TWO QUEUES IN PARALLEL

by

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SUMMARY

A queueing system consisting of two queues, each with single servers (independent negative exponential service time distributions) and correlated bivariate Poisson input is discussed. The joint stationary probabilities for the respective numbers of customers in the two queues in equilibrium are found in the case where we have finite waiting rooms of different sizes for each individual queue. For the model where the waiting room capacities are unlimited, we derive a functional relationship for the joint probability generating function of the equilibrium probabilities. A survey of published research on queues in parallel is also presented.

1. Introduction

In the past, considerable attention has been given to the theory of simple queues. The easy problems have been solved and the methods that have been developed are sufficiently powerful to analyze simple models e.g. those which consist of a group of servers with one waiting line and which have special types of inputs and service mechanisms. A study of the general problem of a network of queues (an arrangement of queues in parallel and in series) is, however, very much in its infancy. To the present time, most interest has been centered around the study of queues in series.

In this paper we consider two queues in parallel, each waiting line having a single server, both assumed to be operating independently with negative exponential service time distributions. The arrivals to the system are assumed to have a correlated bivariate Poisson input distribution. We discuss the problem of finding the joint equilibrium probabilities for the respective number of customers in the two queues under the restriction of finite waiting room capacities and also in the case of unlimited capacities.

Before describing the model in detail, we give a historical background to the general problem of queues in parallel by presenting (Section 2) a survey of the published research in this area. Subsequent sections deal with the obtaining of relationships between the equilibrium probabilities, their solution in the case of finite waiting rooms of different capacities, and a discussion of the problem when we have unlimited capacities. The equilibrium probabilities are found explicitly for certain special cases.

2. Historical Background

The first published attempt at studying two queues in parallel was made by Haight (1958). He considered a model whereby arrivals to the system are assumed to be in the form of a homogeneous Poisson process with parameter λ and that the service time distributions for each queue are independent negative exponential distributions, with parameters μ_1 and μ_2 for queue 1 and queue 2 respectively. Let $X_1(t)$ and $X_2(t)$ denote the lengths at time t of queue 1 and queue 2 respectively. An arrival will join queue 1, if and only if, at his time of arrival $X_1(t) \leq X_2(t)$. If, on the other hand, an arrival finds $X_1(t) > X_2(t)$, he will join queue 2. Thus, queue 1 has a certain advantage in absorbing arrivals in the equally advantageous case when $X_1(t) = X_2(t)$ and is called the "near" queue. Queue 2 is called the "far" queue.

Let $p_{xy}(t) = \Pr\{X_1(t) = x,\ X_2(t) = y\}$. Haight is interested in determining the asymptotic state probabilities i.e. in finding $p_{xy} = \lim_{t\to\infty} p_{xy}(t)$. Using the method of differential-difference equations, he obtains relations between the p_{xy} but is unable to solve for them explicitly. He also finds relationships between the marginal probabilities, the mean queue sizes, and the p_{xy}.

In the event that we permit the queuers to change queues whenever it seems advantageous to do so, the formulation is simplified, and Haight finds explicit expressions.

Wilkins (1960) showed that Haight's results can be extended to the following more general case. If, at the time of arrival of a customer, $X_1(t) = x$ and $X_2(t) = y$, the customer joins queue 1 with prob $W(x, y)$ where

$$W(x, y) = \begin{cases} 1 & x < y , \\ w(x) & x = y , \\ 0 & x > y . \end{cases}$$

Wilkins proceeds to show that some of the relations between the p_{xy} are modified and some remain unchanged. Needless to say, this generalisation did not lead to a solution for the equilibrium probabilities.

However, Kingman (1961), upon making the simplification of symmetry between the two queues (i.e., $\mu_1 = \mu_2$ and $w(x) = \frac{1}{2}$) was able to determine conditions underwhich a state of statistical equilibrium is reached and furthermore was able to express the equilibrium probabilities as an infinite sum of geometric distributions.

Recently, Ghirtis (1966, 1968) considered the imposition of limited waiting rooms for each queue and made certain modifications to the arrival pattern. He assumed that, if queues 1 and 2 had limited waiting rooms of size M and N respectively, an arrival to the system joined queue 1 if there was waiting room and otherwise was assigned to queue 2. If there was no waiting room in queue 2, the customer left the system without returning. Ghirtis was able to find the marginal distributions and indicated how to obtain the general solution of the system of simultaneous equations for the equilibrium state probabilities. In his earlier paper, (1966), he considered the case of $\mu_1 = \mu_2$. The more general case, when $\mu_1 \neq \mu_2$, is considered in his latter paper (1968).

3. Description of the model

The system consists of two queues each having a single server. The service time distributions for each queue are independent negative exponential distributions with parameters μ_1 and μ_2 for queue 1 and queue 2 respectively. Arrivals to each queue are assumed to be governed by a bivariate Poisson process, as follows.

Let $A_i(t)$ = Number of arrivals in the i-th queue during $(0, t]$.

Then $\Pr\{A_1(t) = m, \ A_2(t) = n\}$

$$= e^{-(\lambda_1+\lambda_2+\lambda)t} \sum_{k=0}^{\min(m,n)} \frac{(\lambda_1 t)^{m-k}}{(m-k)!} \frac{(\lambda_2 t)^{n-k}}{(n-k)!} \frac{(\lambda t)^k}{k!} \ . \qquad (3.1)$$

This bivariate Poisson process can be characterised in the following manner. We have three independent Poisson processes with rates λ_1, λ_2, and λ. The input to queue 1 can be regarded as being composed of the sum of Poisson components, of rates λ_1 and λ, while the input to queue 2 consists of the Poisson component of rate λ_2 plus the same Poisson component of rate λ that was present in the input to queue 1.

From equation (3.1), or the above characterisation, it is easily seen that the marginal inputs are each Poisson, rate $\lambda_1 + \lambda$ for queue 1, and rate $\lambda_2 + \lambda$ for queue 2. Furthermore, it can be shown that the covariance between the inputs to each queue is λ.

Haight (1967) presents a collection of useful results and applications pertaining to this process.

One of the basic motivations behind this model, excluding any considerations of its application, was to devise a queueing model where the inputs to the service facilities are correlated in some specific manner. It will be seen in the course of the presentation of my results that there remain many unsolved problems relating to this model. I have not attempted to carry out any investigation to determine the transient solutions.

Concerning applications of this model, one could visualize the following type of situation. Consider two processes, each with components subject to failure. Items that fail in any specific process are serviced by an operator assigned to that process. The failures of items can be of two possible types. Firstly, failures may occur at random in each process independently, and secondly, failures may occur according to some Poisson process in both processes simultaneously (e.g., a failure in both systems due to a power failure).

4. The steady state equations

We first consider our queueing model under the restriction of limited waiting room capacities. Assume that queue 1 has a maximum capacity of M customers and that queue 2 has a maximum capacity of N customers. In the event that an arrival finds his assigned queue at its maximum capacity, he does not join any queue and is lost to the system.

Let $E_{ij}(t) \equiv$ Event that there are i customers in queue 1 and j customers in queue 2 at time t .

In setting up the steady state equations, we shall make use of the method of differential-difference equations. This method is widely used when we have a model consisting of Poisson components. Suppose we have a Poisson process of rate ν , say, generating events; then the probability of one of these events occurring in the time interval $(t, t+\delta t]$ is $\nu\delta t + o(\delta t)$. In our model, we have 5 possible Poisson processes (parameters λ_1, λ_2, λ, μ_1 and μ_2) each acting independently of each other.

We shall, for the present, assume that $M \geq 2$, $N \geq 2$. Consider the possible state changes that may occur in our queueing system during the time interval $(t, t+\delta t]$. Suppose $1 \leq i \leq M-1$, $1 \leq j \leq N-1$ then $E_{ij}(t+\delta t)$ could have arisen from one of the 6 possibilities (to $o(\delta t)$) .

