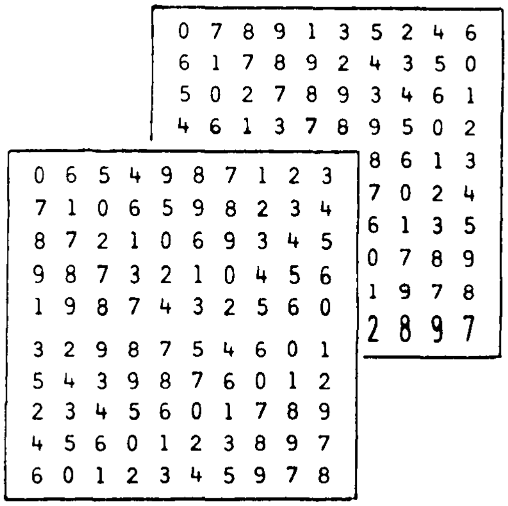

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CONSTRUCTION OF DESIGNS FROM PERMUTATION GROUPS[1]

by

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1. Introduction.

Let a design D have points $a_1, \ldots, a_v$ and blocks $B_1, \ldots, B_b$. An automorphism α is a one-to-one mapping of the points onto themselves preserving incidence. Clearly the automorphisms of D form a group G.

We shall consider only cases in which the group G is transitive both on the points and on the blocks of D. Without any loss of generality, we may identify blocks containing the same points. Hence we may consider G as a permutation group on the points of D, and if we are given any one block, say B_1, as a set of points, then the remaining blocks are the images of B_1 under the action of G.

If G_1 is the subgroup of G fixing a_1, (called the stabilizer of a_1) then $[G: G_1] = v$. Similarly, if S_1 is the stabilizer of the block B_1, then $[G: S_1] = b$. The sets of letters taken into themselves by a subgroup are called the orbits of that group. Clearly B_1 is composed of complete orbits of S_1. For a group difference set $G_1 = S_1 = 1$ the identity. Here we consider cases in which $G_1 = S_1$ so that $v = b$, and even the more special situation in which G is of

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rank 3, meaning in the terms of D. G. Higman [3] that G_1 has orbits $\{a_1\}$ and precisely two others, and we take B_1 to be one of these orbits. More general situations have been considered by the author [2].

2. Designs derived from rank 3 groups.

We suppose that the group G is a permutation group on v letters $\{a_1, \ldots, a_v\} = \Omega$ and that for $a \in \Omega$, G_a has orbits

2.1) $\{a\}, \quad \Delta(a), \quad \Gamma(a)$

where $\Delta(a)$ has length k (for all a) and $\Gamma(a)$ has length ℓ (for all a) and $v = 1 + k + \ell$. We choose our notation so that for $g \in G$ $\Delta(a)^g = \Delta(a^g)$ where for $X \in \Omega$, $g \in G$, X^g is the image of X under g. We also choose the notation so that $k \leq \ell$. Essential to these studies are the intersection numbers λ, μ defined by

2.2) $$|\Delta(a) \cap \Delta(b)| = \begin{cases} \lambda & \text{for } b \in \Delta(a) \\ \mu & \text{for } b \in \Gamma(a). \end{cases}$$

Here λ and μ do not depend on the particular choice of a and b. Immediately

2.3) $$|\Gamma(a) \cap \Gamma(b)| = \begin{cases} \lambda_1 & \text{for } b \in \Gamma(a) \\ \mu_1 & \text{for } b \in \Delta(a) \end{cases}$$

with $\lambda_1 = \ell - k + \mu - 1$, $\mu_1 = \ell - k + \lambda + 1$.

