ON BARTLETT'S TEST AND LEHMANN'S TEST FOR HOMOGENEITY OF VARIANCES

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1. <u>Introduction and summary</u>. The purpose of this paper is to compare two tests due to Bartlett [2] (= M test) and Lehmann [4] (= L test) for homogeneity of variances of k normal populations. The unbiasedness of the M test was established by Pitman [5], whereas the L test is shown to be biased in Section 2. It is well known that these two test statistics have asymptotically the same χ^2 distribution with $k - 1$ degrees of freedom under the hypothesis for large sample sizes. We shall derive the limiting distributions of these test criteria under the sequence of alternative hypotheses with arbitrary rate of convergence to the hypothesis as sample sizes tend to infinity, in Section 3. The asymptotic expansions of two test criteria under fixed alternative hypothesis and also the asymptotic expansion of the L test under the null hypothesis are obtained in Section 4 with some numerical examples. The asymptotic non-null (standardized) distributions of the two test criteria under fixed alternative hypothesis are shown to be normal distributions.

2. <u>Biasedness of L test</u>. Let $X_{i1}, X_{i2}, \ldots, X_{iN_i}$ be a random sample from a normal distribution with mean μ_i and variance σ_i^2 $(i = 1, 2, \ldots, k)$. For testing the hypothesis $H : \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$ against all alternatives $K : \sigma_i^2 \neq \sigma_j^2$ for some i, j $(i \neq j)$ with unspecified mean μ_i, the L test criterion due to Lehmann [4] is given by

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$$(2.1)\qquad L = \frac{1}{2}\sum_{\alpha=1}^{k} n_\alpha[Z_\alpha - \frac{1}{n}\sum_{\beta=1}^{k} n_\beta Z_\beta]^2 ,$$

where $Z_\alpha = \log(S_\alpha/n_\alpha)$ and $S_j = \sum_{\alpha=1}^{N_j} (X_{j\alpha} - \bar{X}_j)^2$ with $\bar{X}_j = \sum_{\alpha=1}^{N_j} X_{j\alpha}/N_j$, $n_j = N_j - 1$ and $n = \sum_{\alpha=1}^{k} n_\alpha$. The M test criterion due to Bartlett [2], without correction factor, is given by

$$(2.2)\qquad M = n\log\left(\sum_{\alpha=1}^{k} S_\alpha/n\right) - \sum_{\alpha=1}^{k} n_\alpha \log(S_\alpha/n_\alpha)$$

with the same notation as above. The L (or M) test rejects the hypothesis H when the observed value of L (or M) is larger than a preassigned constant. It may be remarked that the M test is equivalent to the modified likelihood ratio criterion and the unbiasedness was proved by Pitman [5]. The modification means to change sample size N_α to the degrees of freedom n_α. The following theorem shows that the L test is not always unbiased.

<u>Theorem</u> <u>2.1.</u> In two sample problem (k=2), the L test is unbiased if and only if their sample sizes are equal. In this case $(n_1 = n_2)$, the L test is equivalent to the M test.

Proof. If $k = 2$, $L = (1/2n)n_1 n_2\{\log(n_2 S_1/n_1 S_2)\}^2$ and the acceptance region of the L test is simplified as

$$(2.3)\qquad 1/c \le n_2 S_1/(n_1 S_2) \le c$$

for some constant $c(c > 1)$. Ramachandran [6] showed that the acceptance region

$$(2.4)\qquad c_1 \le n_2 S_1/(n_1 S_2) \le c_2$$

for any constant c_1 and c_2 such that $c_1 < c_2$, gives an unbiased test if and only if the condition

$$(2.5)\qquad c_2^{n_1}\left(1 + \frac{n_1}{n_2} c_2\right)^{-n} = c_1^{n_1}\left(1 + \frac{n_1}{n_2} c_1\right)^{-n}$$

is satisfied. This is proved by putting the derivative of the power function at the null hypothesis to zero. In our case $c_2 = 1/c_1$, the condition is expressed as

$$(2.6) \qquad c_1^{n_1-n_2} = [(n_2 + n_1c_1)/(n_1 + n_2c_1)]^n$$

for $0 < c_1 < 1$. We shall show that this condition is not satisfied unless $n_1 = n_2$. Taking the logarithms on both sides of (2.6), we shall put

$$(2.7) \qquad f(c_1) = (n_1 - n_2) \log c_1 - n \log[(n_2 + n_1c_1)/(n_1 + n_2c_1)],$$

then the derivative of $f(c_1)$ is given by

$$(2.8) \qquad f'(c_1) = (n_1-n_2)n_1n_2(c_1-1)^2/\{c_1(n_1+n_2c_1)(n_2+n_1c_1)\}.$$

Noting that $f(1) = 0$, we can conclude that for any c_1 lying between zero and one, $c_1^{n_1-n_2} < [(n_2 + n_1c_1)/(n_1 + n_2c_1)]^n$ if n_1 is larger than n_2 and the reverse inequality holds if n_1 is less than n_2, which contradicts the condition (2.6). If $n_1 = n_2$, the condition (2.6) is obviously satisfied. In this case $M = (n/2)\log[\frac{1}{4}\{1 + (S_1/S_2)\}^2 (S_2/S_1)]$, the acceptance region of which is equivalent to $1/c \leq S_1/S_2 \leq c$ for some c. Hence our proof is completed.

3. Limiting distributions under sequences of alternative hypotheses.

Since the statistic

$$S_\alpha/\sigma_\alpha^2 = \sum_{\beta=1}^{N_\alpha} (X_{\alpha\beta} - \bar{X}_\alpha)^2/\sigma_\alpha^2$$

has χ^2 distribution with n_α degrees of freedom under alternative K, the statistic $T_\alpha = [(S_\alpha/\sigma_\alpha^2) - n_\alpha]/\sqrt{2n_\alpha}$ is distributed asymptotically according to the standard normal distribution as n_α tends to infinity. We shall express the test statistics L and M given in (2.1) and (2.2) in terms of the statistic T_α $(\alpha = 1, 2, \ldots, k)$. Putting $n_\alpha = \rho_\alpha n$ with $\sum_{\alpha=1}^{k} \rho_\alpha = 1$, we can easily see that

$$(3.1) \qquad Z_\alpha = \log S_\alpha/n_\alpha = \log \sigma_\alpha^2 + \log (1 + \sqrt{2/n_\alpha}\, T_\alpha),$$

