

A Generalization of Moore Graphs of Diameter Two

by

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1. INTRODUCTION. A finite, undirected linear graph G is strongly regular [1] if it is regular of valence n_1 , each adjacent pair of vertices is adjacent to exactly p_{11}^1 other vertices, and each non-adjacent pair of vertices is adjacent to exactly p_{11}^2 other vertices. The notation is that of two-class association schemes, to which strongly regular graphs are abstractly equivalent. As defined by Bose and Shimamoto [3], a two-class association scheme consists of a finite set of v elements together with a scheme of relations among them such that

- (i) any two distinct elements are either first associates or second associates;
- (ii) each element has exactly n_i i -th associates ($i = 1, 2$);
- (iii) if two elements are i -th associates, the number p_{jk}^i of elements which are j -th associates of the first and k -th associates of the second is independent of the original pair, and $p_{jk}^i = p_{kj}^i$ ($i, j, k = 1, 2$).

If the elements of a set on which is defined a two-class association scheme are taken as the vertices of a graph G , with two vertices adjacent (non-adjacent) in G iff the corresponding elements are first

(second) associates, it follows from the above definition that G is strongly regular. Conversely, given a (finite) strongly regular graph G with parameters n_1, p_{11}^1, p_{11}^2 , a two-class association scheme is obtained by associating the elements with vertices and defining two elements to be first (second) associates iff the corresponding vertices are adjacent (non-adjacent). This latter equivalence is a result of the fact that the constancy of the three parameters n_1, p_{11}^1, p_{11}^2 is sufficient to guarantee the constancy of the remaining parameters, and to determine them uniquely through well-known relations. Thus only three of the eleven parameters v, n_i, p_{jk}^i ($i, j, k = 1, 2$) are independent.

The generalization of the above definition to an m -class association scheme is straightforward, but for $m \geq 3$ the graph-theoretic representation is usually lost. Certain classes of graphs, however, do correspond to m -class association schemes in which the association class to which two elements belong is determined by the graph-theoretic distance between the corresponding vertices. Such a graph is said to be metrically regular of diameter m . A connected strongly regular graph ($p_{11}^2 > 0$) is therefore metrically regular of diameter two.

The present paper considers two questions associated with strongly regular graphs. In Section 3, we investigate the consequences of dropping the regularity condition in the definition of a strongly regular graph. We show that the irregular graphs satisfying the remaining two conditions fall into two infinite classes and have a very simple structure.

In Sections 4 and 5, we consider the class of strongly regular graphs with $p_{11}^2 = 1$. This investigation is motivated by the proof of Theorem 1 in Section 3, where some interesting geometric properties of

these graphs are revealed. A characterization in terms of properties of the maximal cliques is given (Theorem 2), which shows that strongly regular graphs with $p_{11}^2 = 1$ provide a generalization of Moore graphs of diameter two [6].

The main results of Section 5 (Theorems 3 and 4) establish necessary conditions for existence of strongly regular graphs with a given value of p_{11}^1 and with $p_{11}^2 = 1$. A consequence of Theorem 4 is that the class of all such graphs is finite if $p_{11}^1 \neq 3$. A table of possible parameter values for $p_{11}^1 \leq 8$ is given.

In Section 6, we define the Moore geometries $MG(r, k, d)$, the graphs associated with which may be regarded as generalizations of Moore graphs of diameter d . However, the investigation of the detailed properties of these geometries and graphs is reserved for a future communication.

2. NOTATION AND TERMINOLOGY. We consider finite, undirected linear graphs. A graph will be denoted $G = (V, E)$, where V is the vertex set and E the edge set. The elements of E are two-element subsets of V . We use the symbol (x, y) to denote an arbitrary two-element subset of V . Two vertices x, y are adjacent or non-adjacent according as $(x, y) \in E$ or $(x, y) \notin E$.

For $x, y \in V$, we define

$$A(x) = \{z: (x, z) \in E\},$$

$$A(x, y) = A(x) \cap A(y).$$

The valence of $x \in V$ will be denoted $d(x)$, equal to the cardinality of $A(x)$. $\Delta(x, y)$ denotes the cardinality of $A(x, y)$.

A graph G is regular if all vertices have equal valence and complete if any two vertices are adjacent. A clique is a set of vertices, any two of which are adjacent. A clique is maximal if it is not properly contained in some other clique.