We define incidence matrices A, B associated with the orbits Δ, Γ respectively, by putting

$$A = [r_{ij}] \qquad i,j = 1, \ldots, v$$

$$r_{ij} = \begin{cases} 1 & \text{if } a_j \in \Delta(a_i) \\ 0 & \text{if } a_j \notin \Delta(a_i) \end{cases}$$

2.4)

$$B = [s_{ij}] \qquad i,j = 1, \ldots, v$$

$$s_{ij} = \begin{cases} 1 & \text{if } a_j \in \Gamma(a_i) \\ 0 & \text{if } a_j \notin \Gamma(a_i) . \end{cases}$$

Then following D. Higman [3], we have the relations, where I is the identity matrix and J is the $v \times v$ matrix with every entry 1, A^T the transpose of A.

2.5)
$$I + A + B = J$$
$$AA^T = kI + \lambda A + \mu B .$$

In addition, the parameters k, ℓ, λ, μ satisfy the relation

2.6) $$\mu\ell = k(k-\lambda-1).$$

If G is of odd order, then $k = \ell$ and $A^T = B$. If G is of even order, then $A^T = A$, $B^T = B$ and either

I. $k = \ell, \quad \mu = \lambda + 1 = k/2$

or

II. $d = (\lambda-\mu)^2 + 4(k-\mu)$ is a square.

In case II, A has the eigenvalue k with multiplicity 1, the integral eigenvalues $\frac{\lambda - \mu \pm \sqrt{d}}{2}$ with multiplicities f_2, f_3 where

$$2.7) \qquad \left\{ \begin{array}{c} f_2 \\ \\ f_3 \end{array} \right\} = \frac{2k + (\lambda-\mu)(k+\ell) \mp \sqrt{d}\,(k+\ell)}{\mp 2\sqrt{d}}$$

If $b \in \Delta(a)$, we say that a and b are first associates, and if $b \in \Gamma(a)$, we say that a and b are second associates. In this terminology, it is not difficult to show that the blocks $\Delta(a_1), \ldots, \Delta(a_v)$ are a partially balanced incomplete block design D with two associate classes in the sense of Bose and Shimamoto [1]. But if $\lambda = \mu$, then D is a symmetric balanced incomplete block design. Also if we define $\Delta^*(a) = \{a, \Delta(a)\}$, then the intersection numbers are

$$2.8) \qquad |\Delta^*(a) \cap \Delta^*(b)| = \begin{cases} \lambda+2 & \text{if} \quad b \in \Delta(a) \\ \mu & \text{if} \quad b \in \Gamma(a) . \end{cases}$$

Hence if $\lambda + 2 = \mu$, the blocks $\Delta^*(a)$ form a symmetric block design.

A computer search has found values with $k \leq 100$ for the parameters k, ℓ, λ, μ such that 2.6) holds $d = (k-\lambda) + 4(k-\mu)$ is a square and f_2, f_3 of 2.7) are integers. If we can find groups G corresponding to such parameters, or take representations of known groups, we will be led to partially balanced designs or symmetric balanced designs.

3. Construction of the symmetric design with $v = 56$, $k = 11$, $\lambda = 2$.

For the construction of the 56, 11, 2 design from LF(3,4), we take several matrices which together generate LF(3,4). We represent the four elements of the field GF(4) by 0, 1, A, B, where necessarily $A^2 = B$, $A^3 = 1$, $A + B = 1$, $1 + 1 = 0$. Matrices generating

LF(3,4) are

3.1) $$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & A & 1 \\ B & 1 & A \\ 1 & B & 0 \end{bmatrix} \quad d = \begin{bmatrix} 0 & A & B \\ B & 0 & A \\ 0 & 0 & 1 \end{bmatrix} .$$

We represent the points of the projective plane $\pi = PG(2,4)$ by the numbers 1, 2, ..., 21 taking 1 = (0, 0, 1). Each of these matrices X determines a collineation of the plane with the action given by $(x, y, z) \to (x, y, z)X$. Thus as (0, 0, 1) c = (1, B, 0), the collineation c maps 1 = (0, 0, 1) onto 18 = (1, B, 0). With the remaining points appropriately numbered, the collineations are given in permutation form as

3.2)

a = (1,10,3,11,7,6,2)(4,13,20,16,15,8,14)(5,12,17,21,19,9,18)
b = (1)(2,10,6)(3,11,7)(4,12,8)(5,13,9)(14)(15,17,16)(18)(19,20,21)
c = (1,18,14)(2,17,21)(3,11,7)(4,8,12)(5)(6,15,20)(9)(10,16,19)(13)
d = (1)(2,9)(3,7)(4,6)(5,8)(10,20)(11,18)(12,19)(13,21)(14)(15)(16)(17).