which implies, for large n with fixed ρ_α $(\rho_\alpha > 0)$,

$$(3.2) \qquad L = (n/2) \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 + n \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma}) \log (1 + \sqrt{2/n_\alpha}\, T_\alpha)$$
$$+ (n/2) \sum_{\alpha=1}^{k} \rho_\alpha\{\log (1 + \sqrt{2/n_\alpha}\, T_\alpha) - \sum_{\beta=1}^{k} \rho_\beta \log(1 + \sqrt{2/n_\beta}\, T_\beta)\}^2$$
$$= (n/2) \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 + \sqrt{2n} \sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, (\tilde{\sigma}_\alpha - \tilde{\sigma}) T_\alpha$$
$$+ \sum_{\alpha=1}^{k} (\tilde{\sigma} - \tilde{\sigma}_\alpha + 1) T_\alpha^2 - (\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, T_\alpha)^2 + o_p(n^{-\frac{1}{2}}),$$

where $\tilde{\sigma}_\alpha = \log \sigma_\alpha^2$ and $\tilde{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \log \sigma_\alpha^2$. In the similar way we can express the test statistic M as

$$(3.3) \qquad M = n(\log \bar{\sigma} - \tilde{\sigma}) + \sqrt{2n} \sum_{\alpha=1}^{k} (\nu_\alpha - 1) \sqrt{\rho_\alpha}\, T_\alpha + \sum_{\alpha=1}^{k} T_\alpha^2$$
$$- (\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, \nu_\alpha T_\alpha)^2 + o_p(n^{-\frac{1}{2}}),$$

with $\bar{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \sigma_\alpha^2$ and $\nu_\alpha = \sigma_\alpha^2/\bar{\sigma}$.

Now we shall specify the sequence of alternatives K_δ as $\sigma_\alpha = \sigma + (\theta_\alpha/n^\delta)$ for $\alpha = 1, 2, \ldots, k$ and $\delta > 0$, where not all θ's are assumed to be equal. If $0 < \delta < \frac{1}{2}$, we can rewrite the expression of L in (3.2) as

$$(3.4) \qquad L = A(\sigma_1, \ldots, \sigma_k) + \frac{2\sqrt{2}\, n^{\frac{1}{2}-\delta}}{\sigma} \sum_{\alpha=1}^{k} (\theta_\alpha - \tilde{\theta}) \sqrt{\rho_\alpha}\, T_\alpha + o_p(n^{\frac{1}{2}-2\delta}) ,$$

where $A(\sigma_1, \ldots, \sigma_k) = (n/2) \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$ substituted by $\sigma_\alpha = \sigma + (\theta_\alpha/n^\delta)$ and $\tilde{\theta} = \sum_{\alpha=1}^{k} \rho_\alpha \theta_\alpha$. Hence we have $A(\sigma_1, \ldots, \sigma_k) = O(n^{1-2\delta})$. This means that the statistic $n^{\delta-\frac{1}{2}}[L - A(\sigma_1, \ldots, \sigma_k)]$ has asymptotically, a normal distribution with mean zero and variance $\tau_L^2 = (8/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2$. If $\delta > \frac{1}{2}$, we can write the statistic L from (3.2) as

$$(3.5) \qquad L = \sum_{\alpha=1}^{k} T_\alpha^2 - (\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, T_\alpha)^2 + o_p(n^{\frac{1}{2}-\delta}) ,$$

which shows that L has asymptotically χ^2 distribution with k-1 degrees of freedom. In this case, the sequence of alternatives K_δ converges so fast to the hypothesis that the limiting distribution of L is unchanged as under the null hypothesis. In the boundary of $\delta = \frac{1}{2}$, we can write

$$(3.6) \qquad L = \sum_{\alpha=1}^{k} \{T_\alpha + \sqrt{2\rho_\alpha}\,(\theta_\alpha - \tilde{\theta})/\sigma\}^2 - (\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, T_\alpha)^2 + o_p(n^{-\frac{1}{2}}).$$

Thus the statistic L has asymptotically noncentral χ^2 distribution with k-1 degrees of freedom and noncentrality parameter $\delta_L^2 = (2/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha (\theta_\alpha - \tilde{\theta})^2$.

Summarizing the above results, we have the following theorem.

<u>Theorem 3.1.</u> Under the sequence of alternatives $K_\delta : \sigma_\alpha = \sigma + (\theta_\alpha / n^\delta)$ for $\alpha = 1, 2, \ldots, k$ where not all θ's are equal, the limiting distribution of the test statistic L given by (2.1) for large n with fixed $\rho_\alpha = n_\alpha / n > 0$ is the following.

(1) If $0 < \delta < \frac{1}{2}$, $[L - A(\sigma_1, \ldots, \sigma_k)]/n^{\frac{1}{2}-\delta}$ is distributed asymptotically according to normal distribution with mean zero and variance $\tau_L^2 = (8/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha (\theta_\alpha - \tilde{\theta})^2$ where $A(\sigma_1, \ldots, \sigma_k) = (n/2) \sum_{\alpha=1}^{k} \rho_\alpha (\tilde{\sigma}_\alpha - \tilde{\sigma})^2 = O(n^{1-2\delta})$ and $\tilde{\theta} = \sum_{\alpha=1}^{k} \rho_\alpha \theta_\alpha$, $\tilde{\sigma}_\alpha = \log \sigma_\alpha^2$, $\tilde{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \tilde{\sigma}_\alpha$.

(2) If $\delta > \frac{1}{2}$, L has asymptotically χ^2 distribution with k-1 degrees of freedom.

(3) If $\delta = \frac{1}{2}$, L has asymptotically noncentral χ^2 distribution with k-1 degrees of freedom and noncentrality parameter $\delta_L^2 = (2/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha (\theta_\alpha - \tilde{\theta})^2$.

The result (3) in the above theorem has often been used in discussing asymptotic relative efficiency of nonparametric tests (Deshpande [3], Sugiura [7] and etc.), however, we have stated it in the theorem for completeness. By the same argument, we have the following results for the test statistic M given in (2.2) from the expression of M in (3.3).

Theorem 3.2. Under the same assumptions as in theorem 3.1., the limiting distribution of the test statistic M under K_δ is the following.

(1) If $0 < \delta < \frac{1}{2}$, $[M - B(\sigma_1, \ldots, \sigma_k)]/n^{\frac{1}{2}-\delta}$ is distributed asymptotically according to normal distribution with mean zero and variance $\tau_M^2 = (8/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2$, where $B(\sigma_1, \ldots, \sigma_k) = n(\log \sum_{\alpha=1}^{k} \rho_\alpha \sigma_\alpha^2 - \sum_{\alpha=1}^{k} \rho_\alpha \log \sigma_\alpha^2) = O(n^{1-2\delta})$.

(2) If $\delta > \frac{1}{2}$, M has asymptotically χ^2 distribution with k-1 degrees of freedom.

(3) If $\delta = \frac{1}{2}$, M has asymptotically noncentral χ^2 distribution with k-1 degrees of freedom and noncentrality parameter $\delta_M^2 = (2/\sigma^2) \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2$.

