proof. f_n, f are bounded by an integrable function; the argument in 3.6. applies and the inequality

$$|\langle f_n - f, x^*(\mu)\rangle_A| \leq \|f_n - f\|_u W_A(\mu, x^*)$$

is evident.

Note that it is sufficient for the convergences and inequality in (i) and (ii) to be (weak) μ - a.e., of course.

An acceptable strong definition implies the weak definition in much the same way that existence of the Bochner (ordinary) integral implies existence and equality of the Pettis integral. For example, using a definition in Dunford-Schwartz [1, p.323],

DEFINITION 3.8. A scalar-valued measurable function f is said to be integrable if there exists a sequence of simple functions $\{f_n\}$ such that

i) $f_n \to f$ μ - a.e. and

ii) $\{\int_A f_n(t)\, \mu(dt)\}$ converges in norm for $A \in \mathcal{Q}$, where

$$\int_A f_n(t)\, \mu(dt) = \Sigma\, \alpha_k\, \mu A_k \quad \text{and} \quad \{A_k\} \quad \text{partitions} \quad A.$$

The limit in (ii) is defined to be $DS\text{-}\int_A f(t)\, \mu(dt)$.

REMARK. The existence of the strong integral $DS\text{-}\int_A f(t)\, \mu(dt)$ implies the existence of the (weak) Pettis-Stieltjes integral $PS\text{-}\langle f, \mu\rangle_A$ and the integrals coincide:

$$x^*(\int_A f_n\, d\mu) = \Sigma\, \alpha_k\, x^*(\mu A_k) = \int_A f_n\, x^*(d\mu) \to \int_A f\, x^*(d\mu) ,$$

the left side converges to $x^*(DS\text{-}\int_A f\, d\mu)$ and the right side is $PS\text{-}\langle f, x^*(\mu)\rangle_A$.

PROPOSITION 3.9. Let f, g be scalar-valued functions on $(T, \mathcal{Q})$ and $\nu(A) = \langle g, \mu\rangle_A$. Then $f \in PS(\nu)$ if and only if $fg \in PS(\mu)$ and for $A \in \mathcal{Q}$

$$\langle f, \nu\rangle_A = \langle fg, \mu\rangle_A .$$

Proof. Let $f \in PS(\nu)$ and $x^* \in \mathcal{X}^*$. Then there exists $\langle f, x^*(\nu)\rangle_A$ and $\{f_n\}$ simple such that $f_n \to f$, $x^*(\mu)$ - a.e. and in $L_1(T)$. (If necessary, consider the non-negative parts of f, g and $x^*(\mu)$ and use the linearity.) $f_n g \to fg$ a.e. and

$$\langle f_m g - f_n g, \mu\rangle_A = \langle f_m - f_n, \nu\rangle_A .$$

Hence $\{f_n g\}$ is Cauchy in $L_1(T)$ with respect to $x^*(\mu)$ and there exists $h^* \in L_1(T)$ such that $f_n g \to h^*$. Consequently, $h^* = fg$ a.e. and

$$\langle f, x^*(\nu)\rangle_A = \langle fg, x^*(\mu)\rangle_A .$$

If $fg \in PS(\mu)$, there exists $\{f_n\}$ simple such that $f_n \leq |f|$ and $f_n \to f$ a.e. So

$$\left|\sup \langle f_n, x^*(\nu)\rangle_A\right| \leq \langle |fg|, W_{at}(\mu, x^*)\rangle$$

and $f \in PS(\nu)$.

4. THE PS-STOCHASTIC INTEGRAL: A REPRESENTATION

In this section, consider the vector measure μ induced by a random function $x: T \to \mathcal{X}$, where $\mathcal{X} = \mathcal{X}(\Omega, \underset{\sim}{F}, \underset{\sim}{P})$ is a Banach space of random variables over a probability space. If $\mathcal{X} = L_1(\Omega)$, then there exists $E|x(t)| < \infty$ for all $t \in T$. Recall the representation for the dual space of $L_1(\Omega)$: given $x(t) \in L_1(\Omega)$ and $x^* \in L_1^*(\Omega)$, there exists $\xi^* \in L_\infty(\Omega)$ such that

$$x^*[x(t)] = E[x(t)\xi^*] .$$

The general definition for the PS-integral becomes in this case,

<u>DEFINITION</u> <u>4.1</u>. The scalar-valued function f on $(T, \mathcal{Q})$ is PS-integrable with respect to the random function $x = \{x(t): t \in T\}$ with values in $L_1(\Omega)$, if for all $A \in \mathcal{Q}$, there exists $x_A \in L_1(\Omega)$ such that

$$E[x_A\xi^*] = \int_A f(t)\, dE[x(t)\xi^*]$$

for all $\xi^* \in L_\infty(\Omega)$. By definition

$$x_A = \text{PS-}\int_A f(t)\, dx(t) \in L_1(\Omega) .$$

To have the induced measure μ be countably additive, it suffices to restrict the scalar function $g^*(\cdot) \equiv E[x(\cdot)\xi^*]$ to be in $BV(T)$, for all $\xi^* \in L_\infty(\Omega)$. Make this assumption in the sequel.