3. IRREGULAR GRAPHS WITH p_{11}^1, p_{11}^2 CONSTANT. In terms of our notation, a strongly regular graph is a graph $G = (V, E)$ such that for fixed non-negative integers n_1, p_{11}^1, p_{11}^2 , the following hold:

$$(A1) \quad d(x) = n_1, \quad x \in V,$$

$$(A2) \quad \Delta(x,y) = p_{11}^1, \quad (x,y) \in E,$$

$$(A3) \quad \Delta(x,y) = p_{11}^2, \quad (x,y) \notin E.$$

We may assume additionally, to exclude trivialities, that strongly regular graphs considered are not complete. Given a strongly regular graph with parameters n_1, p_{11}^1, p_{11}^2 , elementary counting arguments establish the relations

$$(3.1) \quad \begin{aligned} n_2 &= n_1(n_1 - 1 - p_{11}^1)/p_{11}^2, \\ v &= 1 + n_1 + n_2, \end{aligned}$$

where n_2 is the number of vertices non-adjacent to any given vertex, and v is the total number of vertices.

Let $r \geq 1, k \geq 2, s \geq 1$, and define a graph $G_0(r, k, s)$ as follows: $G_0(r,k,s)$ has $rk + s$ vertices and $r + s$ connected components, r of which are complete graphs with k vertices each, and s of which are isolated vertices. Figure 1 illustrates $G_0(2,4,1)$.

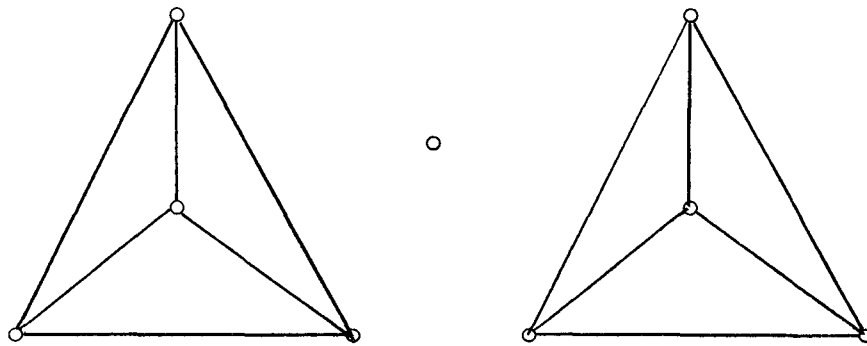


Figure 1. $G_0(2, 4, 1)$.

Evidently $G_0(r, k, s)$ satisfies (A2) and (A3) with $p_{11}^1 = k-2$, $p_{11}^2 = 0$, but is not regular. We define

$$G_0(k) = \{G_0(r, k, s) : r \geq 1, s \geq 1\}.$$

Next suppose $r \geq 2$, $k \geq 2$, and define a graph $G_1(r, k)$ as follows: $G_1(r, k)$ has $r(k-1) + 1$ vertices and consists of r edge-disjoint complete graphs with k vertices each, all of which have a single vertex in common. Figure 2 illustrates $G_1(3, 4)$.

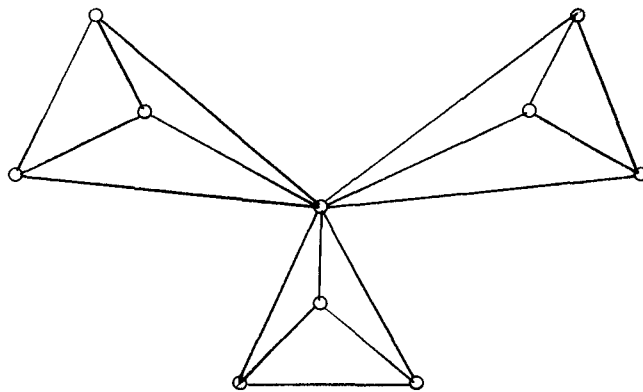


Figure 2. $G_1(3, 4)$.

Note that $G_1(r,k)$ is not regular but satisfies (A2) and (A3) with $p_{11}^1 = k-2$, $p_{11}^2 = 1$. Finally, let

$$G_1(k) = \{G_1(r,k) : r \geq 2\}.$$

We then have

THEOREM 1. Let p_{11}^1, p_{11}^2 be non-negative integers and let $G = (V,E)$ be a graph such that

$$(A2) \quad \Delta(x,y) = p_{11}^1, \quad (x,y) \in E,$$

$$(A3) \quad \Delta(x,y) = p_{11}^2, \quad (x,y) \notin E.$$

Then exactly one of the following holds:

- (a) G is regular; hence strongly regular,
- (b) $p_{11}^2 = 0$ and $G \in G_0(p_{11}^1 + 2)$,
- (c) $p_{11}^2 = 1$ and $G \in G_1(p_{11}^1 + 2)$.

