GRAPHS AND PARTIAL ORDERINGS

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I. Introduction.

Every partial ordering of a set S gives rise to an undirected graph G, called its comparability graph, by identifying the vertices of G with the points of S and joining two vertices of G iff their corresponding points in S are comparable. The problem of characterizing comparability graphs was solved by Wolk [6] for the case of a tree, and later a characterization in the general case in terms of subgraphs G must not contain was given by Gilmore-Hoffman [2] and independently by Ghouilà-Houri [1]. Before we give their solution a few definitions are in order.

Definition: Let G be a finite undirected graph without loops or multiple edges [1)], let V(G), E(G) its vertex-and-edge-set, respectively, then a generalized path (of length k-1) is a progression of vertices $a_1, a_2, \ldots, a_k$, $a_i \in V(G)$, $(a_i, a_{i+1}) \in E(G)$, such that no two ordered pairs (a_i, a_{i+1}), (a_j, a_{j+1}) are the same. (i.e. an edge may be traversed twice, but at most once in either direction). We speak of a generalized cycle, if the vertex-progression is a generalized path with $a_1 = a_k$.

Definition: Given the generalized path $a_1, a_2, \ldots, a_k$, then an edge joining two vertices a_i, a_j with $|j - i| > 1$ is called a chord, in the case $|j - i| = 2$ we speak of a triangular chord. Similar definitions hold for generalized cycles with the understanding that the definitions for chords are given modulo $k - 1$.

The Gilmore-Hoffman theorem now reads: G is a comparability-graph iff G contains no generalized cycles of odd length without triangular chords.

1) For the remainder of this paper, a graph will always be assumed to be of that kind.

In figures 1 and 2, we exhibit two examples G_1, G_2 of graphs G which do not permit partial orderings; they will be seen to be basic for some of the later developments [2].

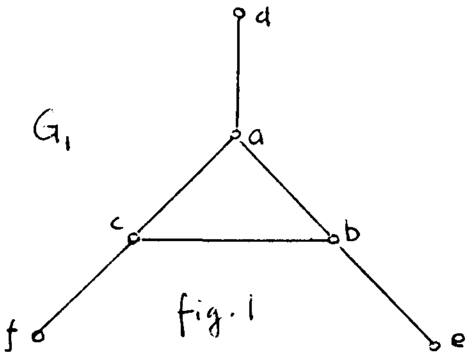

fig. 1

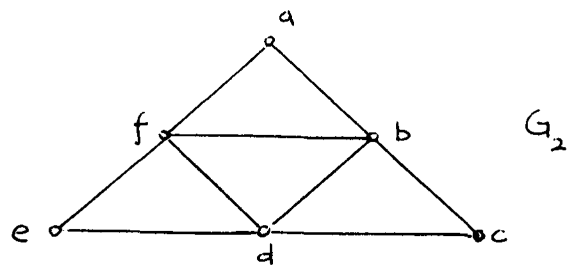

fig. 2

In this paper we investigate some of the structural properties of comparability - and non - comparability - graphs.

In section II, we answer an extremal question by determining all NPO-graphs with a minimum number of lines and section III is devoted to the line-graph of a given graph and problems that arise in connection with partial orderings.

There seems to be an analogy between "planar - non - planar" and "comparability - non - comparability", as first the Kuratowski - type characterization in both instances, and then some theorems in this paper indicate. For example, theorems 3 and 4 have analogues for the planar - non - planar case (Sedlacek [4]), the extremal problem of section II has been solved by Wagner [5] for planar - non planar graphs for the point-version (the line-version being trivial). We hope to further develop this program in a subsequent communication.

2) G_1, G_2 are also two of the fundamental forbidden subgraphs for indifference - systems, cf. Roberts [3].

Terminology:

1. PO-graph stands for comparability-graph, NPO-graph for noncomparability-graph.

2. We denote edges by (a, b), where a, b are the two end-points; $a \sim b$ iff $(a, b) \in E(G)$, $a \not\sim b$ iff $(a, b) \notin E(G)$.

3. By the full subgraph on S of G or subgraph induced by the set $S \subset V(G)$ we mean the subgraph which has S as its vertex-set and includes all edges of E(G) between any two vertices of S.