The above is a definition of a stochastic integral in the weak topology of $\mathcal{X} = L_1(\Omega)$. We will also need another stochastic integral defined pointwise in the scalar topology.

Let x and y be random functions on $(T, \mathcal{Q})$ to $\mathcal{X}(\Omega, \underset{\sim}{F}, \underset{\sim}{P})$.

DEFINITION 4.2. The sample path stochastic integral of y with respect to x over $A \in \mathcal{Q}$ exists and is denoted by

$$SP\text{-}\int_A y(t, \cdot)\, dx(t, \cdot) ,$$

if the scalar-valued integrals

$$(\text{type}) - \int_A y(t, \omega)\, dx(t, \omega)$$

exists for almost all $\omega \in \Omega$, where the type may be one of the Riemann-Stieltjes (RS-) integral definitions or the Lebesgue-Stieltjes (LS-) integral.

Conditions for existence and properties of the SP-integral are discussed in [5]. In both of the above definitions, the integrals are defined on Ω a.s. with respect to $\underset{\sim}{P}$.

Assume that $x : (T, \underset{\sim}{B}(T)) \to L_1(\Omega, \underset{\sim}{F}, \underset{\sim}{P})$ is a non-trivial product measurable random function and T is a finite interval.

THEOREM 4.4. For $f \in BV(T)$ and x of weak bounded variation on T , the Pettis-Stieltjes integral exists and has the representation

$$PS\text{-}\int_A f(t)\, dx(t) = SP\text{-}\int_A x(t, \cdot)\, dm_f(t) \quad \text{a.s.}$$

where $m_f(\cdot) \in CA(T)$ the countably additive scalar-valued set functions on $\underset{\sim}{B}(T)$.

The integral on the right side is the sample path Lebesgue-Stieltjes type.

By the remark following Definition 3.8, the formula is true for strong integrals.

COROLLARY 4.5. When it exists,

$$DS\text{-}\int_T f(t)\ dx(t) \ = \ SP\text{-}\int_T x(t,\ \cdot)\ dm_f(t)\ .$$

The proof of the theorem requires a few preliminary results.

Let $BD_1(T)$ be the space of bounded, scalar-valued functions on T with at most discontinuities of the first kind. For $f \in BD_1(T)$ and $g \in BV(T)$, the modified Stieltjes integral $MS\text{-}\int_T f(t)\ dg(t)$ may be defined; it is a generalization of the Riemann-Stieltjes integral and is discussed in the appendix.

DEFINITION 4.6. Let $\underset{\sim}{C}(T)$ be the field of finite unions of (finite) open intervals and points in T. For $f \in BD_1(T)$, define $m_f(\cdot)$ on $\underset{\sim}{C}(T)$ by

$$m_f(C) \ = \ MS\text{-}\int_T f(t)\ dI_C(t)\ .$$

Let $BA(T)$ be the space of bounded, finitely additive, scalar-valued set functions on T.

LEMMA 4.7.

(i) When $f \in BD_1(T)$, $m_f \in BA(T)$ on $\underset{\sim}{C}(T)$ and

$$m_f((c,\ d)) \ = \ f(c+) - f(d-), \qquad m_f(\{c\}) \ = \ f(c-) - f(c+)\ .$$

(ii) When $f \in BV(T)$, $m_f \in CA(T)$ on $\underset{\sim}{\mathcal{B}}(T)$ and is bounded.

Proof: Let $C \in \underset{\sim}{\mathcal{C}}(T)$. $C = \bigcup_{j=1}^{m} (c_j, d_j) \cup \bigcup_{k=1}^{n} \{e_k\}$, disjoint. (i) follows from Proposition A.6 (appendix), Definition 4.9 and the form of C.

$$\sum_{j=1}^{m} |f(c_j+) - f(d_j-)| + \sum_{k=1}^{n} |f(e_k-) - f(e_k+)| \leq V_T(f) < \infty \quad ;$$

hence, $\sup \{|m_f(C)| : C \in \underset{\sim}{\mathcal{C}}(T)\}$ is finite. Also $f = f_1 - f_2$, f_k increasing, $k = 1, 2$. Therefore,

$$m_f = m_{f_1} - m_{f_2}$$

where $m_{f_k} \in CA(T)$. As a result, m_f may be (uniquely) extended to the sigma field generated by $\underset{\sim}{\mathcal{C}}(T)$, by the Carathéodory Extension theorem, but this is $\underset{\sim}{\mathcal{B}}(T)$.

PROPOSITION 4.8. For $f \in BV(T)$ and x of weak bounded variation on T, a duality formula is obtained:

$$\int_T f(t)\, dg^*(t) = \int_T g^*(t)\, dm_f(t)$$

for all $\xi^* \in L_\infty(\Omega)$, where $m_f \in CA(T)$ and $g^*(\cdot) = E[x(\cdot)\, \xi^*]$.

Proof: We refer to Proposition A.8 in the appendix for an integration by parts result for modified Stieltjes integrals.

$$\text{MS-}\int_T f\, dg = -\,\text{MS-}\int_T g df + [fg]_a^b$$

$$+ \sum_x [f(x-)\{g(x)-g(x-)\} - f(x)\{g(x+)-g(x-)\} + f(x+)\{g(x+)-g(x)\}]$$

where the sum is over the (common) discontinuities of f and g.