The group $\langle b, c, d \rangle$ is isomorphic to A_6. On the points 1, ..., 21, $\langle b, c, d \rangle$ has two orbits, one being {1, 3, 7, 11, 14, 18} and the other the remainig 15 points. The six points 1, 3, 7, 11, 14, 18 form an oval in the plane π. This is to say that no three of these are on a line. Each quadrilateral in π is contained in a unique oval of six points. There are 168 ovals in π. The group LF(3,4) = G is the little projective group of π, that is the group of collineations generated by the elations. The 168 ovals are permuted in three orbits of 56 by G. If we assign the number 1 to the oval {1, 3, 7, 11, 14, 18} then G is represented as a permutation group on 56 letters (a set of 56 ovals). In this notation, the elements a, b, c, d are represented by the following permutations:

a = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21)
(22,23,24,25,26,27,28)(29,30,31,32,33,34,35)
(36,37,38,39,40,41,42)(43,44,45,46,47,48,49)
(50,51,52,53,54,55,56).

b = (1)(45)(2,3,7)(4,5,6)(8,27,42)(9,10,26)(11,38,25)(12,19,37)
(13,21,18)(14,36,20)(15,34,17)(16,35,33)(22,31,53)
(23,51,30)(24,39,50)(28,54,41)(29,52,32)(40,55,56)
(43,44,46)(47,48,49).

3.3) c = (1)(2,8,41)(3,27,28)(4,36,31)(5,20,53)(6,14,22)(7.42,54)
(9,29,34)(10,52,17)(11,24,46)(12,30,48)(13,55,33)
(15,26,32)(16,21,56)(18,40,35)(19,23,49)(25,50,44)
(37,51,47)(38,39,43)(45).

d = (1)(2,34)(3,54)(4,39)(5,13)(6,29)(7,56)(8)(9,44)(10,16)
(11)(12,19)(14)(15,41)(17,55)(18,52)(20,42)(21,24)(22,26)
(23)(25)(27,36)(28,40)(30,47)(31,33)(32,50)(35,43)(37,45)
(38)(46,53)(48)(49,51).

For the stabilizer G_1, the orbits consist of {1} a 10-orbit $\Delta(1)$ = {12,19,23,30,37,45,47,48,49,51} and a 45-orbit $\Gamma(1)$ consisting of the remaining letters. As shown in Section 3, the 56 sets b, $\Delta(b)$ as b = 1, ..., 56 form a symmetric block design with $v = 56$, $k = 11$, $\lambda = 2$. These are listed here with b the first entry followed by the points of $\Delta(b)$.