4. {a, b, ..., c, d} denotes the generalized cycle $a = a_1$, $b = a_2$, ..., $c = a_{k-2}$, $d = a_{k-1}$, with $a_k = a_1 = a$. We speak of the sequence of vertices a, b, ..., to indicate the direction in which we run through the cycle. (e.g. the sequence a, b is different from the edge (a, b)). Subcycles of a given generalized cycle are again indicated by their first and last vertices as long as there is no danger of ambiguity. (e.g. subcycle {b, ...} in the example above means the given cycle minus the first vertex a and the last two vertices c, d, (b, of course, must be adjacent to the vertex immediately preceding c)).

5. Cycles of length k are denoted by C_k; we speak of a simple cycle if no vertex occurs more than once.

II. Minimal Configurations.

According to the Gilmore-Hoffman theorem, an NPO-graph must contain a generalized cycle of odd length without triangular chords which we will call a GH-cycle for the remainder of the paper. In this section the properties of a shortest such cycle in a NPO-graph will be investigated and in theorem 2 all the minimal graphs (in the sense that the deletion of any edge results in a PO-graph) will be determined.

Theorem 1: An arbitrary NPO-graph G must contain a block which together with its outgoing edges does not admit a partial ordering.

Proof: The algorithm designed by Gilmore and Hoffman allows us to start with any particular edge (or for that matter, with any PO-subgraph of G) in order to construct a partial ordering of the points. Hence if all the blocks plus their outgoing edges are PO-graphs, we may start with anyone of them and then keep on orienting the edges. Since by the definition of a block, we never return to the same block once we left it, the algorithm clearly yields a PO-graph.

Remark 1: In theorem 1, we cannot dispose of the condition "with its outgoing edges" as the graph G_1 illustrates.

Remark 2: A trivial corollary of theorem 1 is the fact that all forests can be partially ordered.

In view of theorem 1, we, henceforth, confine ourselves to blocks plus possible outgoing edges. In lemmas 1 - 4, we will study a shortest GH-cycle C of an NPO-graph G. We run through C in one of the two possible directions, but keep this direction fixed once we have chosen it. If the vertex b follows the vertex a, we call a the predecessor of b, b the successor of a, and indicate this fact by a, b.

Lemma 1: Suppose the vertex a appears more than once in C, say, $C = \{a, b, \ldots\ldots, c, a, d, \ldots, e\}$. Suppose w.l.o.g. that $C' = \{a, b, \ldots, c\}$ is of odd length, then

(1) $b \sim c$,

(2) $b \sim d$, $c \sim e$, unless $C = \{a, b, \ldots, c, a, d = e\}$,

(3) $d \sim e$, unless $C'' = \{a, d, \ldots, e\}$ has length 4.

Proof: C' is a generalized cycle of odd length, which together with the hypothesis on C implies (1). If $b \not\sim d$ or $c \not\sim e$ and C" is of length at least 4, then the generalized cycle $\{a, b, \ldots, c, a, d\}$[3] or $\{a, b, \ldots, c, a, e\}$[3], respectively, would contradict the minimality of C. To prove (3) we assume C" has length greater than 4, then by looking at the generalized cycle $\{a, b, \ldots, c, a, d, a, e\}$[3] we infer $d \sim e$, making again use of the hypothesis on C.

Lemma 2: If for two vertices a and b, C contains both sequences a, b and b, a, then they must be consecutive sequences, i.e. C contains a, b, a or b, a, b.

Proof: Let $C = \{\ldots a, b, \ldots, b, a \ldots\}$, then we may assume w.l.o.g. that $C' = \{b, \ldots\}$ and hence $C'' = \{a, b, \ldots, b\}$ are of odd length. But this would clearly contradict (1) in lemma 1, thus the conclusion follows.

Lemma 3: Let G be a minimal NPO-graph and C as before, then every edge of G must appear in C at least once.

Proof: The deletion of any edge not in C would not alter the character of G as to partial orderings, in contradiction to the minimality of G.