PROOF. Let $x \in V$, and suppose $d(x) = n$. Consider the number T of ordered triples (y_1, y_2, z) of distinct vertices of $V - x$, such that $y_1, y_2 \in A(x)$, $z \in A(y_1, y_2)$. If $z \in A(x)$, the pair (y_1, y_2) can be chosen in $p_{11}^1(p_{11}^1 - 1)$ ways, while if $z \notin A(x)$, (y_1, y_2) can be chosen in $p_{11}^2(p_{11}^2 - 1)$ ways. Letting v be the number of vertices in G , we have

$$(3.2) \quad T = np_{11}^1(p_{11}^1 - 1) + (v - n - 1)p_{11}^2(p_{11}^2 - 1).$$

Alternatively, if y_1 is fixed, the ordered pair (y_2, z) can be chosen in $p_{11}^1(p_{11}^1 - 1)$ ways with $y_2 \in A(y_1)$ and in $(n - p_{11}^1 - 1)(p_{11}^2 - 1)$ ways with $y_2 \notin A(y_1)$. Thus

$$(3.3) \quad T = np_{11}^1(p_{11}^1-1) + n(n-p_{11}^1-1)(p_{11}^2-1).$$

Equating (3.2) and (3.3) and simplifying yields

$$(3.4) \quad (p_{11}^2-1)[n^2 - (p_{11}^1-p_{11}^2+1)n - (v-1)p_{11}^2] = 0.$$

If $p_{11}^2 > 1$, (3.4) has a unique non-negative solution for n , so in this case G is regular.

If $p_{11}^2 = 0$, both solutions of (3.4) are non-negative: $n = 0$, $p_{11}^1 + 1$. Since $p_{11}^2 = 0$, every connected component of G must be complete, so that the only possible components are isolated vertices and complete graphs with p_{11}^1+2 vertices. If G is not regular, both types occur, so $G \in G_0(p_{11}^1+2)$.

Before proceeding further, we shall prove the following lemma:

LEMMA 1. If G is a graph satisfying the conditions (A2) and (A3) of Theorem 1, with $p_{11}^2 = 1$, then for any non-adjacent vertices x_0 and y_0 , $d(x_0) = d(y_0)$.

Let $(x,y) \in E$ and define

$$(3.5) \quad K(x,y) = x \cup y \cup A(x,y).$$

If $K(x,y)$ is not a clique, there exist vertices $z_1, z_2 \in A(x,y)$ such that $(z_1, z_2) \notin E$, which contradicts (A3), since $x,y \in A(z_1, z_2)$. Thus $K(x,y)$ is a clique, and clearly maximal. We have

(P1) Every pair x,y of adjacent vertices in G is contained in a unique maximal clique $K(x,y)$ defined by (3.5).

Since $p_{11}^2 = 1 > 0$, G has no isolated vertices, so every maximal clique has the form (3.5). Thus

(P2) Every maximal clique contains exactly $p_{11}^1 + 2$ vertices.

If x, y, z are the vertices of a 3-cycle in G , then $z \in A(x, y)$, so $z \in K(x, y)$.

(P3) The vertices of any 3-cycle in G are contained in a maximal clique.

Suppose w, x, y, z are the vertices of a 4-cycle in G in cyclic order. Since $w, y \in A(x, z)$, (A3) implies that $(x, z) \in E$, so $w, y \in K(x, z)$ and we have

(P4) The vertices of any 4-cycle in G are contained in a maximal clique.