Letting g be g^*, our desired result obtains if we can show that $\int_T g(t)\, dm_f(t)$ expands to become the right hand side of the above **equation.**

From the definition,

$$\int_T g(t)\, dm_f(t) = \lim_D \sum_{k=1}^{n} g(t_k')\, \Delta m_f(t_k)$$

with partitions from $\underset{\sim}{C}(T)$ and the t_k' are arbitrary interior points.

For $t_k' \in (t_{k-1}, t_k)$: $\Delta m_f(t_k) = m_f((t_{k-1}, t_k)) = f(t_{k-1}+) - f(t_k-)$.

For $t_k' = t_k$: $\Delta m_f(t_k) = m_f(\{t_k\}) = f(t_k-) - f(t_k+)$.

Hence $\Sigma\, g(t_k')\, \Delta m_f(t_k) =$

$$\sum_{\substack{\text{intervals}\\ \text{in } D}} g(t_k')[f(t_{k-1}+) - f(t_k-)] + \Sigma\, g(t_k)[f(t_k-) - f(t_k+)]$$

and writing the first square bracket as

$$- [\{f(t_k) - f(t_{k-1})\} + \{f(t_k-) - f(t_k)\} - \{f(t_{k-1}+) - f(t_{k-1})\}] ,$$

we get the following sums

$$S_1 = \sum_D g(t_k')\, \Delta(-f)(t_k) ,$$

$$S_2 = \sum_D g(t_k')[\{f(t_k) - f(t_k-)\} + \{f(t_{k-1}+) - f(t_{k-1})\}] ,$$

$$S_3 = \Sigma\, g(t_k)[f(t_k-) - f(t_k+)] .$$

S_2, S_3 are non-zero only for the discontinuities of f, $\{d_j\}$, a countable set. Therefore, including these in a sequence of increasing partitions and taking the limit,

$$S_1 \to - \int_T g(t)\, df(t) , \quad (+[fg]_a^b \text{ for } f \text{ continuous at endpoints})$$

$$S_2 \to \sum_{j=1}^{\infty} [g(d_j-)\{f(d_j) - f(d_j-)\} + g(d_j+)\{f(d_j+) - f(d_j)\}] ,$$

$$S_3 \to \sum_{j=1}^{\infty} g(d_j)[f(d_j-) - f(d_j+)] .$$

If the endpoints appear in S_3, we get

$$\Sigma\, g(t_k)[f(t_k-) - f(t_k+)] = g(a)[-f(a)] + g(b)[f(b)] = [fg]_a^b .$$

Adding, we get

$$\int_T g(t)\, dm_f(t) = -\int_T g(t)\, df(t) + [fg]_a^b$$

$$+ \sum_{j=1}^{\infty} [f(d_j-)\{g(d_j)-g(d_j-)\} - f(d_j)\{g(d_j+)-g(d_j-)\} + f(d_j+)\{g(d_j+)-g(d_j)\}] .$$

LEMMA 4.9. For $f \in BV(T)$, the Lebesgue integral

$$SP\text{-}\int_T x(t, \cdot)\, dm_f(t)$$

exists.

Proof. Use the Fubini theorem and the fact that $Ex(\cdot)$ is bounded, uniformly in t. (x is product measurable with respect to $(T \times \Omega\ ,\ \underset{\sim}{B}(T) \otimes \underset{\sim}{F}\ ,\ \lambda \otimes \underset{\sim}{P})$, where λ is Lebesgue measure.)

PROPOSITION 4.10. Let $f \in PS(x)$, then

(i) An unsymmetric Fubini theorem obtains,

$$E\left[\int_T f(t)\, dx(t)\right] = \int_T f(t)\, dEx(t) ;$$

in fact,

$$E\left[\int_T f(t)\, dx(t)\ \xi^*\right] = \int_T f(t)\, dE[x(t)\ \xi^*]$$

for all $\xi^* \in L_\infty(\Omega)$.

(ii) Moreover, if $f \in BV(T)$, the above also coincide with

$$E\left[\int_T x(t, \cdot)\, dm_f(t)\ \xi^*\right] = \int_T E[x(t)\ \xi^*]\, dm_f(t)$$

for all $\xi^* \in L_\infty(\Omega)$.

Proof. The equality in (i) follows since $f \in PS(x)$ and that in (ii) from Lemma 4.9 and the ordinary Fubini theorem. Using the duality formula in Proposition 4.8,

$$\int_T f(t)\; dE[x(t)\;\xi^*] \;=\; \int_T E[x(t)\;\xi^*]\; dm_f(t) \;=\; E\left[\int_T x(t,\,\cdot)\; dm_f(t)\;\xi^*\right]$$