3) It may, of course, happen that these cycles are not in conformity with the definition for generalized cycles any more, i.e. the sequences d, a or a, e may occur twice, but these cases are easily seen to yield the same conclusions.

Lemma 4: Let G be a minimal NPO-graph and C a shortest GH-cycle. Suppose that C contains the sequence a, b, c, d with $a \sim d$, then $a = c$ or $b = d$.

Proof: We proceed to prove the assertion by contradiction. Since $a \sim d$, C contains the edge (a, d). Assume first $C = \{a, b, c, d, \ldots, a, d, \ldots\}$. We have to consider two cases depending on whether $C' = \{d, \ldots, a\}$ has odd or even length. In the first case, (2) would imply $a \sim c$, contradicting the fact that C does not possess any triangular chords. In the latter case, we have $C = \{a, d, \ldots, a, b, c, d, \ldots\}$ with $C'' = \{a, d, \ldots\}$ being a generalized cycle of odd length. By appealing to (2) again, we conclude $b \sim d$, a contradiction. The case where C contains d as predecessor of a can be settled in an analogous manner, thus the edge (a, d) must appear within the sequence a, b, c, d, and the lemma follows.

Every simple odd cycle without chords of length at least 5 obviously is a minimal NPO-graph, as are the graphs G_1 and G_2 in section I. The following theorem makes the converse assertion that these are all the minimal graphs.

Theorem 2: The only minimal NPO-graphs are the simple odd cycles of length ≥ 5 without chords and the graphs G_1 and G_2.

Proof: Let G be an arbitrary minimal NPO-graph and C a shortest GH-cycle. If no vertex of G appears more than once in C, then C clearly represents a simple cycle of odd length. Since by lemma 3, C must contain all the edges of G, there can be no chords in C, and we obtain the first class of the above mentioned graphs.

Suppose now there are points that occur at least twice in C, then if x is such a vertex appearing, say, k times, we can think of C as the union of k cycles, each starting and terminating at x. Since C is of odd length, at least one of these cycles must also have odd length; let us denote by $C(x)$ one of these cycles of shortest odd length. In the set of points appearing at least twice, choose the point a such that $C(a) = C'$ is a cycle of shortest length among all $C(x)$, call the complementary cycle C'', and let $C = \{a, b, c, \ldots, d, e, a, f, h, \ldots h', g\}$ with $C' = \{a, b, c, \ldots, d, e\}$ and $C'' = \{a, f, \ldots, g\}$. If we can show that G contains either G_1 or G_2 as a full subgraph, then the theorem will follow. By the construction of C' it is clear that whenever a vertex appears more than once in C', the number of edges between the two occurrences is even, a fact which will be used extensively in the sequel. In C', we have $b \sim e$ (by (1)), and by the minimality of $C' = C(a)$ we infer $c \neq d$, since otherwise $C(c)$ would be shorter than $C(a)$. Thus a, b, c, d, e are 5 distinct vertices.

Case (A): $f = g$. Here lemma 2 implies that $C'' = \{a, f\}$, and by the minimality of $C' = C(a)$ again we have $f \not\sim c$, $f \not\sim d$. Furthermore, we notice $f \neq b$, $f \neq e$ and clearly $f \not\sim b$, $f \not\sim e$, and so the following situation results:

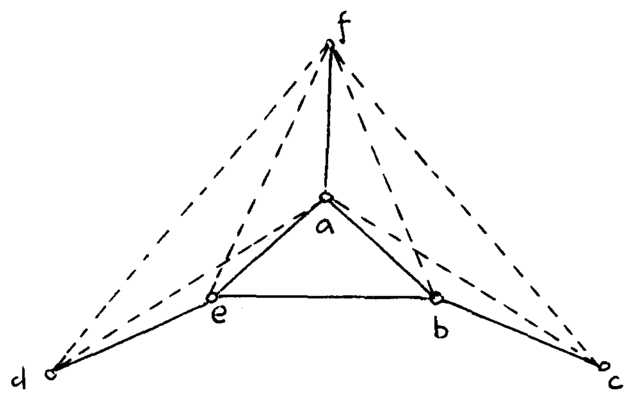

fig. 3

where the dotted lines indicate that these edges are missing. To show that $\{a, b, c, d, e, f\}$ induce G_1, we have to demonstrate the absence of the 3 edges (b, d), (c, e), (c, d). The first two of these edges are missing because they cannot be in C' = C(a) (they would violate the minimality of C'), and C" only consists of a, f, a. Finally, if (c, d) were in G, we could delete it and still retain a NPO-graph, namely G_1, herewith contradicting the hypothesis on G.