Let x_0, y_0 be non-adjacent vertices of G . Let $z_0 = A(x_0, y_0)$, $K_0 = K(x_0, z_0)$, $L_0 = L(y_0, z_0)$. Then $x_0 \in K_0$ and $y_0 \in L_0$. The result is trivial unless one of x_0, y_0 is contained in at least two maximal cliques. Assume there is a second clique K_1 containing x_0 , and let $x_1 \in K_1 - x_0$. By (P1), $x_1 \notin K_0$, and by (P4) $(x_1, y_0) \notin E$. Let $y_1 = A(x_1, y_0)$. Then (P3) and (P4) imply $y_1 \notin L_0$, so there is a second clique $L_1 = K(y_1, y_0)$ containing y_0 . Since $y_1 \neq z_0$, and $y_1 \in A(y_0)$, x_0 and y_1 are non-adjacent by (P4). Hence $x_1 = A(x_0, y_1)$. Thus there is a one-one correspondence between the vertices of $A(x_0) - K_0$ and the vertices of $A(y_0) - L_0$ such that x_1, y_1 correspond iff $(x_1, y_1) \in E$. Since K_0 and L_0 have equal cardinality, it follows that $d(x_0) = d(y_0)$. This completes the proof of Lemma 1.

To complete the proof of Theorem 1, we have to show that if a graph G , satisfying the conditions of Lemma 1, is irregular then $G \in G_1(p_{11}^1 + 2)$. Let x_0 and y_0 be any two non-adjacent vertices of G . Then $d(x_0) = d(y_0) = n_1$, say. Let $z_0 = A(x_0, y_0)$. If $w \in V - z_0$, then by (P4), w is non-adjacent to at least one of x_0 and y_0 .

Hence $d(w) = n_1$. If G is irregular, $d(z_0) \neq n_1$. Hence z_0 is adjacent to every vertex of $V - z_0$, otherwise Lemma 1 would be violated. It follows from (P1) and (P2), that there exists a set of maximal cliques of cardinality $p_{11}^1 + 2$ with z_0 as a common vertex such that any vertex of $V - z_0$ belongs to one and only one of these cliques. Also two vertices of $V - z_0$ belonging to different cliques of the set must be non-adjacent by (P3). Hence $G \in G(p_{11}^1 + 2)$.

COROLLARY. If G satisfies (A2) and (A3) with $p_{11}^2 \geq 2$, then G is strongly regular.

In analogy with Theorem 1, one might ask whether the class of all graphs of diameter two satisfying (A1) and (A2) but not (A3) can be similarly characterized, or the class satisfying (A1) and (A3), but not (A2). Actually these two questions are equivalent, since G has diameter two and satisfies (A1) and (A2) iff the complementary graph \bar{G} satisfies (A1) and (A3), but not (A2), with $\bar{n}_1 = v - 1 - p_{11}^1$, $p_{11}^2 = v - 2n_1 + p_{11}^1$, where v is the number of vertices in G . Several examples of such graphs are known, but no complete characterization is presently available.

4. A CHARACTERIZATION OF STRONGLY REGULAR GRAPHS WITH $p_{11}^2 = 1$.

Properties (P1)-(P4) obtained in the proof of Theorem 1 for the case $p_{11}^2 = 1$ lead to a characterization of strongly regular graphs with $p_{11}^2 = 1$ in terms of properties of the maximal cliques. We present the result in this section (Theorem 2) along with a description of the connection of these graphs with Moore graphs of diameter two.

We consider, therefore, the class of all graphs satisfying (A1)-(A3) with $p_{11}^2 = 1$.

THEOREM 2. A graph $G = (V, E)$ is a strongly regular graph with $p_{11}^2 = 1$ iff there exist integers r, k such that $r \geq k \geq 2$ and the following hold:

- (Q1) Every maximal clique contains k vertices;
- (Q2) Each vertex is contained in r maximal cliques;
- (Q3) Each pair of adjacent vertices is contained in a unique maximal clique;
- (Q4) Given two non-adjacent vertices x, y , there is a unique maximal clique containing x which intersects a maximal clique containing y .

The parameters of G are then given by

$$\begin{aligned} v &= 1 + r(k-1) + r(r-1)(k-1)^2, \\ n_1 &= r(k-1), \\ n_2 &= r(r-1)(k-1)^2, \\ p_{11}^1 &= k - 2, \\ p_{11}^2 &= 1. \end{aligned}$$

PROOF. Suppose G satisfies (Q1)-(Q4) with $r \geq k \geq 2$. Clearly (Q1)-(Q3) imply $d(x) = r(k-1)$ for all $x \in V$.

For $(x, y) \in E$, we denote by $K(x, y)$ the maximal clique containing x, y guaranteed by (Q3). Then if $(x, y) \in E$, the $k-2$ other vertices of the maximal clique $K = K(x, y)$ are clearly in $A(x, y)$. Suppose there is a vertex $z \in A(x, y) - K$. Let $K_1 = K(x, z)$, $L_1 = K(y, z)$. Since $z \notin K$, the maximal cliques K, L_1, K_1 are distinct, and the maximality of K implies the existence of a vertex

$w \in K$ non-adjacent to z . Then K_1, L_1 are two maximal cliques containing z which intersect the maximal clique K containing w , which contradicts (Q4). Hence $\Delta(x,y) = k-2$ for all $(x,y) \in E$.