(i) $E_{ij}(t)$; and in $(t, t + \delta t]$ no μ_1 service, no μ_2 service, no λ_1 arrival, no λ_2 arrival, no λ arrival.

(ii) $E_{i-1,j}(t)$; and in $(t, t + \delta t]$ no μ_1 service, no μ_2 service, one λ_1 arrival, no λ_2 arrival, no λ arrival.

(iii) $E_{i,j-1}(t)$; and in $(t, t + \delta t]$ no μ_1 service, no μ_2 service, no λ_1 arrival, one λ_2 arrival, no λ arrival.

(iv) $E_{i-1,j-1}(t)$; and in $(t, t + \delta t]$ no μ_1 service, no μ_2 service, no λ_1 arrival, no λ_2 arrival, one λ arrival.

(v) $E_{i+1,j}(t)$; and in $(t, t + \delta t]$, one μ_1 service, no μ_2 service, no λ_1 arrival, no λ_2 arrival, no λ arrival.

(vi) $E_{i,j+1}(t)$; and in $(t, t+\delta t]$, no μ_1 service, one μ_2 service, no λ_1 arrival, no λ_2 arrival, no λ arrival.

Let $p_{ij}(t) \equiv \Pr\{E_{ij}(t)\}$.

By the independence of the Poisson processes involved, we have that

$$\begin{aligned} p_{ij}(t+\delta t) = p_{ij}(t) & \left[1 - \{\lambda_1 + \lambda_2 + \lambda + \mu_1 + \mu_2\}\delta t + o(\delta t)\right] \\ & + p_{i-1,j}(t)\left[\lambda_1\, \delta t + o(\delta t)\right] \\ & + p_{i,j-1}(t)\left[\lambda_2\, \delta t + o(\delta t)\right] \\ & + p_{i-1,j-1}(t)\left[\lambda\, \delta t + o(\delta t)\right] \\ & + p_{i+1,j}(t)\left[\mu_1\, \delta t + o(\delta t)\right] \\ & + p_{i,j+1}(t)\left[\mu_2\, \delta t + o(\delta t)\right] . \end{aligned}$$

Since $\lim_{\delta t \to 0} \dfrac{p_{ij}(t+\delta t) - p_{ij}(t)}{\delta t} = \dfrac{dp_{ij}(t)}{dt}$

we deduce that

$$\begin{aligned} \frac{dp_{ij}(t)}{dt} = & -(\lambda_1 + \lambda_2 + \lambda + \mu_1 + \mu_2)p_{ij}(t) \\ & + \lambda_1 p_{i-1,j}(t) + \lambda_2 p_{i,j-1}(t) + \lambda p_{i-1,j-1}(t) \\ & + \mu_1 p_{i+1,j}(t) + \mu_2 p_{i,j+1}(t) . \end{aligned}$$

For statistical equilibrium, $p_{ij} \equiv \lim_{t\to\infty} p_{ij}(t)$. Assuming equilibrium as $t \to \infty$, we obtain a second order difference equation for p_{ij} (equation (4.5) below) by setting $\frac{dp_{ij}(t)}{dt} = 0$.

In a similar manner, considering possible state changes in the interval $(t, t+\delta t]$ for i,j taking on the boundary values of 0, M or 0, N respectively, we obtain the following system of difference equations for the $\{p_{ij}\}$,$(0 \le i \le M$, $0 \le j \le N)$ when $M \ge 2$, $N \ge 2$.

$$(\lambda_1+\lambda_2+\lambda)p_{oo} = \mu_1 p_{1o} + \mu_2 p_{o1} , \tag{4.1}$$

$$(\lambda_1+\lambda_2+\lambda+\mu_1)p_{io} = \lambda_1 p_{i-1,o} + \mu_1 p_{i+1,o} + \mu_2 p_{i,1} , \quad (1 \le i \le M-1), \tag{4.2}$$

$$(\lambda_2+\lambda+\mu_1)p_{Mo} = \lambda_1 p_{M-1,o} + \mu_2 p_{M,1} , \tag{4.3}$$

$$(\lambda_1+\lambda_2+\lambda+\mu_2)p_{oj} = \lambda_2 p_{o,j-1} + \mu_1 p_{1j} + \mu_2 p_{o,j+1} , \quad (1 \le j \le N-1), \tag{4.4}$$

$$(\lambda_1+\lambda_2+\lambda+\mu_1+\mu_2)p_{ij} = \lambda_1 p_{i-1,j} + \lambda_2 p_{i,j-1} + \lambda p_{i-1,j-1} + \mu_1 p_{i+1,j} + \mu_2 p_{i,j+1} , \quad (1 \le i \le M-1,\ 1 \le j \le N-1) , \tag{4.5}$$

$$(\lambda_2+\lambda+\mu_1+\mu_2)p_{Mj} = \lambda_1 p_{M-1,j} + (\lambda_2+\lambda)p_{M,j-1} + \lambda p_{M-1,j-1} + \mu_2 p_{M,j+1} , \quad (1 \le j \le N-1) , \tag{4.6}$$

$$(\lambda_1+\lambda+\mu_2)p_{oN} = \lambda_2 p_{o,N-1} + \mu_1 p_{1N} , \tag{4.7}$$

$$(\lambda_1+\lambda+\mu_1+\mu_2)p_{iN} = (\lambda_1+\lambda)p_{i-1,N} + \lambda_2 p_{i,N-1} + \lambda p_{i-1,N-1} + \mu_1 p_{i+1,N} , \quad (1 \le i \le M-1) , \tag{4.8}$$

$$(\mu_1+\mu_2)p_{MN} = (\lambda_1+\lambda)p_{M-1,N} + (\lambda_2+\lambda)p_{M,N-1} + \lambda p_{M-1,N-1} . \tag{4.9}$$

If $M = 1$ and $N \ge 2$, then equations (4.1), (4.3), (4.4), (4.6), (4.7), and (4.9) hold.

If $M \geq 2$ and $N = 1$, then equations (4.1), (4.2), (4.3), (4.7), (4.8), and (4.9) hold.

If $M = 1$ and $N = 1$, then equations (4.1), (4.3), (4.7) and (4.9) hold.

In the case of unlimited waiting rooms, the $\{p_{ij}\}$ $(i \geq 0,\ j \geq 0)$ satisfy equation (4.1), equations (4.2) for $i \geq 1$, equations (4.4) for $j \geq 1$, and equations (4.5) for $i \geq 1$, $j \geq 1$.

5. Joint probability generating function equation

In an attempt to solve equations (4.1) to (4.9), we first obtain a functional equation involving the joint probability generating function of the equilibrium probabilities.

Define $\pi(s_1, s_2) \equiv \sum_{i=0}^{M} \sum_{j=0}^{N} p_{ij} s_1^i s_2^j$.

Preparatory to deriving an expression for $\pi(s_1, s_2)$ we obtain a set of difference equations for the following univariate generating functions:

$$\varphi_j(s) = \sum_{i=0}^{M} p_{ij}\, s^i .$$

Assume $M \geq 2$, $N \geq 2$. Then

$$\mu_2\varphi_1(s) - \left[\lambda_1(1-s) + \lambda_2 + \lambda + \mu_1(1-\tfrac{1}{s})\right]\varphi_o(s)$$

$$= \mu_1(\tfrac{1}{s}-1)p_{oo} - \lambda_1 s^M(1-s)p_{Mo} , \qquad (5.1)$$

$$\mu_2\varphi_{j+1}(s) - \left[\lambda_1(1-s) + \lambda_2 + \lambda + \mu_1(1-\tfrac{1}{s}) + \mu_2\right]\varphi_j(s) + [\lambda_2+\lambda s]\varphi_{j-1}(s)$$

$$= \mu_1(\tfrac{1}{s}-1)p_{oj} - \lambda_1 s^M(1-s)p_{Mj} - \lambda s^M(1-s)p_{M', j-1} ,$$

$$(j = 1, \ldots, N-1) , \qquad (5.2)$$

$$- \left[(\lambda_1+\lambda)(1-s) + \mu_1(1-\tfrac{1}{s}) + \mu_2\right]\varphi_N(s) + [\lambda_2+\lambda s]\varphi_{N-1}(s)$$

$$= \mu_1(\tfrac{1}{s}-1)p_{oN} - (\lambda_1+\lambda)s^M(1-s)p_{MN} - \lambda s^M(1-s)p_{M', N-1} . \qquad (5.3)$$

Equation (5.1) is obtained from the set of equations (4.1), (4.2), (4.3) by multiplying the i-th equation in the set by s^i and summing over $0 \leq i \leq M$. Similarly for equations (5.2) and (5.3) from the sets (4.4), (4.5), (4.6) and (4.7), (4.8), (4.9) respectively.