Case (B): $f \neq g$. The edge (b, e) must be contained in C', since otherwise the deletion of (b, e) would yield the GH-cycle {a, b, c, ..., d, e}, hence G would not be minimal. Now we may assume w.l.o.g. that e is the successor of b in C', since if we have the sequence y, e, b, y' in C', then the generalized cycles {a, b, y', ... d, e} and {a, b, c, ..., y, e} are both of odd length (because of the minimality of C(a) again) and we either have y = b or y' = e, in which cases e is the successor of b, or $1 \sim y$, $1 \sim y'$ with both edges in C". In the latter case, application of lemma 1 easily yields $y \sim y'$, which in turn implies y = b or y' = e by lemma 4.

Hence, assume C' contains the sequence x, b, e, x'. It is our goal to prove that C' = {a, b, c, b, e, d, e}. To this end it suffices to show c = x and d = x', as lemma 2 clearly indicates. We assume therefore w.l.o.g. $c \neq x$ (the case $d \neq x'$ can be dealt with in an analogous fashion) and proceed to derive a contradiction. We are faced with four possibilities:

(i) $x \neq e$, $x' \neq d$,

(ii) $x = e$, $x' \neq d$,

(iii) $x \neq e$, $x' = d$,

(iv) $x = e$, $x' = d$.

(i). Since $c \neq x$ and $x \neq e$, we infer $a \sim x$ with $(a, x) \in C''$. Applying lemma 1 to (a, x) and (e, a) and recalling $x \not\sim e$, we conclude

$x = f$. Next $x' \neq d$ implies $a \sim x'$, lemma 1 then gives $x' \sim f = x$, hence by lemma 4 $x' = b$. Let us denote by x'' the successor of x' in C' and consider the odd generalized cycle $\{a, b = x', x'', \ldots, d, e\}$. In this cycle we must have $a \sim x''$, otherwise deletion of (b, e) would produce a GH-cycle, contrary to the hypothesis on G. By the same argument as above, we conclude $x'' = f$, which yields the sequence f, b, e, b, f, in violation of lemma 2.

(ii). As before $x' \neq d$ implies $x' \sim a$, and hence $x' = g$ by an analogous argument as in (i). Now consider the odd generalized cycle $\{a, b, c, \ldots, x'', x = e\}$, where x'' is the predecessor of x in C'. As in (i), we infer $a \sim x''$, which in turn yields $x'' = g$. The resulting sequence $g = x''$, $e, b, e, x' = g$ presents the desired contradiction to lemma 2.

(iii). In this case, $c \neq x$ and (2) of lemma 1 imply $e \sim c$. By the minimality of C' the edge $(e, c) \notin C'$ (otherwise either e or c would have to be contained in an even subcycle of C', - cf. the remark at the beginning of the proof). Now we consider the odd generalized cycle $\{c, c', \ldots, x'', x, b, e, x' = d, e, a, e\}$, where c' is c's successor in C' and x'' the predecessor of x. It clearly follows that $e \sim c'$ and $(e, c') \in C'$, for we could delete (e, c'), if it were not in C', without destroying the above cycle, thus obtaining a GH-cycle. The fact that $e \sim c'$ in turn implies $a \sim c'$, and hence by lemma 4 we have $c' = b$ or $x' = d = c'$. If $c' = b$, then $d \sim c$ is easily seen (by lemma 1 applied to (e, c) and (e, d)), and another application of lemma 1 would yield $d \sim b$, a contradiction to the hypothesis $x' = d$. If $d = c'$, then we first establish $x = f$ by the same argument as in (i). Next it is easily seen that $x'' = b$, and we consider the odd generalized cycle $\{c, c', \ldots, x'' = b, x = f\}$, where $c \sim f = x$ follows by means of an application of lemma 1, part (3) to the edges (b, c) and $(x = f, b)$. As $b \neq c'$, this cycle has length at least 5 and we plainly obtain $c' \sim x = f$. Lemma 4 applied to the sequence $x = f, b, e, x' = d = c'$ now presents the desired contradiction.