Noting that two noncentrality parameters δ_L^2 and δ_M^2 in Theorem 3.1 and Theorem 3.2 are equal, we immediately have the following corollary.

Corollary. The Pitman's asymptotic relative efficiency of the L test with respect to the M test is equal to 1.

The limiting distributions of L and M under the sequence of alternatives K_δ make no difference when $\delta \geq \frac{1}{2}$. Even when $0 < \delta < \frac{1}{2}$, the asymptotic variances τ_L^2 and τ_M^2 are equal. Hence we are interested in the asymptotic means $A(\sigma_1, \ldots, \sigma_k)$ and $B(\sigma_1, \ldots, \sigma_k)$ in case (1). We may expect the L test to have the larger asymptotic power when $A(\sigma_1, \ldots, \sigma_k) > B(\sigma_1, \ldots, \sigma_k)$ and the smaller asymptotic power when $A(\sigma_1, \ldots, \sigma_k) < B(\sigma_1, \ldots, \sigma_k)$. We can easily see that

$$A(\sigma_1, \ldots, \sigma_k) = \frac{2n^{1-2\delta}}{\sigma^2} \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2 - \frac{2n^{1-3\delta}}{\sigma^3} \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})\theta_\alpha^{\ 2} + o(n^{1-4\delta})$$

(3.7)

$$B(\sigma_1, \ldots, \sigma_k) = \frac{2n^{1-2\delta}}{\sigma^2} \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})^2 - \frac{2n^{1-3\delta}}{3\sigma^3} \sum_{\alpha=1}^{k} \rho_\alpha(\theta_\alpha - \tilde{\theta})(\theta_\alpha + 2\tilde{\theta})^2 + o(n^{1-4\delta}).$$

Hence the first main terms in the expansion of $A(\sigma_1, \ldots, \sigma_k)$ and $B(\sigma_1, \ldots, \sigma_k)$ are equal. Putting $\delta = \frac{1}{4}$ and $\theta_1 = \theta_2 = \ldots = \theta_{k-1} = \theta$ (equality of first $k-1$ variances), we have

(3.8)

$$A(\sigma_1, \ldots, \sigma_k) - B(\sigma_1, \ldots, \sigma_k) = \frac{4n^{\frac{1}{4}}}{3\sigma^3} \rho_k(1 - \rho_k)(1 - 2\rho_k)(\theta - \theta_k)^3 + o(1).$$

Thus for large n, $A(\sigma_1, \ldots, \sigma_k) > B(\sigma_1, \ldots, \sigma_k)$ when $\rho_k < \frac{1}{2}$ and $\theta > \theta_k$, whereas the reversed inequality holds when $\rho_k < \frac{1}{2}$ and $\theta < \theta_k$. We cannot make our preference of the two tests L and M against all alternatives from the asymptotic powers near hypothesis.

4. Asymptotic expansions of the distributions of L and M.

4.1. Expansion of the null distribution of L and M. If the hypothesis is true, the statistic M given by (2.2) is well known to have χ^2 distribution with $k-1$ degrees of freedom asymptotically, and further this approximation is improved by multiplying the correction factor c to the statistic M due to Bartlett [2], where

$$(4.1) \qquad c = 1 - \frac{1}{3(k-1)} \left(\sum_{\alpha=1}^{k} \frac{1}{n_\alpha} - \frac{1}{n} \right).$$

Putting $m = cn$, we know the following asymptotic expansion of the distribution of M as in Anderson [1];

$$(4.2) \qquad P(cM \geq z) = P(\chi^2_{k-1} \geq z) + \frac{(k-1)(1-c)^2}{4c^2} \{P(\chi^2_{k-1} \geq z) - P(\chi^2_{k+3} \geq z)\} + O(m^{-3}),$$

where the symbol χ^2_f means random variable having the χ^2 distribution with f degrees of freedom.

We shall first derive the asymptotic expansion of the null distribution of L given by (2.1). The statistic L is rewritten as

$$(4.3)\qquad L = \tfrac{1}{2}\Big[\sum_{\alpha=1}^{k} n_\alpha Z_\alpha^2 - \frac{1}{n}\Big(\sum_{\alpha=1}^{k} n_\alpha Z_\alpha\Big)^2\Big].$$

Under the hypothesis H we may assume that S_α has χ^2 distribution with n_α degrees of freedom. Thus the statistic

$$\sqrt{n_\alpha/2}\, Z_\alpha = \sqrt{n_\alpha/2}\, \log(S_\alpha/n_\alpha)$$

has the density function

$$(4.4)\qquad c_{n_\alpha} \exp\Big[\sqrt{\frac{n_\alpha}{2}}\, y - \frac{n_\alpha}{2} e^{\sqrt{2/n_\alpha}\, y}\Big] \qquad -\infty < y < \infty ,$$

where $c_n = (n/2)^{(n-1)/2} \{\Gamma(n/2)\}^{-1}$. We can express the characteristic function of L as

$$(4.5)\qquad C(t) = \Big(\prod_{\alpha=1}^{k} c_{n_\alpha}\Big) \int \exp\Big[it \sum_{\alpha=1}^{k} y_\alpha^2 - it\Big(\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, y_\alpha\Big)^2\Big]$$

$$\times \exp\Big[\sum_{\alpha=1}^{k} \big(\sqrt{n_\alpha/2}\, y_\alpha - \tfrac{1}{2} n_\alpha e^{\sqrt{2/n_\alpha}\, y_\alpha}\big)\Big]\, dy_1 \ldots dy_k .$$

The second exponential part of the integrand in (4.5) is expanded asymptotically for large n by the formula

$$(4.6)\qquad \sum_{\alpha=1}^{k} n_\alpha e^{\sqrt{2/n_\alpha}\, y_\alpha} = n + \sum_{\alpha=1}^{k} \Big(\sqrt{2n_\alpha}\, y_\alpha + y_\alpha^2 + \frac{\sqrt{2}\, y_\alpha^3}{3\sqrt{n_\alpha}} + \frac{y_\alpha^4}{6n_\alpha}\Big) + O(n^{-3/2}).$$

We can write

$$(4.7)\qquad C(t) = \Big(\prod_{\alpha=1}^{k} c_{n_\alpha} e^{-n_\alpha/2}\Big)\Big\{\int \exp\Big[(it-\tfrac{1}{2}) \sum_{\alpha=1}^{k} y_\alpha^2 - it\Big(\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, y_\alpha\Big)^2\Big]$$

$$\times \Big\{1 - \frac{1}{3\sqrt{2}} \sum_{\alpha=1}^{k} \frac{y_\alpha^3}{\sqrt{n_\alpha}} - \frac{1}{12} \sum_{\alpha=1}^{k} \frac{y_\alpha^4}{n_\alpha} + \frac{1}{36}\Big(\sum_{\alpha=1}^{k} \frac{y_\alpha^3}{\sqrt{n_\alpha}}\Big)^2\Big\}\, dy_1 \ldots dy_k + O(n^{-3/2})\Big\}.$$