1	12	19	23	30	37	45	47	48	49	51	29	7	16	20	30	35	37	40	43	53	55
2	13	20	24	31	38	46	48	49	43	52	30	1	17	21	31	29	38	41	44	54	56
3	14	21	25	32	39	47	49	43	44	53	31	2	18	15	32	30	39	42	45	55	50
4	8	15	26	33	40	48	43	44	45	54	32	3	19	16	33	31	40	36	46	56	51
5	9	16	27	34	41	49	44	45	46	55	33	4	20	17	34	32	41	37	47	50	52
6	10	17	28	35	42	43	45	46	47	56	34	5	21	18	35	33	42	38	48	51	53
7	11	18	22	29	36	44	46	47	48	50	35	6	15	19	29	34	36	39	49	52	54
8	4	11	12	17	19	38	39	46	53	55	36	7	12	13	25	26	32	35	38	41	45
9	5	12	13	18	20	39	40	47	54	56	37	1	13	14	26	27	33	29	39	42	46
10	6	13	14	19	21	40	41	48	55	50	38	2	14	8	27	28	34	30	40	36	47
11	7	14	8	20	15	41	42	49	56	51	39	3	8	9	28	22	35	31	41	37	48
12	1	8	9	21	16	42	36	43	50	52	40	4	9	10	22	23	29	32	42	38	49
13	2	9	10	15	17	36	37	44	51	53	41	5	10	11	23	24	30	33	36	39	43
14	3	10	11	16	18	37	38	45	52	54	42	6	11	12	24	25	31	34	37	40	44

3.4)

15	4	11	13	16	21	23	27	31	35	47	43	2	3	4	6	12	18	27	29	41	51
16	5	12	14	17	15	24	28	32	29	48	44	3	4	5	7	13	19	28	30	42	52
17	6	13	8	18	16	25	22	33	30	49	45	4	5	6	1	14	20	22	31	36	53
18	7	14	9	19	17	26	23	34	31	43	46	5	6	7	2	8	21	23	32	37	54
19	1	8	10	20	18	27	24	35	32	44	47	6	7	1	3	9	15	24	33	38	55
20	2	9	11	21	19	28	25	29	33	45	48	7	1	2	4	10	16	25	34	39	56
21	3	10	12	15	20	22	26	30	34	46	49	1	2	3	5	11	17	26	35	40	50
22	7	17	21	24	27	39	40	45	51	52	50	7	10	12	27	28	31	33	49	53	54
23	1	18	15	25	28	40	41	46	52	53	51	1	11	13	28	22	32	34	43	54	55
24	2	19	16	26	22	41	42	47	53	54	52	2	12	14	22	23	33	35	44	55	56
25	3	20	17	27	23	42	36	48	54	55	53	3	13	8	23	24	34	29	45	56	50
26	4	21	18	28	24	36	37	49	55	56	54	4	14	9	24	25	35	30	46	50	51
27	5	15	19	22	25	37	38	43	56	50	55	5	8	10	25	26	29	31	47	51	52
28	6	16	20	23	26	38	39	44	50	51	56	6	9	11	26	27	30	32	48	52	53

4. Designs from the simple group of order 25920.

In the field GF(4) with four elements, 0, 1, A, B, the mapping $x \to x^2 = \bar{x}$ is an automorphism of order 2 fixing 0 and 1 and interchanging A and B. If X is a matrix, we write X^T for its inverse and $\bar{X}$ for the conjugate under the automorphism. The unimodular matrices X of degree 4 satisfying

4.1) $$\bar{X}^T X = I$$

are well known [3] to form a simple group G of order 25920, the unitary group $U_4(4)$. Dickson calls this the hyperorthogonal group HO(4,4).

Here G is generated by the two matrices

4.2) $$t = \begin{bmatrix} 1 & B & A & 0 \\ 0 & B & A & 1 \\ A & 1 & 0 & A \\ B & 0 & 1 & B \end{bmatrix} \qquad v = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Here $t^5 = I$, $v^9 = I$.

If X is a matrix of G and P = (x, y, z, w) is a point of the projective geometry PG(3,4), then X determines a collineation of PG(3,4) by the mapping

4.3) $$P \to PX.$$

The property 4.1) of a matrix X of G is an invariance property of the transformation 4.3). For P = (x, y, z, w), Q = (r, s, t, u), we define an inner product (P, Q) by the rule

4.4) $$(P, Q) = x\bar{r} + y\bar{s} + z\bar{t} + w\bar{u} .$$

The invariance is that for $X \in G$.

4.5) $$(PX, QX) = (P, Q).$$

Points P for which $(P, P) = 0$ are called isotropic points and those for which $(P, P) \neq 0$ are called anisotropic points.