<==> there exists

$$\lim_D E\left[\sum_{j=1}^{m} f(t_j')\;\Delta x(t_j)\;\xi^*\right] \;=\; \lim_D \sum_{j=1}^{m} f(t_j')\;\Delta E[x(t_j)\;\xi^*]$$

$$= E\left[\int_T x(t,\,\cdot)\; dm_f(t)\;\xi^*\right],$$

where the limit exists as a MS-integral. Thus

$$\lim_D E[y_D \xi^*] \;=\; E[y\xi^*]$$

for all $\xi^* \in L_\infty(\Omega)$; i.e., $\lim_D x^*(y_D) = x^*(y)$ for all $x^* \in \mathcal{X}^*$. Hence the weak limit of y_D exists and is y ;

$$\text{weak}\lim_D \Sigma\, f(t_j')\;\Delta x(t_j) \;=\; \int_T x(t,\,\cdot)\; dm_f(t) \in L_1(\Omega) .$$

Denoting the left limit by $x_T = \int_T f(t)dx(t)$, we get the desired relation

$$E[x_T \xi^*] \;=\; \int_T f\; dE[x\xi^*] \;=\; \int_T E[x\xi^*]\; dm_f \;=\; E\left[\int_T x dm_f \xi^*\right] .$$

DEFINITION 4.11. Define $L_1(x)$ to be the closure, in the norm topology, of the (linear) span of $\{x(t) : t \in T\}$.

Proof of Theorem 4.4.

$x = \{x(t) : t \in T\} \neq \{0\} \Rightarrow \dim L_1(x) \geq 1 \Rightarrow L_1(x)$ is a non-trivial

Banach space in $L_1(\Omega)$ => $L_1^*(x)$ is total. Existence of the PS- <f, x> is assured; see, for example, Corollary 5.5. By Proposition 4.10,

$$x^*\left(\int f\, dx\right) = \int f\, dx^*(x) = \int x^*(x)\, dm_f = x^*\left(\int x\, dm_f\right)$$

for all $x^* \in \mathcal{X}^*$. Consequently,

$$\text{PS-}\int f\, dx = \text{SP-}\int x\, dm_f$$

in $L_1(x)$, which means a.s.

In effect, the Pettis-Stieltjes stochastic integral of f with respect to the random function x is represented by an integration by parts formula. The following discussion motivates Theorem 4.4, indicates the origin of Definition 4.6 and outlines the argument supporting the view of the formula as a representation.

By the definition of the PS- integral, existence requires

$$x_f^{**} \in i[L_1(\Omega)] \subset L_1^{**}(\Omega) .$$

Identifying the respective dual spaces,

$$L_1(\Omega) \to L_1^*(\Omega) \cong L_\infty(\Omega) \to L_\infty^*(\Omega) \cong BA(\Omega) .$$

where the equivalences are isometric isomorphisms. Our interest, however, is in the curve $x = \{x(t): t \in T\}$ lying in the space $L_1(\Omega)$ and we want to characterize the functionals

$$E[\cdot\ \xi^*] , \quad \xi^* \in L_\infty(\Omega)$$

which act on $x(t)$, $t \in T$. Rather than considering functionals on $L_1(\Omega)$, we look at $E[\cdot\ \xi^*]$ on the space generated by the curve x, $L_1(x)$.

$L_1(x)$ is a closed, linear subspace of $L_1(\Omega)$ and, consequently, is a Banach space under the relative topology of the $L_1(\Omega)$ norm, $E|\cdot|$. The functionals on $L_1(x)$ look like $g^* + L_1^{\perp}(x)$, where $g^*(\cdot) \equiv E[x(\cdot)\xi^*]$ and $L_1^{\perp}(x)$ is the annihilator of $L_1(x)$ in $L_1^*(\Omega)$, [1, p.72]. On $L_1(x)$, this is just $E[x(\cdot)\xi^*]$ which is in $BV(T)$.

Since the operator topology on $L_1^*(x)$ is the induced $||\cdot||_\infty$ topology on $L_\infty(\Omega)$, consider the $||\cdot||_u$ topology on $BV(T)$ and identify $(BV(T), ||\cdot||_u)^*$. (Note: This is not necessarily an exact isometric isomorphism; the latter holds only in a special case.)

We recall the definition of $BD_1(T)$ and the characterization of its dual space by Hildebrandt, [2, p.873]. (The result allows T to be $\mathcal{R}$.)

<u>PROPOSITION</u> <u>4.12</u>. $(BD_1(T), ||\cdot||_u)^* \cong BA(T)$, where the set functions in $BA(T)$ are defined on the field, $\underset{\sim}{C}(T)$, of subsets of T and the correspondence is an isometric isomorphism.

$BV(T) \subset BD_1(T)$, so the characterization may be used for our purposes; in fact, there is an improvement due to the form of the functionals

$$x_f^{**}(\cdot) = \int_T f(t)\, d(\cdot)$$

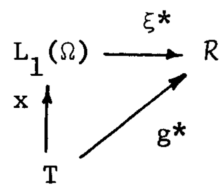
on $(BV(T), ||\cdot||_u)$.