The quadratic form $(it - \frac{1}{2}) \sum_{\alpha=1}^{k} y_\alpha^2 - it\big(\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, y_\alpha\big)^2$ can be written as

$-(\frac{1}{2})y'\Sigma^{-1}y$ where $y' = (y_1, \ldots, y_k)$ and

(4.8)

$$\Sigma = \frac{1}{1-2it}\begin{pmatrix} 1-2\rho_1 it, & -2it\sqrt{\rho_1\rho_2}, & \ldots, & -2it\sqrt{\rho_1\rho_k} \\ -2it\sqrt{\rho_2\rho_1}, & 1-2\rho_2 it, & \ldots, & -2it\sqrt{\rho_2\rho_k} \\ \ldots & \ldots & \ldots, & \ldots \\ -2it\sqrt{\rho_k\rho_1}, & -2it\sqrt{\rho_k\rho_2}, & \ldots, & 1-2\rho_k it \end{pmatrix} .$$

The symmetric matrix Σ has a simple characteristic root of one and $k-1$ roots of $1/(1-2it)$. Noting that all characteristic roots of Σ have positive real parts, we can use the following well-known formulas (based on moments of the k variate normal distribution with mean zero and covariance matrix $\Sigma = (\sigma_{\alpha\beta})$);

$$\int \exp[-\tfrac{1}{2}y'\Sigma^{-1}y]dy = (2\pi)^{k/2}|\Sigma|^{\frac{1}{2}}$$

$$\begin{aligned}\int \exp[-\tfrac{1}{2}y'\Sigma^{-1}y]y_\alpha^\ell\, dy &= 0 && \text{if } \ell = 3 \\ &= 3\sigma_{\alpha\alpha}^2(2\pi)^{k/2}\,|\Sigma|^{\frac{1}{2}} && \text{if } \ell = 4 \\ &= 15\sigma_{\alpha\alpha}^3(2\pi)^{k/2}|\Sigma|^{\frac{1}{2}} && \text{if } \ell = 6\end{aligned}$$

(4.9)

$$\int \exp[-\tfrac{1}{2}y'\Sigma^{-1}y]y_\alpha^3 y_\beta^3 dy = (9\sigma_{\alpha\alpha}\,\sigma_{\beta\beta}\,\sigma_{\alpha\beta} + 6\sigma_{\alpha\beta}^3)(2\pi)^{k/2}\;|\Sigma|^{\frac{1}{2}}.$$

Hence we can simplify the characteristic function of L in (4.7) as

(4.10)

$$C(t) = \Big(\prod_{\alpha=1}^{k} c_{n_\alpha}\; e^{-n_\alpha/2}\Big)(2\pi)^{k/2}(1-2it)^{-(k-1)/2}$$

$$\times [1 - \frac{1}{4n}\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha}\Big(\frac{1-2it\rho_\alpha}{1-2it}\Big)^2 + \frac{5}{12n}\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha}\Big(\frac{1-2it\rho_\alpha}{1-2it}\Big)^3$$

$$+ \frac{1}{4n}\sum_{\alpha\neq\beta}\frac{(1-2it\rho_\alpha)(1-2it\rho_\beta)(-2it)}{(1-2it)^3} + \frac{1}{6n}\Big(\frac{-2it}{1-2it}\Big)^3\sum_{\alpha\neq\beta}\rho_\alpha\rho_\beta + O(n^{-2})].$$

In the above expansion, we can easily see that the term of order $n^{-3/2}$ vanishes because any product moment of 5-th degree from normal population with mean zero always vanishes. Applying the Stirling's formula $\log \Gamma(x) = \log\sqrt{2\pi} + (x - \frac{1}{2}) \log x - 1/(12x) + O(x^{-2})$ to the coefficient c_{n_α} in (4.10), we can obtain

$$(4.11) \qquad \prod_{\alpha=1}^{k} (c_{n_\alpha} e^{-n\alpha/2} \sqrt{2\pi}) = 1 - \frac{1}{6n} \sum_{\alpha=1}^{k} (1/\rho_\alpha) + O(n^{-2}).$$

Arranging the second factor of the characteristic function in (4.10) according to the magnitude of negative power of $(1-2it)$ with the above result, we can get the following asymptotic formula;

(4.12)

$$C(t) = (1-2it)^{-(k-1)/2}\left[1 + \frac{1}{12n}\left\{2(1 - \sum_{\alpha=1}^{k} 1/\rho_\alpha)\right.\right.$$

$$\left.\left. + \frac{3k^2 + 6k - 6 - 3\sum_{\alpha=1}^{k}(1/\rho_\alpha)}{(1-2it)^2} - \frac{3k^2 + 6k - 4 - 5\sum_{\alpha=1}^{k}(1/\rho_\alpha)}{(1-2it)^3}\right\}\right] + O(n^{-2}).$$

Inverting this characteristic function we can get the following theorem.

<u>Theorem</u> <u>4.1.</u> The null distribution of Lehmann's test statistic L given by (2.1), expanded asymptotically in terms of the χ^2 distribution for large n with fixed $\rho_\alpha = n_\alpha/n$ (positive), is

$$(4.13) \qquad P(L \le z) = P(\chi^2_{k-1} \le z) + \frac{1}{12n}\left[2(1-\tilde{\rho})P(\chi^2_{k-1} \le z)\right.$$

$$\left. + (3k^2+6k-6-3\tilde{\rho})P(\chi^2_{k+3} \le z) - (3k^2+6k-4-5\tilde{\rho})P(\chi^2_{k+5} \le z)\right] + O(n^{-2})$$

where $\tilde{\rho} = \sum_{\alpha=1}^{k} (1/\rho_\alpha)$.

From this theorem, we can easily get the asymptotic mean of the statistic L when the hypothesis H is true,

$$(4.14) \quad E[L \mid H] = k-1 + \frac{3}{2}\sum_{\alpha=1}^{k} \frac{1}{n_\alpha} - \frac{k(k+2)}{2n} + O(n^{-2}).$$

This result can also be obtained by calculating directly the asymptotic mean of the statistic Z_α and Z_α^2 defined in (2.1). We shall determine a correction factor d such that $E[dL|H] = k-1$, that is the expectation of dL is equal to the mean of the limiting distribution. We have

$$(4.15) \qquad d = \left[1 + \frac{1}{2(k-1)}\left\{3\sum_{\alpha=1}^{k}\frac{1}{n_\alpha} - \frac{k(k+2)}{n}\right\}\right]^{-1} .$$

Then the statistic dL is expected to show better approximation by χ^2 variate with k-1 degrees of freedom for large n. But we could not choose correction factor such that the term of order n^{-1} in the asymptotic expansion of the null distribution of dL vanishes, as is the case with Bartlett's test shown in (4.2). We shall examine the effectiveness of this correction d in the following example.