Note that $\pi(s_1, s_2) = \sum_{j=0}^{N} \varphi_j(s_1)s_2^j$. Thus by multiplying the j-th equation of the set (5.1), (5.2), (5.3) by s_2^j and summing over $0 \le j \le N$, we obtain upon simplification the following equation.

$$\left[\lambda_1(1-s_1)+ \lambda_2(1-s_2) + \lambda(1-s_1s_2) + \mu_1(1-\frac{1}{s_1}) + \mu_2(1-\frac{1}{s_2})\right]\pi(s_1, s_2)$$

$$= \mu_1(1-\frac{1}{s_1})\pi(0,s_2) + \mu_2(1-\frac{1}{s_2})\pi(s_1,0) + (\lambda_1+\lambda s_2)s_1^M(1-s_1)(\sum_{j=0}^{N} p_{Mj}\, s_2^j)$$

$$+ (\lambda_2 + s_1)s_2^N(1-s_2)(\sum_{i=0}^{M} p_{iN}s_1^i) + \lambda s_1^M s_2^N(1-s_1)(1-s_2)p_{MN} \quad . \tag{5.4}$$

In the case of each waiting line having unlimited waiting room equations (5.1), (5.2) and (5.3) are modified to give for the $\varphi_j(s) = \sum_{i=0}^{\infty} p_{ij}s_1^i$ the following equations.

$$\mu_2\varphi_1(s) - \left[\lambda_1(1-s) + \lambda_2 + \lambda + \mu_1(1-\tfrac{1}{s})\right]\varphi_o(s) = \mu_1(\tfrac{1}{s}-1)\varphi_o(0) , \tag{5.5}$$

$$\mu_2\varphi_{j+1}(s) - \left[\lambda_1(1-s) + \lambda_2 + \lambda + \mu_1(1-\tfrac{1}{s}) + \mu_2\right]\varphi_j(s) + (\lambda_2+\lambda s)\varphi_{j-1}(s)$$

$$= \mu_1(\tfrac{1}{s}-1)\varphi_j(0) , \quad (j = 1, 2, \dots) . \tag{5.6}$$

With $\pi(s_1, s_2) \equiv \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{ij}\, s_1^i\, s_2^j = \sum_{j=0}^{\infty} \varphi_j(s_1)s_2^j$, we obtain from (5.5) and (5.6) the following functional equation for $\pi(s_1, s_2)$.

$$\left[\lambda_1(1-s_1) + \lambda_2(1-s_2) + \lambda(1-s_1s_2) + \mu_1(1-\frac{1}{s_1}) + \mu_2(1-\frac{1}{s_2})\right]\pi(s_1, s_2)$$

$$= \mu_1(1-\frac{1}{s_1})\pi(0, s_2) + \mu_2(1-\frac{1}{s_2})\pi(s_1, 0) . \tag{5.7}$$

In order to solve for the equilibrium probabilities $\{p_{ij}\}$ it is sufficient to determine the appropriate j.p.g.f. $\pi(s_1, s_2)$ satisfying equation (5.4) in the finite waiting room case; or satisfying equation (5.7) in the infinite waiting room case.

Concerning the solution of the 'finite' case, we do not attempt to solve for $\pi(s_1, s_2)$ directly but rather use a matrix theoretic approach (see Section 8). However, such an approach is not applicable for the 'infinite' case and a detailed examination of the functional equation (5.7) above seems desirable. Any attempts to find a j.p.g.f. $\pi(s_1, s_2)$ satisfying equation (5.7) have failed (see Section 10).

6. The marginal distributions - Finite waiting room case

We wish to determine the marginal probabilities for the finite waiting room case (maximum capacities of M and N for queue 1 and 2 respectively).

Define
$$p_{\cdot j} = \sum_{i=0}^{M} p_{ij} , \qquad (j = 0, 1, \ldots, N)$$
$$p_{i\cdot} = \sum_{j=0}^{N} p_{ij} , \qquad (i = 0, 1, \ldots, M) .$$

Putting $s = 1$ in equations (5.1), (5.2), (5.3) and noting that $\varphi_j(1) = p_{\cdot j}$ we have

$$\mu_2 p_{\cdot 1} - (\lambda_2 + \lambda) p_{\cdot o} = 0 , \tag{6.1}$$

$$\mu_2 p_{\cdot j+1} - (\lambda_2 + \lambda + \mu_2) p_{\cdot j} + (\lambda_2 + \lambda) p_{\cdot j-1} = 0 , \quad (j = 1, \ldots, N-1) \tag{6.2}$$

$$- \mu_2 p_{\cdot N} + (\lambda_2 + \lambda) p_{\cdot N-1} = 0 . \tag{6.3}$$

By defining $\rho_2 = \dfrac{\lambda + \lambda_2}{\mu_2}$, these equations can be rewritten as

$$p_{\cdot 1} - \rho_2 p_{\cdot o} = 0 , \tag{6.4}$$

$$p_{\cdot j+1} - (1 + \rho_2) p_{\cdot j} + \rho_2 p_{\cdot j-1} = 0 , \quad (j = 1, \ldots, N-1) , \tag{6.5}$$

$$- p_{\cdot N} + \rho_2 p_{\cdot N-1} = 0 . \tag{6.6}$$

Solving (6.5) by standard difference equation techniques and determining the constants by the boundary conditions we obtain the following expressions for the $p_{\cdot j}$.

$$p_{\cdot j} = \frac{(1 - \rho_2)}{(1 - \rho_2^{N+1})} \rho_2^{\,j} , \; (j = 0, 1, \ldots, N) . \qquad (6.7)$$

Similarly, it can easily be shown that

$$p_{i\cdot} = \frac{(1-\rho_1)}{(1 - \rho_1^{M+1})} \rho_1^{\,i} , \; (i = 0, 1, \ldots, M) ; \qquad (6.8)$$

where $\rho_1 = \frac{\lambda+\lambda_1}{\mu_1}$.

These results are to be expected. Focus attention on queue 1, say. Then we have a single server queue with negative service time distribution, parameter μ_1 , Poisson process input with parameter $\lambda_1 + \lambda$, and a finite waiting room of maximum capacity M . The stationary distribution for such a queueing situation is a truncated geometric distribution, parameter ρ_1 , as is well known.

7. The marginal distributions - Infinite waiting room case.

To derive the marginal probabilities in this case, we shall use the functional relationship given by equation (5.7).

Putting $s_2 = 1$ in this equation gives upon simplification

$$\pi(s_1, 1) = \frac{\pi(0, 1)}{1 - \rho_1 s_1} .$$

Using the fact that $\pi(1, 1) = 1$, since we have a probability generating function, we deduce that

$$\pi(0, 1) = 1 - \rho_1 ,$$

and hence

$$\pi(s_1, 1) = \frac{1 - \rho_1}{1 - \rho_1 s_1} , \quad (\rho_1 < 1) . \qquad (7.1)$$

Thus the marginal distribution in geometric, parameter ρ_1. Similarly

$$\pi(1, s_2) = \frac{1 - \rho_2}{1 - \rho_2 s_2} , \quad (\rho_2 < 1) . \qquad (7.2)$$

As for the finite waiting room case, these results for the marginal distributions are well known, Saaty, (1961).