(iv). By exactly the same argument, we conclude $x'' = g$ (x'' is predecessor of $x = e$ in C'), $e \sim c$, $e \sim c'$ (c' is successor of c) and either $c' = b$ or $x' = d = c'$. In the first case, we again obtain $d \sim c$ and hence the contradiction $b \sim x' = d$. In the latter case, we consider the odd generalized cycle $\{c, c', \ldots, x'' = g, e\}$. Clearly then $c \sim g$ and by lemma 4 applied to the sequence g, a, b, c we would have $g = b$, which is incompatible with the sequence $x'' = g, e, b, e$. All four possibilities lead up to a contradiction, hence we conclude $c = x$ and similarly $d = x'$. The situation at this stage is indicated in fig. 4.

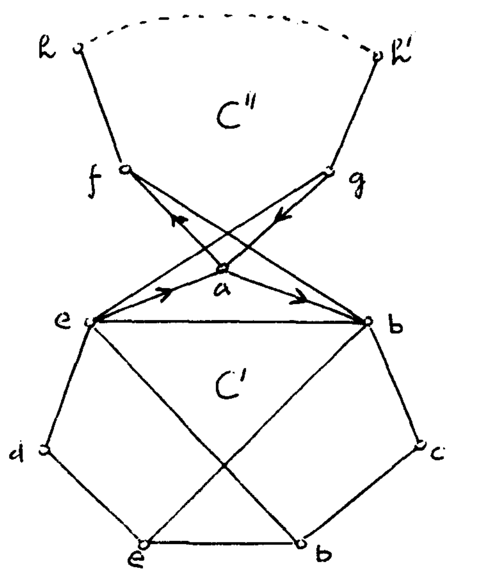

fig. 4

Since $f \neq g$, C'' is of length at least 4. Using lemma 1 we conclude $f \sim b$, $g \sim e$ and if either one of the edges (f, b), (g, e) were outside C', we clearly would obtain a GH-cycle after deleting this edge, thus contradicting the minimality of G. Hence we infer $f = e$ and $g = b$ (as $f \neq c$ and $g \neq d$). Next we notice $h \neq a, b, e$ and $h' \neq a, b, e$, furthermore $h \neq d$ and $h' \neq c$ because of lemma 2, and lastly $h \neq c$ and $h' \neq d$, since the opposite would contradict what we just proved about a shortest odd cycle among all $C(x)$. Lemma 1 applied to the cycle $\{f = e, h, \ldots, c, b, e, d\}$ yields $h \sim d$, and application to the cycle $\{f = e, h, \ldots, c, b\}$ gives $h \sim b$. Similarly we obtain $h' \sim c$ and $h' \sim e$. Now let us finally consider the edge $(h', e) \in C''$. Using once again lemma 1 it is easily seen that either $a \sim h'$ or $h = h'$. As the first possibility cannot occur, we conclude $h = h'$, and the vertices a, b, c, d, e, h induce G_2.

III. The Linegraph.

In this section, the linegraph $L(G)$ of a graph G shall be investigated with respect to partial orderings of the vertex-set.

We begin with two easy lemmas:

Lemma 5: If G contains an odd cycle of length at least 5 (with or without chords), then $L(G)$ does not admit a partial ordering.

Proof: Let C be such an odd cycle in G, say, $C = \{a_1, a_2, \ldots, a_{2k+1}\}$, then the edges (a_1, a_2), $(a_2, a_3), \ldots, (a_{2k}, a_{2k+1})$, (a_{2k+1}, a_1) plainly induce an odd cycle of length $2k+1$ without triangular chords in $L(G)$, i.e. a GH-cycle, hence $L(G)$ is an NPO-graph.

Lemma 6: Let G be one of the minimal graphs of theorem 2 and $L(G)$ its corresponding linegraph, then $L(G)$ is an NPO-graph.