In the projective geometry PG(3,4) a point P has coordinates $(x, y, z, w) \neq (0, 0, 0, 0)$ and if $u \neq 0$, (ux, uy, uz, uw) represents the same point. In PG(3,4), there are 45 isotropic points and 40 anisotropic points.

If we number the isotropic point $(0, 0, 1, 1)$ as 1 then on the isotropic points appropriately numbered we have

4.6)
$$t = (1,19,7,20,37)(2,3,4,17,29,33)(3,9,14,38,15)(4,21,25,22,28)(5,18,27,31,11)(6,35,26,40,42)(8,36,39,12,24)(10,32,13,45,16)(23,43,44,30,41),$$

$$v = (1,5,7)(2,4,8)(3,6,9)(10,18,15,12,17,14,11,16,13)(21,28,40,44,27,30,31,41,26)(22,38,25,39,34,42,35,24,29)(19,37,43,45,36,33,32,23,20).$$

G does not have a doubly transitive representation as has been shown by E. T. Parker. But G has several rank 3 representations. The representation 4.6) is a rank 3 representation with orbit lengths 1, 12, 32 in the stabilizer G_1 and $\lambda = \mu = 3$. With $B_1 = 16, 17, 18, 19, 23, 27, 28, 32, 36, 37, 41, 45$ as an initial block, we find a symmetric design with parameters 45, 12, 3 under the action of the permutation of 4.6).

The representation of G on the 40 anisotropic points, numbering $(0, 0, 0, 1)$ as 1 gives the representation

$t = (1,17,34,3,12)(2,29,38,15,24)(4,39,36,22,21)(5,37,7,13,19)$
$(6,31,27,9,30)(8,11,18,28,40)(10,32,26,25,14)$
$(16,35,33,23,20),$

4.7)

$v = (1,3,2)(4)(5,9,6)(7,13,12)(8,10,11)$
$(14,29,40,22,28,36,18,24,32)(15,23,38,20,13,37,19,27,33)$
$(16,26,39,21,25,35,17,30,34).$

This is also a rank 3 representation of G with parameters $k = 12$, $\ell = 27$, $\lambda = 2$, $\mu = 4$, and characters of degrees 15 and 24. Here the 12 orbit of the stabilizer G_4 consists of 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, the numbers in the three cycles of v.

The 40 12 orbits together with the fixed letters form a symmetric design with $v = 40$, $k = 13$, $\lambda = 4$. It turns out to be a very well known design, namely the planes in the projective space $PG(3,3)$. If we take $1 = (0, 0, 0, 1)$ over $GF(3)$ and coordinatize the remaining points over $GF(3)$, appropriately we have

4.8) $$t = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Furthermore, we have with the matrix A defined below:

4.9) $$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad t^T A t = A, \quad v^T A v = A.$$

The matrices X over $GF(3)$ satisfying $X^T AX = A$ form a group which modulo its center $-I$ is the symplectic group $S_4(3)$, a simple group of order 25920.

The correspondence between the 40 anisotropic points in $PG(3,4)$ with the 40 points of $PG(3,3)$ thus yields a proof of the isomorphism of the unitary group $U_4(4)$ and the symplectic group $S_4(3)$. For t and v as given in 4.2) over $GF(4)$ or as given over $GF(3)$ in 4.8) permute the 40 points by the permutations 4.7). The corresponding points are given the same number below.