Now we see exactly the motivation of Definition 4.6 since

$$m_f(C) = x_f^{**}(I_C) = \int_T f(t)\, dI_C(t) .$$

Combining this last result with the duality formula, the following framework is established for the correspondences between the various spaces used in the analysis:

i) $x(\cdot) \in \Xi = \Xi(T, L_1(\Omega))$, the space of random functions on T to $L_1(\Omega)$ with $Ex(\cdot) \in BV(T)$. $g^*(\cdot) = E[x(\cdot)\ \xi^*] = \xi^*[x(\cdot)] \in BV(T)$, where $\xi^* \in L_\infty(\Omega)$.



ii) $\Xi \overset{\eta}{\to} BV(T)$:

$\eta[x(\cdot)] = g^*(\cdot)$ by $\eta[x(t)] = g^*(t) = \xi^*[x(t)]$.

iii) $\Xi \overset{\eta}{\to} BV(T) \overset{g^{**}}{\to} R$:

$g^{**}(\cdot) = \int_T (\cdot)\ dm_f \in BV^*(T) \cong CA(T)$. $(g^{**} \circ \eta)[x(\cdot)]$
$= g^{**}\{\eta[x(\cdot)]\} = g^{**}[g^*(\cdot)] = \int_T g^*(t)\ dm_f(t) = \int_T \xi^*[x(t)dm_f(t)$.

Thus $g^{**} \circ \eta: \Xi \to R$ and $g^{**} \circ \eta$ is a bounded, linear functional on Ξ , $g^{**} \circ \eta \in \Xi^*$.

iv) $\Xi \overset{\zeta}{\to} L_\infty(\Omega)$:

$\zeta[x(\cdot)] = \xi^*$, where ξ^* defines g^* .

v) $\Xi \overset{\zeta}{\to} L_\infty(\Omega) \overset{\xi^{**}}{\to} R$:

$\xi^{**}(\cdot) = \int_T fd(\cdot) \in L_\infty^{**}(\Omega) \cong BA(\Omega)$. $(\xi^{**} \circ \zeta)\ [x(\cdot)]$
$= \xi^{**}\{\zeta[x(\cdot)]\} = \xi^{**}[\xi^*(\cdot)] = \int_T f(t)\ d\xi^*[x(t)] = \int_T f(t)dg^*(t)$.

Hence $\xi^{**} \circ \zeta: \Xi \to R$ and $\xi^{**} \circ \zeta$ is a bounded, linear functional on Ξ , $\xi^{**} \circ \zeta \in \Xi^*$.

Comparing the above under the appropriate conditions, we get the representation $\xi^{**} \circ \zeta = g^{**} \circ \eta$ or, symbolically, the duality formula from 4.8, $f \circ \xi^* = m_f \circ \xi^*$. The relationship is illustrated in Figure 1.

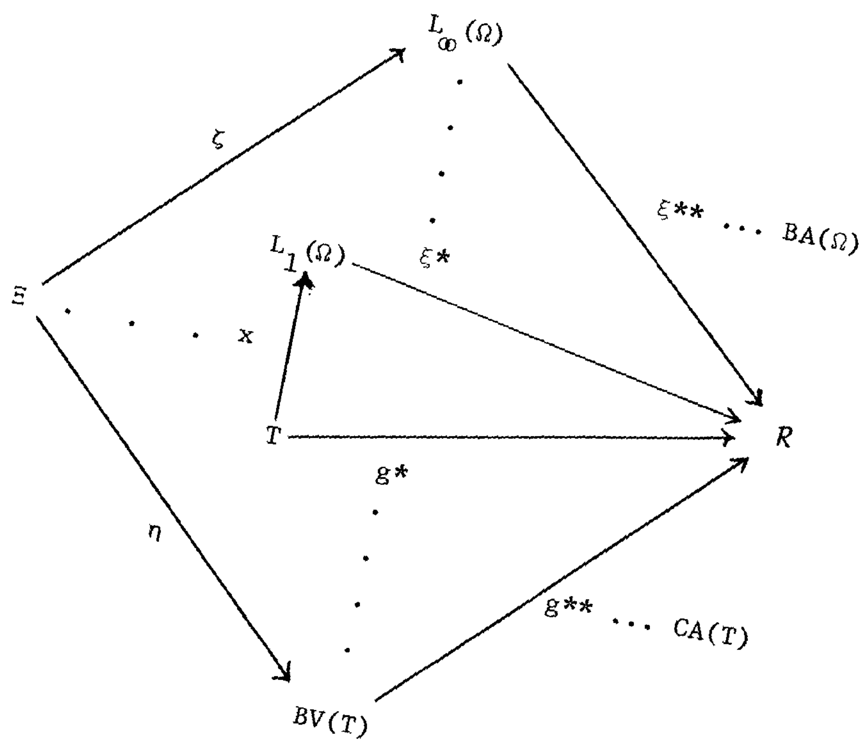


FIGURE 1

We record the following from [4, p.77].

DEFINITION 4.13. The function x on $(T, \mathcal{Q})$ to the Banach space $\mathcal{X}$ is Pettis-integrable if for all $A \in \mathcal{Q}$, there exists $x_A \in \mathcal{X}$ such that

$$x^*(x_A) = \int_A x^*[x(t)] \, d\lambda(t)$$

for all $x^* \in \mathcal{X}^*$, where the integral on the right is the scalar-valued Lebesgue integral. By definition

$$x_A = P\text{-}\int_A x(t) \, d\lambda(t) .$$

The result of Theorem 4.4 holds for the Pettis integral.