Suppose $(x,y) \notin E$. Let K be the unique maximal clique containing x which intersects a maximal clique containing y . If K intersects two maximal cliques containing y , then by reversing the roles of x and y in (Q4) we obtain a contradiction. Thus there is a unique maximal clique L containing y which intersects K . By (Q3), $K \cap L$ has cardinality one, so $\Delta(x,y) = 1$ if $(x,y) \notin E$. It follows that G is strongly regular with $n_1 = r(k-1)$, $p_{11}^1 = k-2$, $p_{11}^2 = 1$. The expressions for v and n_2 follow from (3.1).

Conversely, if G is strongly regular with $p_{11}^2 = 1$, properties (P1)-(P4) of Section 3 hold. In addition, since G is regular, every vertex is contained in the same number r of maximal cliques. Putting $k = p_{11}^1 + 2$, (Q1)-(Q3) follow. Then (Q4) is an immediate consequence of (Q3) and the condition $p_{11}^2 = 1$.

It remains to establish the inequality $r \geq k$. Since G is not complete, $r \geq 2$ and there exists $(x_0, y_0) \notin E$. Let $z_0 = A(x_0, y_0)$, $K_0 = K(x_0, z_0)$, $L_0 = K(y_0, z_0)$ as in the proof of Lemma 1. Let K_1 be a second maximal clique containing x_0 and let L_1, L_2, \dots, L_{r-1} be the remaining $r-1$ maximal cliques containing y_0 . For each vertex $x_1 \in K_1 - x_0$, there is a unique vertex $y_1 \in A(y_0) - L_0$ such that $(x_1, y_1) \in E$. If for two vertices x_1, x'_1 in $K_1 - x_0$ the corresponding vertices y_1, y'_1 are in the same maximal clique L_j ($1 \leq j \leq r-1$), then we contradict (P3) if $y_1 = y'_1$ and (P4) if $y_1 \neq y'_1$. Hence the $k-1$ vertices y_1 are distinct, and no two appear in the same maximal clique L_j ($1 \leq j \leq r-1$). Thus $k-1 \leq r-1$, and the proof is complete.

In view of Theorem 2, we may denote an arbitrary strongly regular graph with $p_{11}^2 = 1$ by $G(r,k)$, where $r \geq k \geq 2$ and r, k have the meanings attached by (Q2),(Q1) respectively. It is, of course, possible that there may exist several non-isomorphic graphs satisfying (Q1)-(Q4) for given values of r, k , so that the notation $G(r,k)$ does not uniquely define the graph.

Strongly regular graphs with $p_{11}^2 = 1$ are a generalization of Moore graphs of diameter two [6]. A Moore graph $M(r,d)$ is a regular graph of valence r and diameter d for which the number v of vertices attains the upper bound

$$1 + r + r(r-1) + \dots + r(r-1)^{d-1}$$

imposed by these requirements. In particular, for $M(r,2)$,

$$v = 1 + r + r(r-1) .$$

Thus $G(r,2)$ is a Moore graph of diameter two, and the converse can also be easily established. In [6], it is shown that $M(r,2)$, hence $G(r,2)$, exists only for $r = 2$ (pentagon), 3 (Peterson graph), 7, and possibly $r = 57$. That these are the only possibilities for r will follow as a special case of Theorem 4 below.

Moore graphs of diameter two have girth five, hence no 3-cycles or 4-cycles. A similar structure is exhibited by $G(r,k)$ if we define girth in terms of minimal cycles, no three vertices of which lie in the same maximal clique.

5. NECESSARY CONDITIONS FOR EXISTENCE OF STRONGLY REGULAR GRAPHS

WITH $p_{11}^2 = 1$. While the existence of three Moore graphs of diameter two

$(G(r,2)$ for $r = 2, 3, 7)$ is known, no strongly regular graphs $G(r,k)$ with $k > 2$ are presently known. In this section, we derive necessary conditions for the existence of $G(r,k)$. Our first result is an easy consequence of Theorem 2.