Anisotropic points in PG(3,4) Points of PG(3,3)

	(0	0	0	1)	1	(0	0	0	1)
	(0	0	1	0)	2	(0	0	1	1)
	(0	1	0	0)	3	(0	0	1	-1)
	(1	0	0	0)	4	(0	0	1	0)
	(0	1	1	1)	5	(0	1	0	0)
	(0	1	1	A)	6	(0	1	0	1)
	(0	1	1	B)	7	(0	1	0	-1)
	(0	1	A	1)	8	(0	1	1	1)
	(0	1	A	A)	9	(0	1	1	-1)
	(0	1	A	B)	10	(0	1	1	0)
	(0	1	B	1)	11	(0	1	-1	-1)
	(0	1	B	A)	12	(0	1	-1	0)
	(0	1	B	B)	13	(0	1	-1	1)
	(1	0	1	1)	14	(1	1	0	0)
	(1	0	1	A)	15	(1	1	0	1)
	(1	0	1	B)	16	(1	1	0	-1)
	(1	0	A	1)	17	(1	1	-1	-1)
	(1	0	A	A)	18	(1	1	-1	0)
	(1	0	A	B)	19	(1	1	-1	1)
4.10)	(1	0	B	1)	20	(1	1	1	1)
	(1	0	B	A)	21	(1	1	1	-1)
	(1	0	B	B)	22	(1	1	1	0)
	(1	1	0	1)	23	(1	-1	0	0)
	(1	1	0	A)	24	(1	-1	0	-1)
	(1	1	0	B)	25	(1	-1	0	1)
	(1	A	0	1)	26	(1	-1	-1	1)
	(1	A	0	A)	27	(1	-1	-1	0)
	(1	A	0	B)	28	(1	-1	-1	-1)
	(1	B	0	1)	29	(1	-1	1	-1)
	(1	B	0	A)	30	(1	-1	1	1)
	(1	B	0	B)	31	(1	-1	1	0)
	(1	1	1	0)	32	(1	0	0	0)
	(1	1	A	0)	33	(1	0	1	1)
	(1	1	B	0)	34	(1	0	-1	-1)
	(1	A	1	0)	35	(1	0	1	-1)
	(1	A	A	0)	36	(1	0	-1	0)
	(1	A	B	0)	37	(1	0	0	1)
	(1	B	1	0)	38	(1	0	-1	1)
	(1	B	A	0)	39	(1	0	0	-1)
	(1	B	B	0)	40	(1	0	1	0)

Here G has three further rank three representations, respectively on 27, 36, and 40 letters. The 27 letters are the classical 27 lines on the general cubic surface. The second representation on 40 letters again gives orbits 1, 12, 27 and the $\Delta^*(a)$ give a block design with parameters 40, 13, 4, but it is not the geometry given by the representation of 4.7). Here are permutations a and b generating G and α an outer automorphism of order 2.

$g = 25920$

On 27 letters

a = (1 4 12 25 10 22 7 21 6 18 27 3)(2 8 20 5 19 16 13 14 15 24 9 23)(11 17 26)

b = (1)(2 4 5 6 7)(3 8 9 10 11)(12)(13 19 17 23 24)(14 18 22 15 21)(16 20 25 26 27)

α = (1)(2)(3)(4 8)(5 6)(7 11)(9 10)(12 15)(13 17)(14 18)(16 19)(20 27)(21 25)(22 26)(23 24)

On 36 letters

a = (1 2 3 4 5 6 7 8 9 10 11 12)(13 14 15 16 17 18 19 20 21 22 23 24)(25 26 27 28 29 30)(31 32 33 34 35 36)

4.11) b = (1)(2 13 9 25 14)(3 11 31 27 32)(4 24 33 19 34)(5 16 12 15 29)(6 7 8 26 28)(10 35 22 36 17)(18 20 21 23 30)

α = (1 36)(2 9)(3 22)(4)(5 20)(6)(7 19)(8 16)(10 35)(11 27)(12 26)(13 31)(14 25)(15 18)(17 32)(21 30)(23)(24 34)(28 33)(29)

On 40 letters

a = (1 2 3 4 5 6)(7 8)(9 10 11 12 13 14 15 16 17 18 19 20)(21 22)(23 24 25 26 27 28 29 30 31 32 33 34)(35 36 37 38 39 40)

b = (1 7 9 8 6)(2 16 21 14 5)(3 4 23 22 28)(10 33 35 30 20)(11 12 18 19 38)(13 24 36 37 31)(15 29 25 26 34)(17 32 39 40 27)