8. Solution for equilibrium probabilities - Finite waiting room

In this section, we outline a procedure for solving equations (4.1) through (4.9). Rather than attempt to solve for the joint probability generating function we use a matrix theoretic approach.

Assume $M \geq 2$, $N \geq 2$.

Let $\underline{p}_j' = [p_{oj}, p_{ij}, \ldots, p_{Mj}]$, $(j = 0, ., \ldots, N)$.

Define $A(x, y, z)$, an $(M+1)\times(M+1)$ matrix, the values of the elements depending on $x, y, z,$ by

$$A(x, y, z) = \begin{bmatrix} x-z & -z & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -y & x & -z & \ddots & & & & & \cdot \\ 0 & -y & x & \ddots & \ddots & & & & \cdot \\ \cdot & \ddots & \ddots & \ddots & \ddots & \ddots & & & \cdot \\ \cdot & & \ddots & \ddots & \ddots & \ddots & \ddots & & \cdot \\ \cdot & & & \ddots & \ddots & \ddots & \ddots & \ddots & \cdot \\ \cdot & & & & \ddots & \ddots & x & -z & 0 \\ \cdot & & & & & \ddots & -y & x & -z \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -y & x-y \end{bmatrix} .$$

Note that all the non-zero elements of $A(x, y, z)$ appear on the diagonal or the off-diagonals. Such a matrix is known as a "continuant".

We define

$$A_o \equiv A\left(\frac{\lambda_1 + \lambda_2 + \lambda + \mu_1}{\mu_2}, \frac{\lambda_1}{\mu_2}, \frac{\mu_1}{\mu_2}\right),$$

$$A_1 \equiv A\left(\frac{\lambda_1 + \lambda_2 + \lambda + \mu_1 + \mu_2}{\mu_2}, \frac{\lambda_1}{\mu_2}, \frac{\mu_1}{\mu_2}\right),$$

$$A_2 \equiv A\left(\frac{\lambda_1 + \lambda + \mu_1 + \mu_2}{\mu_2}, \frac{\lambda_1 + \lambda}{\mu_2}, \frac{\mu_1}{\mu_2}\right),$$

$$B_1 \equiv A\left(\frac{\lambda_2}{\mu_2}, \frac{-\lambda}{\mu_2}, 0\right).$$

With these definitions, it is easy to show that equations (4.1), (4.2), (4.3) can be expressed by the matrix equation (8.1) below. Similarly (4.4), (4.5), (4.6) can be expressed by equation (8.2) and (4.7), (4.8), (4.9) by (8.3).

$$A_o \underset{\sim}{p}_o = \underset{\sim}{p}_1 , \qquad (8.1)$$

$$A_1 \underset{\sim}{p}_j = \underset{\sim}{p}_{j+1} + B_1 \underset{\sim}{p}_{j-1} , \quad (j = 1, \ldots, N-1) , \qquad (8.2)$$

$$A_2 \underset{\sim}{p}_N = B_1 \underset{\sim}{p}_{N-1} . \qquad (8.3)$$

Since $A(x_1, y_1, z_1) + A(x_2, y_2, z_2) = A(x_1 + x_2, y_1 + y_2, z_1 + z_2)$, and noting that $A(1, 0, 0) = I$, we have the following relationships

$$A_1 = I + A_o , \qquad (8.4)$$

$$A_1 = A_2 + B_1 . \qquad (8.5)$$

The problem now reduces to one of finding $\underset{\sim}{p}_o$, since once an expression for $\underset{\sim}{p}_o$ is found, expressions for the $\underset{\sim}{p}_j$ $(j = 1, \ldots, N)$ are readily determined by (8.2).

Let us define

$$\underset{\sim}{p} \equiv \sum_{j=0}^{N} \underset{\sim}{p}_j = \begin{bmatrix} p_{o\cdot} \\ p_{1\cdot} \\ \\ p_{M\cdot} \end{bmatrix} \qquad (8.6)$$

where the $p_{i.}$ are expressed by equation (6.8) .

The technique we use to solve the second order difference equation (8.2) for the vector $\underset{\sim}{p}_j$ is based upon a technique used for solving systems of second order differential equations. We reduce equation (8.2) to a first order difference equation in the following manner.

We define, for $j = 0, \ldots, N-1$, the $(2M + 2) \times 1$ vector $\underset{\sim}{z}_j$ by

$$\underset{\sim}{z}_j = \begin{bmatrix} \underset{\sim}{p}_j \\ \underset{\sim}{p}_{j+1} \end{bmatrix} .$$

Then from equation (8.2) it is easy to see that

$$\underset{\sim}{z}_j = D \underset{\sim}{z}_{j-1} , \quad (j = 1, \ldots, N-1) ; \tag{8.7}$$

where D is a $(2M + 2) \times (2M + 2)$ matrix given by

$$D = \left[\begin{array}{c|c} 0 & I \\ \hline -B_1 & A_1 \end{array} \right] . \tag{8.8}$$

From equation (8.7) we obtain, by recursion,

$$\underset{\sim}{z}_j = D^j \underset{\sim}{z}_o , \quad j = 1, \ldots, N-1 . \tag{8.9}$$

Let us consider solving for $\underset{\sim}{z}_o$. First note that

$$\underset{\sim}{z}_o + \cdots + \underset{\sim}{z}_{N-1} = \begin{bmatrix} \underset{\sim}{p} - \underset{\sim}{p}_N \\ \underset{\sim}{p} - \underset{\sim}{p}_o \end{bmatrix} . \tag{8.10}$$

Also

$$\begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \underset{\sim}{z}_o = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{p}_o \end{bmatrix} \tag{8.11}$$

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \underset{\sim}{z}_{N-1} = \begin{bmatrix} \underset{\sim}{p}_N \\ \underset{\sim}{0} \end{bmatrix} . \tag{8.12}$$

Thus adding equations (8.10), (8.11), and (8.12) we have

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \underset{\sim}{z}_o + \underset{\sim}{z}_1 + \ldots + \underset{\sim}{z}_{N-2} + \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \underset{\sim}{z}_{N-1} = \begin{bmatrix} \underset{\sim}{p} \\ \underset{\sim}{p} \end{bmatrix} . \tag{8.13}$$

Using the results of equations (8.9), equation (8.13) can be rewritten as

$$H \underset{\sim}{z}_o = \begin{bmatrix} \underset{\sim}{p} \\ \underset{\sim}{p} \end{bmatrix} , \tag{8.14}$$

where

$$H = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} + D + D^2 + \ldots + D^{N-2} + \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} D^{N-1} . \tag{8.15}$$

Let us examine H in a little more detail. We shall see that H is in fact a singular matrix and hence we are unable to use matrix inversion to solve (8.14). However, in the course of our examination of H, we find that it is possible to solve for $\underset{\sim}{p}_o$ explicitly. Let

$$H = \left[\begin{array}{c|c} H_1 & H_2 \\ \hline H_3 & H_4 \end{array}\right] .$$

Also, let

$$D^k = \left[\begin{array}{c|c} {}_kD_1 & {}_kD_2 \\ \hline {}_kD_3 & {}_kD_4 \end{array}\right] , \qquad k \geq 1 .$$

Thus, we first observe that

$$D = \left[\begin{array}{c|c} {}_1D_1 & {}_1D_2 \\ \hline {}_1D_3 & {}_1D_4 \end{array}\right] = \left[\begin{array}{c|c} 0 & I \\ \hline B_1 & A_1 \end{array}\right] . \tag{8.16}$$