COROLLARY 4.14. Suppose x is of weak bounded variation and $f \in PS(x)$. Then

$$PS\text{-}\int_A f(t) \, dx(t) = P\text{-}\int_A x(t) \, dm_f(t) .$$

Proof. Let $x_A = PS\text{-}\langle f, x\rangle$. Then for $x^* \in \mathcal{X}^*$

$$x^*(x_A) = \int_A f dx^*(x) = \int_A x^*(x) \, dm_f = x^*(P\text{-}\int_A x dm_f)$$

by the duality formula. (Existence of the Pettis integral is insured by existence of the sample path integral, Lemma 4.9, and the usual Fubini theorem.)

REMARK. Let $T = [0, 1]$ and $f \in BD_1(T)$, but not in $BV(T)$, be defined by

$$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} I_{A_n}(t) , \quad t \neq 0; \; f(0) = 0 .$$

where $A_n = (\frac{1}{n+1}, \frac{1}{n}]$. Then

$$V_T(f) \geq \sum_{k=1}^{n} |f(\tfrac{1}{k}) - f(\tfrac{1}{k+1})| = \sum_{k=1}^{n} (\tfrac{1}{k} + \tfrac{1}{k+1}) \underset{n}{\to} \infty .$$

Therefore, even for a random function as simple as $x(t) = K \neq 0$, $\int_T x \, dm_f = \infty$. In fact, should we restrict f to be continuous on T but not in $BV(T)$, we can't define $SP\text{-}\int_T x(t, \cdot) \, dm_f(t)$ for, say, x being the Brownian motion process, since the paths are a.s. not of bounded variation. As a result, $f \in BV(T)$ is a necessary hypothesis for the representation.

Although the representation for arbitrary continuous functions cannot be obtained in general, a strong approximation is available.

PROPOSITION 4.15. Let f be continuous on T. For $\varepsilon > 0$, there exists a polynomial p on T such that

$$\left|\left| \int_T f(t) \, dx(t) - \int_T x(t, \cdot) \, dm_p(t) \right|\right| < \varepsilon .$$

Proof. The Weierstrass approximation theorem insures the existence of a polynomial p with $||f - p||_u < \varepsilon/4V_T(x)$. Since $p \in BV(T)$, the integrals exist and the representation holds for $PS\text{-}\int_T p \, dx$. Let $||x^*|| \leq 1$ and $y = \int_T f \, dx - \int_T p \, dx$, then using Proposition 2.3

$$|x^*(y)| \leq \int_T |f - p| dW_{at}(x, x^*) < {}^{\varepsilon}/_{2}$$

which means that $||y|| < \varepsilon$.

5. OTHER PROPERTIES

Restricting x to be of weak bounded variation on T is, in a sense, a weakest possible assumption, since we shall usually want to integrate, at the least, all continuous functions.

PROPOSITION 5.1. If $PS\text{-}\int_T f(t)\,dx(t)$ exists for all f continuous on T , then x is of weak bounded variation on T .

Proof. $MS\text{-}\int_T f(t)\,dx^*[x(t)]$ exists for all $x^* \in \mathcal{X}^*$ and equals $RS\text{-}\int_T f(t)\,dx^*[x(t)]$ for f continuous, since the oscillation over (r, t) and [r, t] is the same. (See the appendix.) For RS-integrals, the proposition is known, [3, p.271]. So $x^*[x(\cdot)] \in BV(T)$, for all $x^* \in \mathcal{X}^*$.

Let $ID\text{-}\int_T f(t)\,dx(t)$ be the well-known Ito-Doob stochastic integral in $L_2(\Omega, \underset{\sim}{F}, \underset{\sim}{P})$.

PROPOSITION 5.2. When the following stochastic integrals exist, the $PS\text{-}\int_T f(t)\,dx(t)$ exists and coincides with them:

i) $SP\text{-}\int_T f(t)\,dx(t, \cdot)$, f bounded, LS-type in $L_1(\Omega)$.

ii) $ID\text{-}\int_T f(t)\,dx(t)$, where $Ex(\cdot) \in BV(T)$.

Proof. When $y = SP\int_T f\,dx$ exists, f is measurable and x is a.s. of bounded variation on T , hence $Ex(\cdot) \in BV(T)$. See [5]. Let $\xi^* \in L_\infty(\Omega)$,

$E[y\,\xi^*] = \int_T f(t)\,dE[x(t)\,\xi^*]$, hence $y = PS\text{-}\int_T f\,dx$. The argument for the ID-integral is similar to the remark following Definition 3.8.

Returning to the strong topology on $\mathcal{X}$, the standard definition of a Riemann-Stieltjes integral in a Banach space, [4, p.62], may be slightly generalized.

DEFINITION 5.3. If $\lim_{D} \sum_{k=1}^{n} f(t_k') \, \Delta x(t_k)$ exists in the norm topology with t_k' arbitrary in (t_{k-1}, t_k) and partitions are successively finer, then denote the limit by

$$MS\text{-}\int_T f(t) \, dx(t) \ .$$

PROPOSITION 5.4. Let $f \in BD_1(T)$ and $x\colon T \to \mathcal{X}$ such that x is of weak bounded variation on T. Then $MS\text{-}\int_T f(t) \, dx(t)$ exists (in the norm topology).