Proof: We only have to verify the cases $G = G_1$ and $G = G_2$, since simple cycles without chords are isomorphic to their linegraphs. $L(G_1) = G_2$ and the fact that G_2 contains a 5 - cycle now prove the assertion.

Theorem 3: If G is an NPO-graph, then $L(G)$ is an NPO-graph.

Proof: We consider a shortest GH-cycle in G, denote it by C, and keep it fixed throughout the argument. If no edge of G appears more than once in C, i.e. C is an ordinary cycle, then lemma 5 clearly verifies the assertion. Suppose now there are edges which appear in C in both directions, then by lemma 2 they must be consecutive sequences in C. Let $a, b, c_1, b, c_2, \ldots, c_t, b, d$ for $t \geq 1$ such a sequence. If $t \geq 2$, then appealing to lemma 1 we obtain $a \sim c_t$, $d \sim c_1$, $a \sim d$, and hence the 5 - cycle $\{a, c_t, b, c_1, d\}$ results. Let us assume therefore $t = 1$ for all such sequences.

<u>Case (A)</u>: Every sequence b, c, b is embedded in the larger sequence a, b, c, b, d, e, x with $d \neq x$, i.e. a, ..., x are distinct vertices of G. In this case we construct the following sequence of edges in L(G): (a, b), (b, c), (a, b), (a, d), (d, e), (e, x), using the edge (a, d) whose existence is assured by lemma 1. Replacing every such sequence of vertices in C by the indicated sequence of edges in L(G), we clearly arrive at a GH-cycle in L(G), thus proving the theorem.

<u>Case (B)</u>: There exists a sequence a, b, c, b, d, e, d, f, g, h, i in G, where a, b, c, d, e are distinct vertices as we have shown in the paragraph preceding case (A). By lemma 1, we infer $a \sim d$, $b \sim f$, hence $g \neq a, e, b, d$. If $f \neq a$, then $g \not\sim a$ (since otherwise we would obtain the 5 - cycle {a, b, d, f, g}), and furthermore, $f \not\sim c$ (otherwise 5 - cycle {f, d, a, b, c}), which in turn implies $g \neq c$. But now the six vertices b, c, d, e, f, g induce G_1 (fig. 5), and lemma 6 completes the proof.

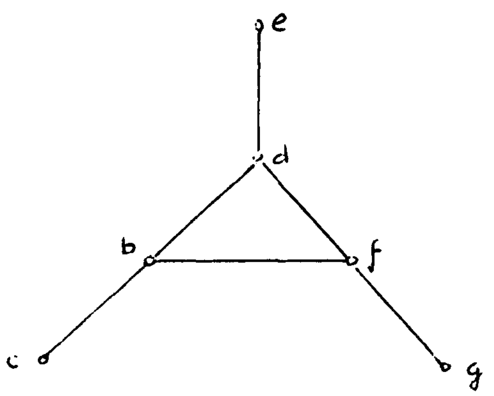

fig. 5

Finally if $f = a$, then fig. 5 again represents G_1 with $f = a$, unless $g \sim b$. Assume now $g \sim b$, then in the case where $h \neq a$, $h \not\sim b$, we construct the GH-cycle {... a, b, c, b, g, h ...}, thus contradicting the minimality of C, and in the case $h \neq a$, $h \sim b$, we find the 5 - cycle { h, b, d, a, g} and appeal to lemma 5. As to the only remaining possibility

$h = a$, we apply to the sequence $b, d, e, d, f = a, g, h = a, i$ what we have just shown above (note that because of $g \sim b$, the GH-cycle C must be longer than $\{a, b, c, b, d, e, d, a, g\}$) and conclude $b = i$, which is impossible since the sequence a, b already occurred in C.

In view of theorem 3, we now wish to determine all PO-graphs G whose line-graphs $L(G)$, too, admit a partial ordering of the vertex-set by some orientation of the edges. Since connected components of G correspond to connected components of $L(G)$, we may confine ourselves to connected PO-graphs G. We observe first that in the graph G_1, as displayed in fig. 1, we may add the edges joining the outer points d, e, f (or some of these edges) and the resulting graph will still be a NPO-graph; in other words, in order to preserve the NPO-character, we only have to make sure that no outer point is adjacent to an inner point. This gives rise to the following definition:

Definition: A graph T as in fig. 6 is called a big triple centered at x iff $a_1 \neq b_1 \neq c_1 \neq a_1$ and no outer point a_2, b_2, c_2 coincides with an inner point a_1, b_1, c_1. (outer points, however, may coincide).