Furthermore, since $D^{k+1} = D^k D = D D^k$, we obtain, for $k \geq 1$, by equating the respective matrices in the block multiplication,

$$_{k+1}D_1 = - {_k}D_2 B_1 = {_k}D_3 , \tag{8.17}$$

$$_{k+1}D_2 = {_k}D_1 + {_k}D_2 A_1 = {_k}D_4 , \tag{8.18}$$

$$_{k+1}D_3 = -{_k}D_4 B_1 = - B_{1k}D_1 + A_{1k}D_3 , \tag{8.19}$$

$$_{k+1}D_4 = {_k}D_3 + {_k}D_4 A_1 = - B_{1k}D_2 + A_{1k}D_4 . \tag{8.20}$$

By making use of equations (8.17) and (8.18), equation (8.15) can be split into its component matrices to yield

$$H_1 = H_3 = I + \sum_{k=1}^{N-1} {_k}D_3 = I + \sum_{k=1}^{N} {_k}D_1 , \tag{8.21}$$

$$H_2 = H_4 = I + \sum_{k=1}^{N-1} {_k}D_4 = \sum_{k=1}^{N} {_k}D_2 . \tag{8.22}$$

Thus we observe that the matrix H is singular. However, equation (8.14) becomes

$$\left[\begin{array}{c|c} H_1 & H_2 \\ \hline H_1 & H_2 \end{array}\right] \begin{bmatrix} p_o \\ p_1 \end{bmatrix} = \begin{bmatrix} p \\ p \end{bmatrix} . \tag{8.23}$$

Upon simplification, equation (8.23) gives two identical equations, namely

$$H_1 p_o + H_2 p_1 = p . \tag{8.24}$$

Using the boundary condition equation (8.1), equation (8.24) reduces to

$$C\, p_o = p , \tag{8.25}$$

where $$C = H_1 + H_2 A_o .$$

Simplification of C is possible as follows:

$$C = I + \sum_{k=1}^{N} \{{}_kD_1 + {}_kD_2A_o\} \qquad \text{using (8.21), (8.22),}$$

$$= I + \sum_{k=1}^{N} \{{}_kD_1 + {}_kD_2A_1 - {}_kD_2\} \qquad \text{using (8.4),}$$

$$= I + \sum_{k=1}^{N} \{{}_{k+1}D_2 - {}_kD_2\} \qquad \text{using (8.18),}$$

$$= {}_{N+1}D_2 \qquad \text{using (8.16).}$$

Thus

$${}_{N+1}D_2\, p_o = p \,.$$

We have not been able to show that ${}_{N+1}D_2$ is non-singular for all M, N. However, for any simple case considered direct evaluation has shown the determinant of this matrix to be non-zero. Note that ${}_1D_2$ and ${}_2D_2$ are non-singular for all M; (${}_2D_2 = A_1$; $A(x, y, z)$ is non-singular when $x - y - z > 0$, $x > 0$). Furthermore, computor studies have not shown ${}_{N+1}D_2$ to be singular for moderate values of M and N. Therefore, we may assume

$$p_o = {}_{N+1}D_2^{-1}\, p \,. \tag{8.26}$$

Having determined p_o, we can now find p_j, $(j = 1, \ldots, N)$, as follows. From equation (8.9), $(1 \leq j \leq N-1)$,

$$\begin{bmatrix} p_j \\ p_{j+1} \end{bmatrix} = \left[\begin{array}{c|c} {}_jD_1 & {}_jD_2 \\ \hline {}_jD_3 & {}_jD_4 \end{array}\right] \begin{bmatrix} p_o \\ p_1 \end{bmatrix} . \tag{8.27}$$

Thus, for $j = 1, \ldots, N-1$,

$$\begin{aligned} \underset{\sim}{p}_j &= {}_jD_1 \, \underset{\sim}{p}_o + {}_jD_2 \, \underset{\sim}{p}_1 \\ &= \{{}_jD_1 + {}_jD_2 \, A_o\} \, \underset{\sim}{p}_o & \text{using (8.1)} \\ &= \{{}_jD_1 + {}_jD_2 \, A_1 - {}_jD_2\} \, \underset{\sim}{p}_o & \text{using (8.4)} \\ &= \{{}_{j+1}D_2 - {}_jD_2\} \, \underset{\sim}{p}_o & \text{using (8.18).} \end{aligned}$$

Also, from equation (8.27), (taking $j = N-1$) ,

$$\begin{aligned} \underset{\sim}{p}_N &= {}_{N-1}D_3 \, \underset{\sim}{p}_o + {}_{N-1}D_4 \, \underset{\sim}{p}_1 \\ &= \{{}_{N-1}D_3 + {}_{N-1}D_4 \, A_o\} \, \underset{\sim}{p}_o & \text{using (8.1),} \\ &= \{{}_ND_1 + {}_ND_2 \, A_o\} \, \underset{\sim}{p}_o & \text{using (8.18),} \\ &= \{{}_{N+1}D_2 - {}_ND_2\} \, \underset{\sim}{p}_o \ . \end{aligned}$$

Hence, for $j = 1, 2, \ldots, N$,

$$\underset{\sim}{p}_j = \{{}_{j+1}D_2 - {}_jD_2\} \, \underset{\sim}{p}_o \ . \qquad (8.28)$$

In order that this procedure be useful in solving for the equilibrium probabilities, we require an easy technique for obtaining ${}_jD_2$. Using equations (8.16) through (8.20), we have the following simple iterative procedure.

<u>Lemma 8.1</u>:

$$\begin{aligned} {}_1D_2 &= I \ , \\ {}_2D_2 &= A_1 \ , \end{aligned}$$

and for $j \geq 2$,

$${}_{j+1}D_2 = \begin{cases} {}_jD_2 \, A_1 - {}_{j-1}D_2 \, B_1 \ , \\ A_1 \, {}_jD_2 - B_1 \, {}_{j-1}D_2 \ . \end{cases} \qquad (8.29)$$

The results of this section may be conveniently summarised by the following theorem.

Theorem 8.2: For $M \geq 2$, $N \geq 2$, the matrix equations (8.1), (8.2) and (8.3) have the following solution.

$$p_o = {}_{N+1}D_2^{-1}\, p ,$$

$$p_j = \{{}_{j+1}D_2 - {}_jD_2\}\, p_o , \qquad (j = 1, \ldots, N) ,$$

where the ${}_jD_2$ are given by equation (8.29).

This method of solving for the equilibrium probabilities in the finite waiting room case has been programmed for an IBM System 360, model 75 computor. Note that the recursive generation of the ${}_jD_2$ enables us to set up a very efficient computor program.

In any numerical studies it is useful to have checks on computation details. Besides the checks provided by the marginal probabilities, we have the following result that states, that for any given ${}_{j+1}D_2$ the sum of the elements of each column are constant.

Let $\xi' = (1, 1, \ldots, 1)$, a $1 \times (M+1)$ row vector.

Lemma 8.3: For $j \geq 0$,

$$\xi'\, {}_{j+1}D_2 = \left[\frac{1 - \rho_2^{j+1}}{1 - \rho_2} \right] \xi' . \tag{8.30}$$

Proof: First note that

$$\xi' A(x, y, z) = (x - y - z)\xi' ,$$

and thus

$$\xi' A_1 = (\rho_2 + 1)\xi' ,$$

$$\xi' B_1 = \rho_2\, \xi' .$$

Furthermore, from equations (8.29),

$$\underset{\sim}{\xi}'\ {}_1D_2 = \underset{\sim}{\xi}' ,$$

$$\underset{\sim}{\xi}'\ {}_2D_2 = (\rho_2 + 1)\underset{\sim}{\xi}' .$$

Thus the lemma is true for $j = 0, 1$. Using induction, we assume that equation (8.20) is true for $j = 0, 1, \ldots, k$. Then by equations (8.29)

$$\begin{aligned}\underset{\sim}{\xi}'\ {}_{k+2}D_2 &= \underset{\sim}{\xi}'\ {}_{k+1}D_2\, A_1 - \underset{\sim}{\xi}'\ {}_kD_2\, B_1 \\ &= \left[\left\{ \frac{1 - \rho_2^{k+1}}{1 - \rho_2} \right\} (\rho_2 + 1) - \left\{ \frac{1 - \rho_2^{k}}{1 - \rho_2} \right\} \rho_2 \right] \underset{\sim}{\xi}' \\ &= \left[\frac{1 - \rho_2^{k+2}}{1 - \rho_2} \right] \underset{\sim}{\xi}'\end{aligned}$$

Thus equation (8.30) is true for $j = k+1$ and hence in general.