Proof. For $\varepsilon > 0$, select a partition D_ε of T such that $osc(f) < \varepsilon/4M$ over any open subinterval of any $D \supset D_\varepsilon$, where $M = V_T(x) < \infty$. Recall that $W(x, x^*) \le 2M\|x^*\|$ for $x^* \in \mathcal{X}^*$ and let $D, D' \supset D_\varepsilon$.

$$|x^*[\sum_D f(t_j') \, \Delta x(t_j) - \sum_{D'} f(t_k') \, \Delta x(t_k)]| \le \sum_{D \cup D'} \sum_{j,k} osc(f) |\Delta x^*[x(t_{jk})]| < \|x^*\| \, {}^{\varepsilon}/_{2}$$

where osc is over $(t_{j-1}, t_j) \cap (t_{k-1}, t_k)$ and $\Delta g(t_{jk}) = g(\min\{t_j, t_k\}) - g(\max\{t_{j-1}, t_{k-1}\})$. So

$$\|\sum_D f \, \Delta x - \sum_{D'} f \, \Delta x\| < \varepsilon$$

and the limit exists.

COROLLARY 5.5. For $f \in BD_1(T)$ and x of weak bounded variation on T, $f \in PS(x)$ and

$$PS\text{-}\int_T f(t) \, dx(t) = MS\text{-}\int_T f(t) \, dx(t) \ .$$

Proof. $x^*[MS\text{-}\int_T f(t)\, dx(t)] = \int_T f(t)\, dx^*[x(t)]$.

When the p-th moments of $x(t)$ exist, $t \in T$, the computation of the p-th moments of the PS-integral falls out from the definition. Let $\mathcal{X} = L_1(\Omega)$, then

$$Ex_A = \int_A f(t)\, dEx(t).$$

When $\mathcal{X} = L_2(\Omega)$, take $\xi^* = x_A \in L_2(\Omega)$ and

$$Ex_A^2 = \int_A f(t)\, E[dx(t)\, x_A]$$

$$= \int_A \int_A f(s)\, f(t)\, E[dx(s)\, dx(t)]$$

where we assume $\Gamma(\cdot, \cdot) \in BV(T^2)$, $\Gamma(s, t) = E[x(s)\, x(t)]$. If x has orthogonal increments

$$Ex_A^2 = \int_A |f(t)|^2\, E|dx(t)|^2 .$$

In general, if $\mathcal{X} = L_p(\Omega)$, $1 \le p < \infty$, take $\xi^* = x_A^{p-1} \in L_q(\Omega)$, $q = p/(p-1)$, since $E|x_A^{p-1}|^q = E|x_A|^p < \infty$ and

$$Ex_A^p = \int_A f(s)\, E[dx(s)\, \{\int_A f(t)\, dx(t)\}^{p-1}] .$$

For p an integer,

$$Ex_A^p = \int_A \cdots \int_A f(s_1) \ldots f(s_p)\, E[dx(s_1) \ldots dx(s_p)] ,$$

again assuming $\Gamma \in BV(T^p)$, $\Gamma(s_1, \ldots, s_p) = E[x(s_1) \ldots x(s_p)]$.

6. EXAMPLES

6.1. To illustrate Theorem 4.4, the representation for the PS-integral, consider a Poisson process on $T = [0, b]$ with parameter $\lambda > 0$. $V_T(x)$ is finite, hence the PS -integral may be defined (see below).

For this process, almost all sample paths are increasing step functions, integer-valued with jumps of magnitude one and continuous from the left. Also, there are only a finite number of discontinuities in any finite interval.

Let f be continuous on T. Then the SP-integral exists and is

$$\int_T f(t)\, dx(t) = \sum_{k=1}^{N} f(d_k)\, [x(d_k+) - x(d_k)] = \sum_{k=1}^{N} f(d_k)$$

where $N = N_T$ is a random variable representing the number of discontinuities, $\{d_k(\cdot)\}$, of the sample path functions on T. So the (stochastic) integral of f with respect to the Poisson process is a random sum of random variables.

For $D = \{t_j\}_{j=1}^{m}$ such that $\max_{1 \le j \le m} \Delta t_j < \max_{1 \le k \le N(\omega)} \Delta d_k(\omega)$

and $D \supset \{d_k(\omega)\}_{k=1}^{N(\omega)}$,

$$\sum_{j=1}^{m} x(t_j', \omega)\, \Delta m_f(t_j) = \sum_{j=2}^{m} (j-1)\, \{\Sigma' \, [f(t_{j-1}) - f(t_j)]\}$$

where $t_j' \in (t_{j-1}, t_j)$ and Σ' is the sum of differences Δm_f on each subinterval $\Delta d_k(\omega)$. But $x(\cdot, \omega)$ is constant on these subintervals, so the telescoping Σ' reduces to $f[d_{k-1}(\omega)] - f[d_k(\omega)]$ and

$$\sum_{j=1}^{m} x(t_j', \omega)\, \Delta m_f(t_j) = \sum_{k=1}^{N(\omega)} f[d_k(\omega)] .$$

that is, $\text{PS-}\int_T f\, dx = \text{SP-}\int_T x\, dm_f = \sum_{1}^{N} f(d_k)$.