Example 4.1. When $k = 2$ and $n_1 = n_2 = 50$, correction factor d is given by 1/1.02. Five percent point of the χ^2 distribution with one degrees of freedom is 3.84146. From Theorem 4.1. and the formula (4.2) for Bartlett's test, we have

$$(4.16) \qquad \begin{aligned} P(L \geqq 3.84146) &= 0.0526 + O(n^{-2}) \\ P(dL \geqq 3.84146) &= 0.0503 + O(n^{-2}) \\ P(cM \geqq 3.84146) &= 0.0500 + O(m^{-3}). \end{aligned}$$

Thus the value of 3.84146/d may be used as an approximate five percent point for the L test in this case.

4.2. Expansion of the non-null distribution of L. We shall now consider the asymptotic expansion of the non-null distribution of L under a fixed alternative. Putting

$$(4.17) \qquad L' = L - (n/2)\sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$$

in (3.2), we can easily see that $(L'/\sqrt{n}) - \sum_{\alpha=1}^{k} \sqrt{2\rho_\alpha}(\tilde{\sigma}_\alpha - \tilde{\sigma})\, T_\alpha = o_p(n^{-\frac{1}{2}})$.
Hence the statistic $L'/\sqrt{n}$ converges in law to the normal distribution with mean zero and variance $\tau_L^2 = \sum_{\alpha=1}^{k} 2\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$. More precisely we have

$$(4.18) \qquad n^{-\frac{1}{2}} L' = \ell_0(T) + n^{-\frac{1}{2}}\ell_1(T) + n^{-1}\ell_2(T) + o_p(n^{-3/2}),$$

where each term is given by

$$\ell_0(T) = \sum_{\alpha=1}^{k} \sqrt{2\rho_\alpha}\, (\tilde{\sigma}_\alpha - \tilde{\sigma})\, T_\alpha$$

$$(4.19) \qquad \ell_1(T) = \sum_{\alpha=1}^{k} a_\alpha T_\alpha^2 - \Big(\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\, T_\alpha\Big)^2$$

$$\ell_2(T) = \sum_{\alpha=1}^{k} a'_\alpha\, T_\alpha^3 + \Big(\sum_{\alpha=1}^{k} \sqrt{2\rho_\alpha}\, T_\alpha\Big)\Big(\sum_{\alpha=1}^{k} T_\alpha{}^2\Big)$$

with $a_\alpha = \tilde{\sigma} - \tilde{\sigma}_\alpha + 1$ and $a'_\alpha = \sqrt{2}\, \rho_\alpha{}^{-\frac{1}{2}} \{(2/3)(\tilde{\sigma}_\alpha - \tilde{\sigma}) - 1\}$. Hence the characteristic function of $L'/(\sqrt{n}\, \tau_L)$ $(\tau_L > 0)$ is expressed as

$$(4.20) \qquad c_L(t) = E\Big[e^{it\ell_0(T)/\tau_L}\Big\{1 + n^{-\frac{1}{2}} it\ell_1(T)/\tau_L + n^{-1}[it\ell_2(T)/\tau_L + \tfrac{1}{2}(it)^2\ell_1(T)^2/\tau_L^2]\Big\}\Big] + o(n^{-3/2}).$$

Since $T_\alpha = (\chi^2_{n_\alpha} - n_\alpha)/\sqrt{2n_\alpha}$, we can easily obtain the following formulas from the χ^2 distribution.

$$(1) \quad E[e^{tT_\alpha}] = (1 - \sqrt{2/n_\alpha}\, t)^{-n_\alpha/2} \exp(-\sqrt{n_\alpha/2}\, t)$$
$$= \Big\{1 + \tfrac{1}{3}\sqrt{2/n_\alpha}\, t^3 + \tfrac{1}{n_\alpha}\big(\tfrac{1}{2}t^4 + \tfrac{1}{9}t^6\big)\Big\} e^{t^2/2} + o(n^{-3/2})$$

$$(2) \quad E[T_\alpha e^{tT_\alpha}] = t(1 - \sqrt{2/n_\alpha}\, t)^{-1} E[e^{tT_\alpha}]$$
$$= \Big\{t + \sqrt{2/n_\alpha}\big(\tfrac{1}{3}t^4 + t^2\big)\Big\} e^{t^2/2} + o(n^{-1})$$

(4.21)

$$(3) \quad E[T_\alpha^2\, e^{tT_\alpha}] = (1+t^2)(1-\sqrt{2/n_\alpha}\, t)^{-2} E[e^{tT_\alpha}]$$
$$= \{(1+t^2) + \sqrt{2/n_\alpha}\big(\tfrac{1}{3}t^5 + \tfrac{7}{3}t^3 + 2t\big)\} e^{t^2/2} + o(n^{-1})$$

(4) $E[T_\alpha^3 e^{tT_\alpha}] = (t^3 + 3t)e^{t^2/2} + o(n^{-\frac{1}{2}})$

(5) $E[T_\alpha^4 e^{tT_\alpha}] = (t^4 + 6t^2 + 3)e^{t^2/2} + o(n^{-\frac{1}{2}})$.

Applying the first formula (1) to the first term in (4.20) with the abbreviated notation $b_\alpha = \sqrt{2\rho_\alpha}\,(it/\tau_L)(\tilde{\sigma}_\alpha - \tilde{\sigma})$ in $\ell_o(T)$, we have

$$E[e^{it\ell_o(T)/\tau_L}] = e^{-t^2/2}\Big[1 + \frac{1}{3}\sqrt{2/n}\sum_{\alpha=1}^{k} b_\alpha^3/\sqrt{\rho_\alpha} \tag{4.22}$$
$$+ n^{-1}\Big\{\frac{1}{9}\Big(\sum_{\alpha=1}^{k} b_\alpha^3/\sqrt{\rho_\alpha}\Big)^2 + \frac{1}{2}\sum_{\alpha=1}^{k} b_\alpha^4/\rho_\alpha\Big\}\Big] + o(n^{-3/2}).$$