Actually this is an indirect check on the marginals, since from equation (8.28)

$$p_{\cdot j} = \underset{\sim}{\xi}'\ \underset{\sim}{p}_j = \underset{\sim}{\xi}' \left[{}_{j+1}D_2 - {}_jD_2 \right] \underset{\sim}{p}_o$$

$$= \rho_2^{\,j}\ \underset{\sim}{\xi}'\ \underset{\sim}{p}_o = \rho_2^{\,j}\ p_{\cdot o} . \qquad (8.31)$$

Also, since ${}_{N+1}D_2\ \underset{\sim}{p}_o = \underset{\sim}{p}$,

$$\underset{\sim}{\xi}'\ {}_{N+1}D_2\ \underset{\sim}{p}_o = \left[\frac{1 - \rho_2^{N+1}}{1 - \rho_2} \right] \underset{\sim}{\xi}'\ \underset{\sim}{p}_o = \underset{\sim}{\xi}'\ \underset{\sim}{p} = 1 .$$

Thus, $$\underset{\sim}{\xi}'\ \underset{\sim}{p}_o = \frac{1 - \rho_2}{1 - \rho_2^{N+1}} ,$$

and from equation (8.31)

$$p_{\cdot j} = \left[\frac{1 - \rho_2}{1 - \rho_2^{N+1}} \right] \rho_2^{\,j} \quad ; \quad j = 0, 1, \ldots, N .$$

as already obtained (equation 6.7).

9. Special cases

In Section 8, we outlined a method for obtaining the equilibrium probabilities in the finite waiting room case when $M \geq 2$ and $N \geq 2$. In this section, we look at some special cases when M and N take on specific values (not necessarily ≥ 2).

(a) $M = 1$, $N = 1$.

For this case, we wish to solve equations (4.1), (4.3), (4.7) and (4.9) for p_{ij} $(i = 0, 1 ; j = 0, 1)$. These equations can be reduced to a matrix form as was done in Section 8. Let

$$A(x, y, z) = \begin{bmatrix} x-z & -z \\ -y & x-y \end{bmatrix}.$$

Then

$$A_o \underset{\sim}{p}_o = \underset{\sim}{p}_1 , \tag{9.1}$$

$$A_2 \underset{\sim}{p}_1 = B_1 \underset{\sim}{p}_o , \tag{9.2}$$

where A_o, A_2, and B_1 are determined using the same substitutions for x, y, z as in Section 8.

Defining $A_1 = A_o + I$, as in equation (8.4), we have from (9.1)

$$(A_1 - I)\underset{\sim}{p}_o = \underset{\sim}{p}_1 .$$

Thus
$$A_1 \underset{\sim}{p}_o = \underset{\sim}{p}_o + \underset{\sim}{p}_1 = \underset{\sim}{p} .$$

Hence
$$\underset{\sim}{p}_o = A_1^{-1} \underset{\sim}{p} , \tag{9.3}$$

where

$$\underset{\sim}{p} = \begin{bmatrix} p_{o\cdot} \\ \\ p_{1\cdot} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1 - \rho_1}{1 - \rho_1^2} \right) \\ \\ \left(\dfrac{1 - \rho_1}{1 - \rho_1^2} \right) \rho_1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1 + \rho_1} \\ \\ \dfrac{\rho_1}{1 + \rho_1} \end{bmatrix} .$$

Let us define $\tau = \lambda_1 + \lambda_2 + \lambda + \mu_1 + \mu_2$. Then

$$A_1 = \begin{bmatrix} \frac{\tau - \mu_1}{\mu_2} & -\frac{\mu_1}{\mu_2} \\ -\frac{\lambda_1}{\mu_2} & \frac{\tau - \lambda_1}{\mu_2} \end{bmatrix}$$

$$\det A_1 = \frac{\tau}{\mu_2}(1 + \rho_2) = \Delta_1 ,$$

$$A_1^{-1} = \frac{1}{\Delta_1} \begin{bmatrix} \frac{\tau - \lambda_1}{\mu_2} & \frac{\mu_1}{\mu_2} \\ \frac{\lambda_1}{\mu_2} & \frac{\tau - \mu_1}{\mu_2} \end{bmatrix} .$$

From (9.3), we can solve for $\underline{p}_0' = (p_{00}, p_{10})$. From the relation $\underline{p}_0 + \underline{p}_1 = \underline{p}$, $\underline{p}_1' = (p_{01}, p_{11})$ can then be determined to yield,

$$p_{00} = \frac{1}{(1+\rho_1)(1+\rho_2)}\left[1 + \frac{\lambda}{\tau}\right],$$

$$p_{01} = \frac{1}{(1+\rho_1)(1+\rho_2)}\left[\rho_2 - \frac{\lambda}{\tau}\right],$$

$$p_{10} = \frac{1}{(1+\rho_1)(1+\rho_2)}\left[\rho_1 - \frac{\lambda}{\tau}\right],$$

$$p_{11} = \frac{1}{(1+\rho_1)(1+\rho_2)}\left[\rho_1\rho_2 + \frac{\lambda}{\tau}\right].$$

Note that when $\lambda = 0$ (two independent queues in parallel) p_{ij} is given by the product of the marginals, as one would expect.

(b) $M = 1$, $N \geq 2$.

In this case, we wish to solve equations (4.1), (4.3), (4.4), (4.6), (4.7), and (4.9). These equations reduce to the same matrix equations (8.1), (8.2) and (8.3) as in Section 8 but with the change that

$$A(x, y, z) = \begin{bmatrix} x - z & -z \\ -y & x - y \end{bmatrix}.$$

The $\underline{p}'_j = (p_{oj}, p'_{1j})$, $j = 0, 1, \ldots, N$, are thence solved as in Section 8.

As a special case, we solve these equations explicitly for $M = 1$, $N = 2$.

$$\underline{p}_o = {}_3D_2^{-1}\, \underline{p} \ , \tag{9.5}$$

$$\underline{p}_1 = \{{}_2D_2 - {}_1D_2\}\, \underline{p}_o \ , \tag{9.6}$$

$$\underline{p}_2 = \underline{p} - \underline{p}_o - \underline{p}_1 \ , \tag{9.7}$$

where
$${}_1D_2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$${}_2D_2 = A_1 = \begin{bmatrix} \dfrac{\tau - \mu_1}{\mu_2} & -\dfrac{\mu_1}{\mu_2} \\ -\dfrac{\lambda_1}{\mu_2} & \dfrac{\tau - \lambda_1}{\mu_2} \end{bmatrix},$$

$${}_3D_2 = A_1^2 - B_1$$

$$= \frac{1}{\mu_2^2} \begin{bmatrix} (\tau-\mu_1)^2+\mu_1\lambda_1-\mu_2\lambda_2 & -2\mu_1\tau+\mu_1^2+\mu_1\lambda_1 \\ -2\lambda_1\tau+\lambda_1^2+\mu_1\lambda_1-\mu_2\lambda & (\tau-\lambda_1)^2+\mu_1\lambda_1-\mu_2\lambda-\mu_2\lambda_2 \end{bmatrix}.$$