Here, the SP- , PS- and ID- integrals all exist and coincide.

6.2. Define a process x_γ on $T = [0, b]$ by letting the increments $\Delta x_\gamma(t)$ be independent and normally distributed with mean $\gamma\sigma\Delta t$ and variance $\sigma^2\Delta t$; $\gamma, \sigma > 0$. x_γ is a shift of the Brownian motion for $\gamma > 0$ and $V_T(x_\gamma)$ is finite.

Here the PS- and ID- integrals may be defined, but not the SP-integral.

Using the above three examples and others, a table may be set up displaying all possible combinations of existence for the sample path, Pettis-Stieltjes and Ito-Doob integrals.

REMARK. The Brownian motion shift is an example of a process which induces a measure of finite variation on $\underset{\sim}{\mathcal{B}}(T)$ (countably additive) but not of finite strong variation.

$$E|\Delta x_1(t)| = \left[\sqrt{2\Delta t}\int_0^{\sqrt{\Delta t/2}} e^{-\xi^2}\, d\xi + e^{-\Delta t/2}\right] E|\Delta x(t)|$$

where x is (zero-mean) Brownian motion. Hence,

$$E|\Delta x_1(t)| \geq E|\Delta x(t)| = \sigma\sqrt{2\Delta t/\pi}$$

and

$$S_T(x_1) \geq S_T(x) \geq \sigma\sqrt{2/\pi}\sum_{k=1}^{n}\sqrt{\Delta t_{kn}} \to \infty .$$

Recalling that $W_T(\mu) = \beta V_T(\mu)$ where μ: $(T, \mathcal{Q}) \to \mathcal{X}$ and $1 \le \beta \le 2$, the Poisson process provides an example for which the equality is obtained $(\beta = 1)$; let $||x^*|| \le 1$,

$$W_T(x, x^*) \le \sup \sum_{k=1}^{n} E[\Delta x(t_{kn})] = \lambda b .$$

But

$$V_T(x) = \sup E|\Sigma [x(t_{kn}) - x(s_{kn})]|$$

$$= EN_T = Ex(b) = \lambda b ,$$

where N_T is defined in 6.1. Therefore $W_T(x) \le V_T(x)$, so $\beta = 1$. This may also be seen from the fact that

$$\mathcal{S}_T(x) = \sup \Sigma E|\Delta x(t_{kn})| = \lambda b ,$$

so $V_T(x) = W_T(x) = \mathcal{S}_T(x)$. In fact, these equalities obtain for any process in $L_1(\Omega)$ with nondecreasing (or nonincreasing) sample paths.

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APPENDIX

The properties of the scalar-valued modified Stieltjes integral are summarized here for reference. The integral is due to B. Dushnik and some results are listed in [3, p.273]. Proofs are omitted.

DEFINITION A.1. Let f and g be real-valued functions on T. The modified Stieltjes integral is defined as

$$MS\text{-}\int_T f(t)\, dg(t) = \lim_D \sum_{k=1}^{n} f(t_k')\, \Delta g(t_k)$$

where the limit is taken over successively finer partitions D of T. The t_k' are arbitrary interior points of the subintervals (t_{k-1}, t_k).

Note that the MS-integral is more general than the usual Riemann-Stieltjes (RS-) integral.

DEFINITION A.2. osc(f) is the oscillation of f over a prescribed interval T and equals $\sup \{|f(s) - f(t)| : s,t \in T\}$.

LEMMA A.3. Let $f \in BD_1(T)$, then for every $\varepsilon > 0$, there exists a partition D_ε of T such that $osc(f) < \varepsilon$ over any open subinterval of D_ε and, hence, over any open subinterval of $D \supset D_\varepsilon$.

PROPOSITION A.4. When $f \in BD_1(T)$ and $g \in BV(T)$, there exists $MS\text{-}\int_T f(t)\, dg(t)$.

PROPOSITION A.5. Let $f \in BD_1(T)$ and g continuous in BV(T). Then there exists

$$RS\text{-}\int_T f(t)\, dg(t) = MS\text{-}\int_T f(t)\, dg(t)$$

PROPOSITION A.6. Let $f \in BD_1(T)$ and $g \in BV(T)$. Then

$$MS\text{-}\int fdg = RS\text{-}\int fdg_c + MS\text{-}\int fdg_d$$

$$= RS\text{-}\int fdg_c + \sum_t \{f(t+)[g(t+) - g(t)] + f(t-)[g(t) - g(t-)]\}$$

where g_c is the continuous part of g and g_d exhibits the discontinuities.

PROPOSITION A.7. For $f \in BD_1(T)$ and $g \in BV(T)$, the Lebesgue-Stieltjes (LS-) integral exists and

$$LS\text{-}\int fdg = MS\text{-}\int fdg .$$

PROPOSITION A.8. When both f and g are in $BV(T)$, a type of integration by parts theorem holds.

$$\int fdg + \int gdf = [gf]_a^b$$

$$+ \sum_x [f(x-)\{g(x)-g(x-)\} - f(x)\{g(x+)-g(x-)\} + f(x+)\{g(x+)-g(x)\}]$$

where the sum is taken over the (common) discontinuities of f and g.

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