Noting that $\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}b_\alpha = 0$, we can write each expectation in (4.20) by formula (4.21) as

$$E[\ell_1(T)e^{it\ell_o(T)/\tau_L}] = e^{-t^2/2}\Big[\Sigma\, a_\alpha b_\alpha^2 + \Sigma\, a_\alpha - 1 \tag{4.23}$$
$$+ \sqrt{2}\, n^{-\frac{1}{2}}\Big\{\frac{1}{3}(\Sigma\, b_\alpha^3/\sqrt{\rho_\alpha})(\Sigma\, a_\alpha b_\alpha^2) + \frac{1}{3}(\Sigma\, b_\alpha^3/\sqrt{\rho_\alpha})(\Sigma\, a_\alpha - 1)$$
$$+ 2\Sigma\, a_\alpha b_\alpha^3/\sqrt{\rho_\alpha} + 2\Sigma\, a_\alpha b_\alpha/\sqrt{\rho_\alpha}\Big\}\Big] + o(n^{-1})$$

$$E[\ell_2(T)e^{it\ell_o(T)/\tau_L}] = e^{-t^2/2}[\Sigma\, a'_\alpha b_\alpha^3 + 3\Sigma\, a'_\alpha b_\alpha] + o(n^{-\frac{1}{2}}) \tag{4.24}$$

$$E[\ell_1(T)^2\, e^{it\ell_o(T)/\tau_L}] = e^{-t^2/2}\Big[(\Sigma\, a_\alpha b_\alpha^2)^2 + 4\Sigma\, a_\alpha^2 b_\alpha^2 \tag{4.25}$$
$$+ 2\Sigma\, a_\alpha b_\alpha^2(\Sigma\, a_\alpha - 1) + 2\Sigma\, a_\alpha^2 + (\Sigma a_\alpha)^2 - 4\Sigma\, a_\alpha\rho_\alpha - 2\Sigma a_\alpha + 3\Big] + o(n^{-\frac{1}{2}}),$$

where the symbol Σ means the summation $\sum_{\alpha=1}^{k}$. It follows that the characteristic function of $L'/\sqrt{n}\tau_L$ is expanded asymptotically as

$$C_L(t) = e^{-t^2/2}\Big[1 + n^{-\frac{1}{2}}\Big\{(it/\tau_L)\Big(\sum_{\alpha=1}^{k}[\tilde{\sigma} - \tilde{\sigma}_\alpha] + k-1\Big) \tag{4.26}$$
$$+ (it/\tau_L)^3\Big(\tau_L^2 - \frac{2}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3\Big)\Big\} + n^{-1}\sum_{\alpha=1}^{3}(it/\tau_L)^{2\alpha}\, g_{2\alpha} ,$$

where coefficients g_2, g_4, g_6 are given by

$$g_2 = \Sigma(\tilde{\sigma}_\alpha - \tilde{\sigma})^2 + \frac{1}{2}\{\Sigma(\tilde{\sigma}_\alpha - \tilde{\sigma})\}^2 - (k+3)\ \Sigma(\tilde{\sigma}_\alpha - \tilde{\sigma}) + \frac{1}{2}(k^2-1)$$

$$g_4 = \frac{2}{3}\Sigma\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^4 + \frac{2}{3}\Sigma(\sigma_\alpha - \tilde{\sigma})\ \Sigma\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3 - \frac{2}{3}(k+5)\ \Sigma\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3$$

(4.27)

$$- \Sigma(\tilde{\sigma}_\alpha - \tilde{\sigma})\ \tau_L^2 + (k+1)\ \tau_L^2$$

$$g_6 = \frac{2}{9}\{\Sigma\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3\}^2 - \frac{2}{3}\tau_L^2\ \Sigma\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3 + \frac{1}{2}\tau_L^4 .$$

Inverting this characteristic function, we have the following theorem.

Theorem 4.2. Under fixed alternative K, the distribution of the statistic $L' = L - (n/2) \sum_{\alpha=1}^{k} \rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$, where L is given by (2.1) with $\tilde{\sigma}_\alpha = \log\sigma_\alpha^2$ and $\tilde{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha\tilde{\sigma}_\alpha$, is expanded asymptotically for large n as

(4.28)

$$P(L'/(\sqrt{n}\tau_L) \leq z) = \Phi(z) - n^{-\frac{1}{2}}[\Phi^{(1)}(z)\tau_L^{-1}\{\sum_{\alpha=1}^{k}(\tilde{\sigma} - \tilde{\sigma}_\alpha) + k-1\}$$

$$+ \Phi^{(3)}(z)\tau_L^{-3}\{\tau_L^2 - \frac{2}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^3\}] + n^{-1}\sum_{\alpha=1}^{3}\Phi^{(2\alpha)}(z)g_{2\alpha}/\tau_L^{2\alpha} + O(n^{-3/2}),$$

where $\tau_L^2 = 2\sum_{\alpha=1}^{k}\rho_\alpha(\tilde{\sigma}_\alpha - \tilde{\sigma})^2$ and $\Phi^{(j)}(z)$ means the j-th derivative of the standard normal distribution function $\Phi(z)$. The coefficients $g_{2\alpha}$ are given by (4.27).

4.3. Expansion of the non-null distribution of cM. We shall now consider the asymptotic expansion of the distribution of cM test (= Bartlett's test) under a fixed alternative. Putting $cn_\alpha = m_\alpha$ for $\alpha = 1, 2, \ldots, k$, where correction factor c is given by (4.1), we can write

$$cM = m\log(\sum_{\alpha=1}^{k} S_\alpha/m) - \sum_{\alpha=1}^{k} m_\alpha\log(S_\alpha/m_\alpha)$$

with $m = \sum_{\alpha=1}^{k} m_\alpha$. Let $U_\alpha = [(S_\alpha/\sigma_\alpha^2) - m_\alpha]/\sqrt{2m_\alpha}$, then cM is expressed by U_α as

$$(4.29)\qquad cM = m(\log \bar{\sigma} - \tilde{\sigma}) + \sqrt{m}\,q_0(U) + q_1(U) + m^{-\frac{1}{2}}q_2(U) + o_p(m^{-1}) ,$$

where $\bar{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \sigma_\alpha^2$, $\tilde{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \log \sigma_\alpha^2$ and

$$q_0(U) = \sum_{\alpha=1}^{k} \sqrt{2\rho_\alpha}(\nu_\alpha - 1)U_\alpha$$

$$(4.30)\qquad q_1(U) = \sum_{\alpha=1}^{k} U_\alpha^2 - \Big(\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\,\nu_\alpha U_\alpha\Big)^2$$

$$q_2(U) = \tfrac{2}{3}\sqrt{2}\Big\{\Big(\sum_{\alpha=1}^{k} \sqrt{\rho_\alpha}\,\nu_\alpha U_\alpha\Big)^3 - \sum_{\alpha=1}^{k} U_\alpha^3/\sqrt{\rho_\alpha}\Big\}$$