Also

$$\det {}_3D_2 = (\rho_2^2 + \rho_2 + 1) \left[\frac{\tau^2 - \mu_2\lambda_2}{\mu_2^2} \right] = \Delta_2 \ ,$$

$${}_3D_2^{-1} = \frac{1}{\mu_2^2\Delta_2} \begin{bmatrix} (\tau-\lambda_1)^2 + \mu_1\lambda_1 - \mu_2\lambda - \mu_2\lambda_2 & 2\mu_1\tau - \mu_1^2 - \mu_1\lambda_1 \\ 2\lambda_1\tau - \lambda_1^2 - \mu_1\lambda_1 + \mu_2\lambda & (\tau-\mu_1)^2 + \mu_1\lambda_1 - \mu_2\lambda_2 \end{bmatrix}.$$

Since $\underset{\sim}{p} = \begin{bmatrix} p_{o\cdot} \\ p_{1\cdot} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\rho_1} \\ \frac{\rho_1}{1+\rho_1} \end{bmatrix}$,

we solve equation (9.5) to yield

$$p_{oo} = \frac{1}{(1+\rho_1)(\rho_2^2 + \rho_2 + 1)} \left[1 + \frac{\lambda(\tau + \mu_2\rho_2)}{\tau^2 - \mu_2\lambda_2} \right],$$

$$p_{1o} = \frac{1}{(1+\rho_1)(\rho_2^2 + \rho_2 + 1)} \left[\rho_1 - \frac{\lambda(\tau + \mu_2\rho_2)}{\tau^2 - \mu_2\lambda_2} \right].$$

From equation (9.6) ,

$$\underset{\sim}{p}_1 = \begin{bmatrix} \rho_2 + \frac{\lambda_1}{\mu_2} & -\frac{\mu_1}{\mu_2} \\ -\frac{\lambda_1}{\mu_2} & \rho_2 + \frac{\mu_1}{\mu_2} \end{bmatrix} \underset{\sim}{p}_o ,$$

we solve for $\underset{\sim}{p}_1$ to yield,

$$p_{o1} = \frac{1}{(1+\rho_1)(\rho_2^2 + \rho_2 + 1)} \left[\rho_2 - \frac{\lambda\{\tau(1-\rho_2) + \lambda\}}{\tau^2 - \mu_2\lambda_2} \right]$$

$$p_{11} = \frac{1}{(1+\rho_1)(\rho_2^2 + \rho_2 + 1)} \left[\rho_1\rho_2 + \frac{\lambda\{\tau(1-\rho_2) + \lambda\}}{\tau^2 - \mu_2\lambda_2} \right].$$

Finally, from (9.7),

$$p_{o2} = \frac{1}{(1+\rho_1)(\rho_2^2 + \rho_2 + 1)} \left[\rho_2^2 - \frac{\lambda(\tau\rho_2 + \lambda_2)}{\tau^2 - \mu_2\lambda_2} \right],$$

$$p_{12} = \frac{1}{(1+\rho_1)(\rho_2^2 + \rho_2 + 1)} \left[\rho_1\rho_2^2 + \frac{\lambda(\tau\rho_2 + \lambda_2)}{\tau^2 - \mu_2\lambda_2} \right].$$

(c) $\underline{M \geq 2}$, $\underline{N = 1}$

The equations (4.1), (4.2), (4.3), (4.7), (4.8), and (4.9) that we wish to solve can be expressed in matrix form by equations (9.1) and (9.2) with $A(x, y, z)$ the $(M+1) \times (M+1)$ matrix as defined in Section 8.

Consider determination of the p_{ij} for the case where $M = 2$, $N = 1$.

$$\underset{\sim}{p}_o = A_1^{-1}\, \underset{\sim}{p} , \tag{9.8}$$

$$\underset{\sim}{p}_1 = \underset{\sim}{p} - \underset{\sim}{p}_o , \tag{9.9}$$

where

$$\underset{\sim}{p} = \begin{bmatrix} p_{o\cdot} \\ p_{1\cdot} \\ p_{2\cdot} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\rho_1^2 + \rho_1 + 1} \\ \dfrac{\rho_1}{\rho_1^2 + \rho_1 + 1} \\ \dfrac{\rho_1^2}{\rho_1^2 + \rho_1 + 1} \end{bmatrix} ,$$

and

$$A_1 = \begin{bmatrix} \dfrac{\tau - \mu_1}{\mu_2} & -\dfrac{\mu_1}{\mu_2} & 0 \\ -\dfrac{\lambda_1}{\mu_2} & \dfrac{\tau}{\mu_2} & -\dfrac{\mu_1}{\mu_2} \\ 0 & -\dfrac{\lambda_1}{\mu_2} & \dfrac{\tau - \lambda_1}{\mu_2} \end{bmatrix} .$$

Thus

$$\det A_1 = (\rho_2 + 1)\left[\frac{\tau^2 - \mu_1\lambda_1}{\mu_2^2} \right] = \Delta_3 ,$$

and

$$A_1^{-1} = \frac{1}{\mu_2^2\Delta_3}\begin{bmatrix} \tau(\tau-\lambda_1)-\mu_1\lambda_1 & \mu_1(\tau-\lambda_1) & \mu_1^2 \\ \lambda_1(\tau-\lambda_1) & (\tau-\mu_1)(\tau-\lambda_1) & \mu_1(\tau-\mu_1) \\ \lambda_1^2 & \lambda_1(\tau-\mu_1) & \tau(\tau-\mu_1)-\mu_1\lambda_1 \end{bmatrix} .$$

We now use equation (9.8) to find $p_0' = (p_{oo}, p_{1o}, p_{2o})$ and also equation (9.9) to find $p_1' = (p_{o1}, p_{11}, p_{21})$ yielding:

$$p_{oo} = \frac{1}{(\rho_1^2 + \rho_1 + 1)(\rho_2 + 1)}\left[1 + \frac{\lambda(\tau + \mu_1\rho_1)}{\tau^2 - \mu_1\lambda_1}\right],$$

$$p_{1o} = \frac{1}{(\rho_1^2 + \rho_1 + 1)(\rho_2 + 1)}\left[\rho_1 - \frac{\lambda\{\tau(1-\rho_1) - \lambda\}}{\tau^2 - \mu_1\lambda_1}\right],$$

$$p_{2o} = \frac{1}{(\rho_1^2 + \rho_1 + 1)(\rho_2 + 1)}\left[\rho_1^2 - \frac{\lambda(\tau\rho_1 + \lambda_1)}{\tau^2 - \mu_1\lambda_1}\right].$$

Also

$$p_{o1} = \frac{1}{(\rho_1^2 + \rho_1 + 1)(\rho_2 + 1)}\left[\rho_2 - \frac{\lambda(\tau + \mu_1\rho_1)}{\tau^2 - \mu_1\lambda_1}\right],$$

$$p_{11} = \frac{1}{(\rho_1^2 + \rho_1 + 1)(\rho_2 + 1)}\left[\rho_1\rho_2 + \frac{\lambda\{\tau(1-\rho_1) - \lambda\}}{\tau^2 - \mu_1\lambda_1}\right],$$

$$p_{21} = \frac{1}{(\rho_1^2 + \rho_1 + 1)(\rho_2 + 1)}\left[\rho_1^2\rho_2 + \frac{\lambda(\tau\rho_1 + \lambda_1)}{\tau^2 - \mu_1\lambda_1}\right].$$

Note that we could have derived these same results from the earlier solution when $M = 1$, $N = 2$ by interchanging the roles of μ_1 and μ_2; and λ_1 and λ_2.

(d) To conclude this section, we investigate the derivation of the equilibrium probabilities in the case $M = 2$ and $N = 2$.