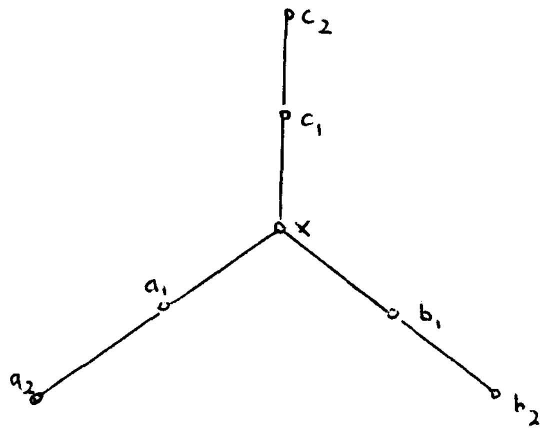

fig. 6

Since the linegraph $L(T)$ of any big triple T is isomorphic to G_1 plus possible edges joining outer points in G_1, we at once have a necessary condition for G with $L(G)$ a PO-graph, namely, that it contain no big triple.

Theorem 4: The linegraph $L(G)$ of a graph G admits a partial ordering iff

(i) G contains no odd cycle of length ≥ 5,

(ii) G contains no triangle with edges leading from the three vertices into distinct blocks, or with two paths of length at least two leading from two of the vertices into distinct blocks, or with an edge leading from one of the three vertices into a different block, if the triangle is embedded in a complete graph on four points,

(iii) G contains no big triple[4].

Proof: The necessity of (i) and (iii) is clear. As to (ii), suppose G contains a triangle with three edges attached to it as specified in the theorem, then these six edges clearly induce G_2 in $L(G)$. If the triangle has two paths of length at least two attached to it, then these seven edges are readily seen to induce a 5 - cycle with two chords and two outgoing edges opposite the two chords, which is an NPO-graph. Finally it is easily verified that the linegraph of a complete graph on 4 points plus an edge does not admit a partial ordering, thus completing the proof of the necessity-part.

Suppose then G satisfies the conditions of the theorem, and let C_k be a longest cycle in G.

Case (A): $k \geq 6$, even. By (i) and (iii) C_k contains no chords, and again by (iii) there can only be claws[5] attached to C_k, centered at the

4). It is interesting to note the analogy between the above conditions and the conditions in Sedlacek [4] for a graph to have a planar linegraph.

5). A claw or stargraph consists of a vertex, called the center, plus outer points and all the edges joining the center with every outer point.

vertices of C_k, and any two claws must be point-disjoint, as (i) clearly implies. The linegraph of such a graph is easily seen to be a PO-graph.

<u>Case (B)</u>: $k = 4$. Assume first there are no chords in C_4, then (iii) readily provides the answer. Next assume there is exactly one chord in C_4. Let $C_4 = \{a, b, c, d\}$ with the chord (a, c). Again applying (iii), we conclude there can only be claws attached to C_4, centered at the vertices of C_4. (ii) now implies that we either have claws centered at a and c, or at b and d. The linegraphs of such graphs are PO-graphs. Finally if we have both chords in C_4, we appeal to (ii) again, noting that the linegraph of the complete graph on 4 points admits a partial ordering.

<u>Case (C)</u>: $k = 3$. By (i) two triangles cannot be in the same block unless they have an edge in common. The hypothesis $k = 3$ and (iii) now imply that there are at most two triangles, joined by a path (possibly of length zero, in which case the triangles have a common vertex). (ii) and (iii) applied to the triangles and the connecting path now settle this case.

<u>Case (D)</u>: $k = 1$, i.e. G is a tree. Making use of (iii) once more, we readily recognize G to be a path with point-disjoint claws attached to it, the linegraph of which is a PO-graph.

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