with $\nu_\alpha = \sigma_\alpha^2/\bar{\sigma}$ for abbreviation. Note that since the random variables $U_1, \ldots, U_k$ are independent and each of them has asymptotically the standard normal distribution as $m \to \infty$, the statistic $M'/\sqrt{m} = \{cM - m(\log \bar{\sigma} - \tilde{\sigma})\}/\sqrt{m}$ is distributed asymptotically according to the normal distribution with zero mean and variance $\tau_M^2 = 2\sum_{\alpha=1}^{k} \rho_\alpha(\nu_\alpha - 1)^2$. Further the characteristic function of $M'/(\sqrt{m}\tau_M)$ $(\tau_M > 0)$ can be expressed as

$$(4.31)\qquad c_M(t) = E\Big[e^{itq_0(U)/\tau_M}\Big\{1 + m^{-\frac{1}{2}}\, itq_1(U)/\tau_M + m^{-1}[itq_2(U)/\tau_M + \tfrac{1}{2}(it)^2 q_1(U)^2/\tau_M^2]\Big\}\Big] + o(m^{-3/2}) .$$

Corresponding to the formulas (4.21) for T_α , we have

$$(1)\quad E[e^{tU_\alpha}] = e^{t^2/2}\Big[1 + m_\alpha^{-\frac{1}{2}}\{\tfrac{1}{2}\sqrt{2}\,\Delta\rho_\alpha t + \tfrac{1}{3}\sqrt{2}t^3\} + m_\alpha^{-1}\{\tfrac{1}{2}t^4 + \tfrac{1}{2}\Delta\rho_\alpha t^2 + (\tfrac{1}{3}t^3 + \tfrac{1}{2}\Delta\rho_\alpha t)^2\}\Big] + o(m^{-3/2})$$

$$(2)\quad E[U_\alpha e^{tU_\alpha}] = e^{t^2/2}\Big[t + m_\alpha^{-\frac{1}{2}}\sqrt{2}\{\tfrac{1}{3}t^4 + t^2(1 + \tfrac{1}{2}\Delta\rho_\alpha) + \tfrac{1}{2}\Delta\rho_\alpha\}\Big] + o(m^{-1})$$

(4.32)

$$(3)\quad E[U_\alpha^2 e^{tU_\alpha}] = e^{t^2/2}\Big[t^2 + 1 + m_\alpha^{-\frac{1}{2}}\sqrt{2}\{\tfrac{1}{3}t^5 + (\tfrac{7}{3} + \tfrac{1}{2}\Delta\rho_\alpha)t^3 + (\tfrac{3}{2}\Delta\rho_\alpha + 2)t\}\Big] + o(m^{-1})$$

$$(4)\quad E[U_\alpha^3 e^{tU_\alpha}] = e^{t^2/2}(t^3 + 3t) + o(m^{-\frac{1}{2}})$$

(5) $E[U_\alpha^4 e^{tU_\alpha}] = e^{t^2/2}(t^4 + 6t^2 + 3) + O(m^{-\frac{1}{2}})$,

where $\Delta = n(1-c) = O(1)$. If we put $\Delta = 0$ and change m_α to n_α in these formulas, we have the same results as in (4.21). After some computation with the abbreviated notation $b_\alpha = \sqrt{2\rho_\alpha}(\nu_\alpha - 1)it/\tau_M$ in $q_o(U)$ and $\Sigma a_\alpha = \sum_{\alpha=1}^{k} a_\alpha$, we have

$$E[e^{itq_o(U)/\tau_M}] = e^{-t^2/2}\Big[1 + \frac{1}{3}\sqrt{2/m}\ \Sigma b_\alpha^3/\sqrt{\rho_\alpha} + m^{-1}\Big\{\frac{1}{9}(\Sigma b_\alpha^3/\sqrt{\rho_\alpha})^2 + \frac{1}{2}\Sigma b_\alpha^4/\rho_\alpha + \frac{1}{2}\Delta\Sigma b_\alpha^2\Big\}\Big] + O(m^{-3/2}) . \tag{4.33}$$

Putting $a_\alpha = \sqrt{\rho_\alpha}\ \nu_\alpha$ in $q_1(U)$ and $q_2(U)$ in (4.30), we have

$$E[q_1(U)e^{itq_o(U)/\tau_M}] = e^{-t^2/2}\Big[\Sigma b_\alpha^2 - (\Sigma a_\alpha b_\alpha)^2 + k - \Sigma a_\alpha^2 + m^{-\frac{1}{2}}\sqrt{2}\Big\{\frac{1}{3}(\Sigma b_\alpha^3/\sqrt{\rho_\alpha})(\Sigma b_\alpha^2 - [\Sigma a_\alpha b_\alpha]^2) + (\Sigma b_\alpha^3/\sqrt{\rho_\alpha})(\frac{1}{3}k + 2 - \frac{1}{3}\Sigma a_\alpha^2) - 2(\Sigma a_\alpha b_\alpha^2/\sqrt{\rho_\alpha})(\Sigma a_\alpha b_\alpha) + 2\Sigma b_\alpha(1-a_\alpha^2)/\sqrt{\rho_\alpha} - \Delta\Sigma a_\alpha\sqrt{\rho_\alpha}\ \Sigma a_\alpha b_\alpha\Big\}\Big] + O(m^{-1}) \tag{4.34}$$

$$E[q_2(U)e^{itq_o(U)/\tau_M}] = e^{-t^2/2}\ \frac{2}{3}\sqrt{2}\Big\{(\Sigma a_\alpha b_\alpha)^3 - \Sigma b_\alpha^3/\sqrt{\rho_\alpha} + 3\Sigma a_\alpha^2\ \Sigma a_\alpha b_\alpha - 3\Sigma b_\alpha/\sqrt{\rho_\alpha}\Big\} + O(m^{-\frac{1}{2}}) \tag{4.35}$$

$$E[q_1(U)^2e^{itq_o(U)/\tau_M}] = e^{-t^2/2}\Big[\{\Sigma b_\alpha^2 - (\Sigma a_\alpha b_\alpha)^2\}^2 + 2\Sigma b_\alpha^2(k + 2 - \Sigma a_\alpha^2) + 2(\Sigma a_\alpha b_\alpha)^2(3\Sigma a_\alpha^2 - k - 4) + 3(\Sigma a_\alpha^2)^2 - 2(k+2)\Sigma a_\alpha^2 + k(k+2)\Big] + O(m^{-\frac{1}{2}}) , \tag{4.36}$$

which implies the following asymptotic formula of the characteristic function of $M'/(\sqrt{m}\ \tau_M)$.