To solve for $\underset{\sim}{p}_0$, $\underset{\sim}{p}_1$, and $\underset{\sim}{p}_2$ where

$$A_0\underset{\sim}{p}_0 = \underset{\sim}{p}_1 , \tag{9.10}$$

$$A_1\underset{\sim}{p}_1 = \underset{\sim}{p}_2 + B_1\underset{\sim}{p}_0 , \tag{9.11}$$

$$A_2\underset{\sim}{p}_2 = B_1\underset{\sim}{p}_1 , \tag{9.12}$$

where

$$A(x, y, z) = \begin{bmatrix} x - z & -z & 0 \\ -y & x & -z \\ 0 & -y & x - y \end{bmatrix} .$$

The solution is given by

$$\underset{\sim}{p}_0 = {}_3D_2^{-1}\underset{\sim}{p} = \left[A_1^2 - B_1\right]^{-1}\underset{\sim}{p} , \tag{9.13}$$

$$\underset{\sim}{p}_1 = A_0\underset{\sim}{p}_0 , \tag{9.14}$$

$$\underset{\sim}{p}_2 = \underset{\sim}{p} - \underset{\sim}{p}_0 - \underset{\sim}{p}_1 , \tag{9.15}$$

where

$$\underset{\sim}{p} = \begin{bmatrix} p_{0\cdot} \\ p_{1\cdot} \\ p_{2\cdot} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\rho_1^2 + \rho_1 + 1} \\ \dfrac{\rho_1}{\rho_1^2 + \rho_1 + 1} \\ \dfrac{\rho_1^2}{\rho_1^2 + \rho_1 + 1} \end{bmatrix} .$$

Now

$$A_1^2 - B_1 = \frac{1}{\mu_2^2}\begin{bmatrix} (\tau-\mu_1)^2 + \mu_1\lambda_1 - \mu_2\lambda_2 & -\mu_1(2\tau-\mu_1) & \mu_1^2 \\ -\lambda_1(2\tau-\mu_1) - \mu_2\lambda & \tau^2 + 2\mu_1\lambda_1 - \mu_2\lambda_2 & -\mu_1(2\tau-\lambda_1) \\ \lambda_1^2 & -\lambda_1(2\tau-\lambda_1) - \mu_2\lambda & (\tau-\lambda_1)^2 + \mu_1\lambda_1 - \mu_2^2\rho_2 \end{bmatrix},$$

$$\det(A_1^2-B_1) = \frac{1}{\mu_2^4}(\rho_2^2 + \rho_2 + 1)\left[\tau^4 - 2\tau^2(\mu_1\lambda_1+\mu_2\lambda_2) - 2\tau\lambda\mu_1\mu_2 + (\mu_1\lambda_1-\mu_2\lambda_2)^2\right]$$

$$= \Delta_4 \qquad (9.16)$$

It can be easily shown (by expanding τ^4) that Δ_4 is positive, and hence $A_1^2 - B_1$ is non-singular.

Because of the tedious algebra involved, we do not solve for $\underline{p}_0$, $\underline{p}_1$, or $\underline{p}_2$. However, theoretical expressions for the p_{ij} ($i = 0, 1, 2$; $j = 0, 1, 2$) can be derived from equations (9.13), (9.14) and (9.15).

Before concluding this section, we note that equation (9.16) can be expressed as follows,

$$\Delta_4 = \frac{1}{\mu_2^4}(\rho_2^2 + \rho_2 + 1)\left[\sigma_4 - 2\mu_1\mu_2\tau\lambda\right]$$

where

$$\sigma_4 = (\tau - \sqrt{\mu_1\lambda_1} - \sqrt{\mu_2\lambda_2})(\tau - \sqrt{\mu_1\lambda_1} + \sqrt{\mu_2\lambda_2})(\tau + \sqrt{\mu_1\lambda_1} - \sqrt{\mu_2\lambda_2})(\tau + \sqrt{\mu_1\lambda_1} + \sqrt{\mu_2\lambda_2}).$$

It may be that this form of Δ_4 is an exhibition of some trend that may appear in the determinant of ${}_{N+1}D_2$. This possibility has not been investigated.

10. Solution for equilibrium probabilities - Infinite waiting room

As pointed out in Section 5, any attempts to find a joint probability generating function $\pi(s_1, s_2)$ satisfying equation (5.7) have failed.

Since the marginal distributions are both geometric (special case of a negative binomial distribution), we have that $\pi(s_1, s_2)$ is the j.p.g.f. of some bivariate negative binomial distribution.

We attempted to choose the constants a_0, a_1, a_2, a_{12} so that a j.p.g.f. of the form $[a_0 + a_1 s_1 + a_2 s_2 + a_{12} s_1 s_2]^{-1}$ would be a suitable candidate for $\pi(s_1, s_2)$. It was found that, in general, it is impossible to determine such constants for equations (5.7) to be satisfied.

Any approach at solving for the equilibrium probabilities using the matrix theoretic approach as in Section 8 does not appear to give any fruitful results.

One may be interested in holding M fixed, and letting $N \to \infty$, and apply the approach of Section 8. However, in this case, we also run into difficulties. Firstly, suppose we consider equations (8.14) and (8.15). One would suspect simplification using $\sum_{k=0}^{\infty} D^k = [I - D]^{-1}$. However, for this to be true, we require $[I - D]$ to be non-singular which is not the case.

$$I - D = \left|\begin{array}{c|c} I & -I \\ \hline B_1 & I - A_1 \end{array}\right| .$$

Note that

$$\xi' I = \xi' , \qquad \xi'(-I) = -\xi' ,$$

$$\xi' B_1 = \rho_2 \xi' , \quad \xi'(I - A_1) = -\rho_2 \xi' .$$

Thus the column sums of the first (M+1) rows are

$$(1, \ldots, 1, -1, \ldots, -1) ,$$

and the column sums of the last (M+1) rows are

$$(\rho_2, \ldots, \rho_2, -\rho_2, \ldots, -\rho_2) .$$

Hence $\det(I - D) = 0$ and $I - D$ is not of full rank.

Secondly, if one wishes to use Theorem 8.2 when $N \to \infty$, one has to evalute $\lim_{N\to\infty} {}_{N+1}D_2 = {}_{\infty}D_2$ (if it exists) and then investigate determination of ${}_{\infty}D_2^{-1}$. However, we show that if ${}_{\infty}D_2$ exists, then it is singular.

Under the assumption of the existence of ${}_{\infty}D_2$, we have from equation (8.29)

$${}_{\infty}D_2 = {}_{\infty}D_2 A_1 - {}_{\infty}D_2 B_1 .$$

From equation (8.5)

$$0 = {}_{\infty}D_2 [I - A_2] . \tag{10.1}$$

Now $A_2 = A(x, y, z)$ where $x = \dfrac{\lambda_1 + \lambda + \mu_1}{\mu_2} + 1$, $y = \dfrac{\lambda_1 + \lambda}{\mu_2}$, $z = \dfrac{\mu_1}{\mu_2}$.

Thus $x - 1 = y + z$.

$$I - A_2 = \begin{vmatrix} -y & z & 0 & \cdot & \cdot & 0 \\ y & -y-z & z & \ddots & & \cdot \\ 0 & y & -y-z & z & \ddots & \cdot \\ \cdot & \ddots & \ddots & \ddots & \ddots & \cdot \\ \cdot & & \ddots & \ddots & \ddots & 0 \\ & & \ddots & y & -y-z & z \\ 0 & \cdot & \cdot & 0 & y & -z \end{vmatrix}$$

Let ${}_{\infty}D_2 = [d_{ij}]$; $i, j = 1, \ldots, M+1$. From equation (10.1), we have for each $i = 1, \ldots, M+1$,

$$- d_{i1}\, y + d_{i2}\, y = 0 ,$$

$$d_{i,j-1}\, z - d_{ij}\, (y+z) + d_{i,j+1}\, y = 0 , \qquad (j = 2, \ldots, M) ,$$

$$d_{iM}\, z - d_{i,M+1}\, z = 0 .$$

The only solution for these equations is $d_{ij} = d_i$ $(j = 1, \ldots, M+1)$ and thus ${}_{\infty}D_2$ is singular (in fact rank 1).

To investigate this result, the computor program (mentioned in Section 8) was constructed so that ${}_{N+1}D_2$ was printed out for selected values of (M, N). No limiting tendencies appeared evident.

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