α = (1 38)(2 3)(4 18)(5)(6 11)(7 25)(8 35)(9 22)(10 12)(13)(14 19)(15 28)(16 36)(17)(20 27)(21 26)(23 40)(24 31)(29)(30 39)(32)(33)(34 37)

25920 on 40 points (not geometry)

4.12)

1	3	4	5	8	9	10	15	16	23	29	37	40
2	4	5	6	7	10	11	16	17	24	30	35	38
3	1	5	6	8	11	12	17	18	25	31	36	39
4	1	2	6	7	12	13	18	19	26	32	37	40
5	1	2	3	8	13	14	19	20	27	33	35	38
6	2	3	4	7	9	14	15	20	28	34	36	39
7	2	4	6	8	21	22	23	25	27	29	31	33
8	1	3	5	7	21	22	24	26	28	30	32	34
9	1	6	13	14	16	17	22	23	25	26	28	35
10	1	2	14	15	17	18	21	24	26	27	29	36
11	2	3	15	16	18	19	22	25	27	28	30	37
12	3	4	16	17	19	20	21	26	28	29	31	38
13	4	5	9	17	18	20	22	27	29	30	32	39
14	5	6	9	10	18	19	21	28	30	31	33	40
15	1	6	10	11	19	20	22	29	31	32	34	35
16	1	2	9	11	12	20	21	23	30	32	33	36
17	2	3	9	10	12	13	22	24	31	33	34	37
18	3	4	10	11	13	14	21	23	25	32	34	38
19	4	5	11	12	14	15	22	23	24	26	33	39
20	5	6	12	13	15	16	21	24	25	27	34	40
21	7	8	10	12	14	16	18	20	22	35	37	39
22	7	8	9	11	13	15	17	19	21	36	38	40
23	1	7	9	16	18	19	24	27	31	34	38	39
24	2	8	10	17	19	20	23	25	28	32	39	40
25	3	7	9	11	18	20	24	26	29	33	35	40
26	4	8	9	10	12	19	25	27	30	34	35	36
27	5	7	10	11	13	20	23	26	28	31	36	37
28	6	8	9	11	12	14	24	27	29	32	37	38
29	1	7	10	12	13	15	25	28	30	33	38	39
30	2	8	11	13	14	16	26	29	31	34	39	40
31	3	7	12	14	15	17	23	27	30	32	35	40
32	4	8	13	15	16	18	24	28	31	33	35	36
33	5	7	14	16	17	19	25	29	32	34	36	37
34	6	8	15	17	18	20	23	26	30	33	37	38
35	2	5	9	15	21	25	26	31	32	37	38	39
36	3	6	10	16	22	26	27	32	33	38	39	40
37	1	4	11	17	21	27	28	33	34	35	39	40
38	2	5	12	18	22	23	28	29	34	35	36	40
39	3	6	13	19	21	23	24	29	30	35	36	37
40	1	4	14	20	22	24	25	30	31	36	37	38

g = 25920 on 36 letters

15 orbits

1	3	6	7	8	11	18	20	21	23	26	27	28	30	31	32
2	4	7	8	9	12	19	21	22	24	27	28	29	25	32	33
3	5	8	9	10	1	20	22	23	13	28	29	30	26	33	34
4	6	9	10	11	2	21	23	24	14	29	30	25	27	34	35
5	7	10	11	12	3	22	24	13	15	30	25	26	28	35	36
6	8	11	12	1	4	23	13	14	16	25	26	27	29	36	31
7	9	12	1	2	5	24	14	15	17	26	27	28	30	31	32
8	10	1	2	3	6	13	15	16	18	27	28	29	25	32	33
9	11	2	3	4	7	14	16	17	19	28	29	30	26	33	34
10	12	3	4	5	8	15	17	18	20	29	30	25	27	34	35
11	1	4	5	6	9	16	18	19	21	30	25	26	28	35	36
12	2	5	6	7	10	17	19	20	22	25	26	27	29	36	31