THEOREM 3. A necessary condition for the existence of a graph $G(r,k)$ is that k divide $r(r-1)^2$.

PROOF. Let b be the number of maximal cliques in $G(r,k)$. Then by counting vertex-maximal clique incidences, we have $bk = vr$, so k divides vr . Since

$$\begin{aligned} v &= 1 + r(k-1) + r(r-1)(k-1)^2 \\ &= [r(r-1)k - r(2r-3)]k + (r-1)^2, \end{aligned}$$

k divides vr iff k divides $r(r-1)^2$.

In deriving stronger necessary conditions for the existence of $G(r,k)$, we shall make use of a theorem on two-class association schemes due to Bose and Mesner [2]. We state the theorem here in its equivalent form for strongly regular graphs.

THEOREM (Bose-Mesner). Let G be a strongly regular graph with parameters $v, n_1, n_2, p_{11}^1, p_{11}^2$, and let $A = A(G)$ be the adjacency matrix of G . Then n_1 is a simple characteristic root of A , and there are two other distinct characteristic roots θ_1, θ_2 with multiplicities

$$\alpha_1, \alpha_2 = \frac{1}{2}[(n_1 + n_2) \pm N/\sqrt{\Delta}],$$

where

$$\begin{aligned}
N &= (n_1 - n_2) + \gamma(n_1 + n_2), \\
\gamma &= p_{11}^1 - p_{11}^2 - 1, \\
\beta &= 2n_1 - p_{11}^1 - p_{11}^2 - 1, \\
\Delta &= \gamma^2 + 2\beta + 1.
\end{aligned}$$

The requirement that α_1, α_2 be integral imposes necessary conditions on the parameters of G . Applying the theorem to $G(r, k)$, we have, from Theorem 2,

$$\begin{aligned}
(5.1) \quad N &= r(k-1)^2 [1 + (r-1)(k-3)] \\
\gamma &= k - 2, \\
\beta &= 2rk - 2r - k, \\
\Delta &= (k-1)^2 + 4(r-1)(k-1).
\end{aligned}$$

If Δ is not a square, then clearly N must vanish, which implies $r = k = 2$. The graph $G(2, 2)$ is, of course, the pentagon. We assume below, therefore, that $(r, k) \neq (2, 2)$.

Then Δ must be a square, so we set

$$(5.2) \quad (k-1)^2 + 4(r-1)(k-1) = (k-1+a)^2.$$

Noting that $k-1$ and $k-1+a$ must be of the same parity, we put $a = 2b$ in (5.2) and simplify, obtaining

$$(5.3) \quad r - 1 = b + b^2/(k-1).$$

Thus $k-1$ divides b^2 , say $b^2 = m(k-1)$. Define integers c, f by

$$(5.4) \quad k - 1 = c^2 f,$$

where f is square-free. Then $b^2 = c^2 f m$, and since f is square

free, m has the form $m = t^2 f$ for some integer t . Thus $b = cft$, so by (5.3),

$$(5.5) \quad r - 1 = (t+c)tf.$$

Substituting (5.4) and (5.5) into the expression for Δ given by (5.1), we obtain

$$(5.6) \quad \sqrt{\Delta} = cf(2t+c).$$

Consider now the expression for N given in (5.1). For k fixed, we may regard N as a second degree polynomial in r . Substituting (5.4) and (5.5) into N gives a fourth degree polynomial in t with integral coefficients determined by k . Factoring out the constant factor $(k-1)^2$, we have

$$N = c^4 f^2 P_k(t),$$

where $P_k(t)$ is a fourth degree polynomial in t with integral coefficients determined by k . Then by (5.6),

$$(5.7) \quad N/\sqrt{\Delta} = c^3 f P_k(t)/(2t + c).$$

By the Bose-Mesner Theorem, $N/\sqrt{\Delta}$ must be integral. We consider two cases.

CASE 1. $k \not\equiv 1 \pmod{4}$: In this case c is odd, so $2t + c$ is odd. We may therefore multiply $c^3 f P_k(t)$ by any power of two without affecting its divisibility (or lack of divisibility) by $2t + c$. Choosing the factor 2^4 (or any higher power) permits us to carry out

the polynomial division in (5.7) to obtain a quotient polynomial with integral coefficients, plus a remainder term. We omit the calculations here. The result is

$$2^4 c^3 f P_k(t)/(2t+c) = Q_k(t) + R_k/(2t+c),$$

where $Q_k(t)$ is a cubic polynomial in t with integral coefficients, and

$$R_k = c(k-1)(k-5)(k^2 - 4k - 1).$$

Hence, if $G(r,k)$ exists, $2t + c$ must divide R_k .