$$C_M(t) = e^{-t^2/2}\Big[1 + m^{-\frac{1}{2}}\Big\{(it/\tau_M)(k - \sum_{\alpha=1}^{k}\rho_\alpha\nu_\alpha^2) + (it/\tau_M)^3 \times (\frac{4}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\nu_\alpha - 1)^3 + \tau_M^2 - \frac{1}{2}\tau_M^4\Big\} + m^{-1}\sum_{\alpha=1}^{3}(it/\tau_M)^{2\alpha}\ h_{2\alpha}\Big] , \tag{4.37}$$

where the coefficients h_2, h_4 and h_6 are given by

$$h_2 = \frac{11}{2}(\Sigma\rho_\alpha\nu_\alpha^2)^2 - 4\Sigma\rho_\alpha\nu_\alpha^3 - (k+2)\Sigma\rho_\alpha\nu_\alpha^2 + k(k+2)/2 - \frac{1}{2}\Delta\tau_M^2$$

$$h_4 = 2\Sigma\rho_\alpha(\nu_\alpha-1)^4 + \frac{4}{3}\Sigma\rho_\alpha(\nu_\alpha-1)^3(k + 4 - \Sigma\rho_\alpha\nu_\alpha^2)$$

(4.38)

$$+ \tau_M^2\left\{k + 1 - 4\Sigma\rho_\alpha\nu_\alpha(\nu_\alpha-1)^2\right\} + \frac{1}{2}\tau_M^4(3\Sigma\rho_\alpha\nu_\alpha^2 - k - 5) + \frac{1}{3}\tau_M^6$$

$$h_6 = \frac{8}{9}\{\Sigma\rho_\alpha(\nu_\alpha-1)^3\}^2 + \frac{2}{3}\Sigma\rho_\alpha(\nu_\alpha-1)^3(2 - \tau_M^2)\ \tau_M^2 + \frac{1}{8}\tau_M^4(2 - \tau_M^2)^2 .$$

Inverting this characteristic function, we have the following theorem.

Theorem 4.3. Under fixed alternative K, the distribution of the statistic $M' = cM - m(\log\bar{\sigma}-\tilde{\sigma})$, where cM is Bartlett's test statistic given by (2.2) and (4.1) with $\bar{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha\sigma_\alpha^2$ and $\tilde{\sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \log \sigma_\alpha^2$, can be expanded asymptotically for large $m(= nc)$ as

$$(4.39)\qquad P(M'/(\sqrt{m}\ \tau_M) \le z) = \Phi(z) - m^{-\frac{1}{2}}\Big[\Phi^{(1)}(z)\tau_M^{-1}(k - \sum_{\alpha=1}^{k}\rho_\alpha\nu_\alpha^2)$$

$$+ \Phi^{(3)}(z)\tau_M^{-3}\left\{\frac{4}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\nu_\alpha-1)^3 + \tau_M^2 - \frac{1}{2}\tau_M^4\right\}\Big] + m^{-1}\sum_{\alpha=1}^{3}\Phi^{(2\alpha)}(z)h_{2\alpha}/\tau_M^{2\alpha} + O(m^{-3/2})$$

where $\tau_M^2 = 2\sum_{\alpha=1}^{k}\rho_\alpha(\nu_\alpha-1)^2$ with $\nu_\alpha = \sigma_\alpha^2/\bar{\sigma}$ and $h_{2\alpha}(\alpha = 1, 2, 3)$ are given by (4.38) with $\Delta = n(1-c)$.

The limiting distribution of the statistic M in multivariate model has been obtained by Sugiura [8] and coincides with the first term of the formula (4.39) in Theorem 4.3. Since asymptotic variances τ_L^2 and τ_M^2 vanish when the hypothesis is true, these asymptotic formulas for the distribution of L and M do not give good approximation, when the alternative hypothesis K is near to the null hypothesis.

4.4 Numerical example. We shall finally show some numerical values of the asymptotic power of Lehmann's test (= dL) and Bartlett's test (= cM) in the following special cases.

Example 4.2. When $k = 2$ and $n_1 = n_2 = 50$, the L test and the M test are equivalent by Theorem 2.1. From asymptotic formula (4.28) and (4.39) with the result in Example 4.1, we have the following approximate powers when $\sigma_1^2 = 2\sigma_2^2$.

	$P_K(dL \geq 3.84146)$	$P_K(cM \geq 3.84146)$
first term	0.6649	0.6642
second term	0.0135	0.0124
third term	0.0014	0.0020
approx. power	0.680	0.679

These two powers should be equal, because $k = 2$ and $n_1 = n_2$, within the accuracy of five percent point of two tests given in Example 4.1. Thus our result gives a reasonable approximation to this problem.

Example 4.3. When $k = 2$ and $n_1 = 4$, $n_2 = 20$, exact values of the power of cM test for some alternatives have been given by Ramachandran [6]. From his table, we can also obtain exact five percent point of cM as 3.795. Formula (4.2) shows $P_H(cM \geq 3.795) = 0.0502 + O(m^{-3})$. Formula (4.39) shows the following approximate powers of cM test for the alternatives $K : \sigma_2{}^2 = \delta\sigma_1{}^2$.

	$P_\delta(cM \geq 3.795)$		
	$\delta = 10$	$\delta = 5$	$\delta = 10/3$
first term	0.6572	0.3224	0.1398
second term	0.0748	0.0804	0.1145

third term	0.0001	-0.0013	-0.0170
approx. power	0.732	0.402	0.237
exact power	0.729	0.397	0.230

When δ is less than $10/3$, the first term becomes smaller than the second term. Thus we cannot apply our formula effectively for alternatives near H.

Example 4.4. When $k = 3$ and $n_1 = 50$, $n_2 = 100$, $n_3 = 150$, we have $P_H(dL \geqq 5.99147) = 0.0507 + O(n^{-2})$ from (4.13) and $P_H(cM \geqq 5.99147) = 0.0500 + O(m^{-3})$ from (4.2) We shall specify the alternatives K as $\sigma_2{}^2 = \delta\sigma_1{}^2$ and $\sigma_3{}^2 = \delta^2\sigma_1{}^2$. Then the formulas (4.28) and (4.39) give the following approximate powers.

	$P_\delta(dL \geqq 5.99147)$		$P_\delta(cM \geqq 5.99147)$	
	$\delta = 1.5$	$\delta = 0.7$	$\delta = 1.5$	$\delta = 0.7$
first term	0.8483	0.7562	0.8430	0.7658
second term	0.0783	0.0556	0.0700	0.0615
third term	-0.0014	0.0070	0.0028	0.0077
approx. power	0.925	0.819	0.916	0.835

This example seems to show that for $\delta = 1.5$ the power of Lehman's test is larger than that of Bartlett's test and for $\delta = 0.7$ the reverse inequality holds though the differences are small.

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