4.13)

13	3	5	6	8	14	15	19	23	24	25	26	32	33	34	36
14	4	6	7	9	15	16	20	24	13	26	27	33	34	35	31
15	5	7	8	10	16	17	21	13	14	27	28	34	35	36	32
16	6	8	9	11	17	18	22	14	15	28	29	35	36	31	33
17	7	9	10	12	18	19	23	15	16	29	30	36	31	32	34
18	8	10	11	1	19	20	24	16	17	30	25	31	32	33	35
19	9	11	12	2	20	21	13	17	18	25	26	32	33	34	36
20	10	12	1	3	21	22	14	18	19	26	27	33	34	35	31
21	11	1	2	4	22	23	15	19	20	27	28	34	35	36	32
22	12	2	3	5	23	24	16	20	21	28	29	35	36	31	33
23	1	3	4	6	24	13	17	21	22	29	30	36	31	32	34
24	2	4	5	7	13	14	18	22	23	30	25	31	32	33	35
25	2	4	5	6	8	10	11	12	13	18	19	24	28	31	34
26	3	5	6	7	9	11	12	1	14	19	20	13	29	32	35
27	4	6	7	8	10	12	1	2	15	20	21	14	30	33	36
28	5	7	8	9	11	1	2	3	16	21	22	15	25	34	31
29	6	8	9	10	12	2	3	4	17	22	23	16	26	35	32
30	7	9	10	11	1	3	4	5	18	23	24	17	27	36	33
31	1	6	7	12	14	16	17	18	20	22	23	24	25	28	34
32	2	7	8	1	15	17	18	19	21	23	24	13	26	29	35
33	3	8	9	2	16	18	19	20	22	24	13	14	27	30	36
34	4	9	10	3	17	19	20	21	23	13	14	15	28	25	31
35	5	10	11	4	18	20	21	22	24	14	15	16	29	26	32
36	6	11	12	5	19	21	22	23	13	15	16	17	30	27	33

The 36, 15, 6 design here may be used to construct a Hadamard matrix H_{36} of size 36. We simply define $H_{36} = [h_{ij}]$ putting $h_{ij} = +1$ if $a_i \in B_j$ in the design and $h_{ij} = -1$ otherwise. An automorphism of H_{36} is given by monomial permutation matrices P and Q with one entry ±1 in each row and column, the rest zero, such that

4.14) $$PH_{36}Q = H_{36} .$$

In this sense, with Q a permutation in G and $P = Q^{-1}$, the permutations of G are automorphisms of H. But normalizing H_{36} by changing signs of rows and columns to make the first row and first column of H_{36} consist entirely of +1's, we find that there is a further automorphism with $P^{-1} = Q = d$ where

4.15) $$d = (1)(2,12,11,21,33,32,26,25)(3,30,29,28,14,16,7,4)(5,6)(8,15,27,31)(9,36,34,20,19,18,17,13)(10,24,23,22)(35).$$

Here $\langle a, b, d\rangle$ generate a larger group of permutations (ignoring the monomial ±1's) which is of order 1,451,520 and is isomorphic to the simple group $S_6(2)$ the six dimensional symplectic group over GF(2).

REFERENCES

[1] R.C. Bose and T. Shimamoto, "Classification and analysis of partially balanced incomplete block designs with two associate classes." J. Amer. Stat. Assn. 47 (1952), 151-184.

[2] Marshall Hall, Jr., "Designs with transitive automorphism groups." Proceedings of the Symposium on Combinatorics UCLA, November, 1968.

[3] D.G. Higman, "Finite permutations groups of rank 3." Math. Zertsch. 86 (1964), 145-156.