CASE 2. $k \equiv 1 \pmod{4}$: In this case, $c = 2d$ is even, and (5.7) becomes

$$N/\sqrt{\Delta} = 4d^3 f P_k(t)/(t+d).$$

The polynomial division yields

$$4d^3 f P_k(t)/(t+d) = Q_k(t) - R'_k/(t+d),$$

where $Q_k(t)$ is a cubic polynomial in t with integral coefficients, and

$$R'_k = d(k-1)(k-5)(k^2+3)/16.$$

Thus the existence of $G(r,k)$ implies that $t + d$ divides R'_k , or equivalently, that $2t + c$ divides $R_k = 2R'_k$. Our results are summarized in

THEOREM 4. Let $k \geq 2$ and let c, f be positive integers defined by

$$k - 1 = c^2 f,$$

where f is square-free. Then a necessary condition for the existence of a graph $G(r,k)$, other than $G(2,2)$, is that

$$r - 1 = (t+c)tf,$$

where t is an integer such that $2t + c$ divides

$$R_k = \begin{cases} c(k-1)(k-5)(k^2-4k-1), & k \not\equiv 1 \pmod{4}, \\ c(k-1)(k-5)(k^2+3)/16, & k \equiv 1 \pmod{4}. \end{cases}$$

A consequence of Theorem 4 is the

COROLLARY. Let $k \geq 2$ and let $G(k)$ be the class of all strongly regular graphs with $p_{11}^1 = k-2$, $p_{11}^2 = 1$. Then if $k \neq 5$, $G(k)$ is finite.

PROOF. In order that $G(k)$ be infinite, it is necessary and sufficient that there exist a graph $G(r,k)$ for an infinite number of values of r , since for any fixed r there are a finite number of non-isomorphic graphs $G(r,k)$. By Theorem 4, this can occur iff R_k vanishes, which is true iff $k = 5$.

We list below all the values of r which satisfy $r \geq k$ and the necessary conditions of Theorems 3 and 4, for each value of k in the range $2 \leq k \leq 10$. The prime power factorization of R_k and the values of c, f are also given in the table. Known graphs are marked with an asterisk. Note that for $k = 3$ there are no graphs, since R_k is a power of two and c , hence $2t + c$, is odd. This result was obtained by Erdős, Rényi, and Sós [5], and a different proof appears in Bose and Shrikhande [4]. Note, also, that for the cases $k = 4$, $k = 7$ there is

at most one possible value of r . We may pose as unsolved problems the determination of further necessary conditions for the existence of $G(r,k)$ and/or the construction of any of these graphs, apart from the three presently known, which are unique.

Values of r, k satisfying Theorems 3 and 4, $2 \leq k \leq 10$.

k	c	f	R_k	Possible values of r
2	1	1	$3 \cdot 5$	2^* , 3^* , 7^* , 57
3	1	2	2^4	none
4	1	3	3	7
5	2	1	0	s^2 , where $s \equiv -1, 0, 1 \pmod{5}$
6	1	5	$5 \cdot 11$	31, 151, 3781
7	1	6	$2^4 \cdot 3 \cdot 5$	337
8	1	7	$3 \cdot 7 \cdot 31$	85, 1681, 82405
9	2	2	$2^4 \cdot 3 \cdot 7$	31, 97, 127, 391, 1567, 56447
10	3	1	$3^2 \cdot 5 \cdot 59$	55, 505, 7831, 21755, 195805, 1762255.

6. MOORE GEOMETRIES. We may give a geometric interpretation to Theorem 2, characterizing strongly regular graphs with $p_{11}^2 = 1$.

Let there be two types of elements 'points' and 'lines', and a relation of 'incidence', such that a point may or may not be incident with a line. Then we have a Moore geometry $MG(r, k, 2)$ if there exist integers r, k such that $r \geq k \geq 2$ and the following axioms hold.

- (1) Each line is incident with k points, $k \geq 2$.
- (2) Each point is incident with r lines, $r \geq k$.

- (3) Any two distinct points are incident with at most one line.
- (4) Given two non-collinear points x and y , there is a unique line incident with x , which intersects a line incident with y (two lines may be said to intersect if they are incident with a common point).

Then we get a graph $G(r,k)$ from a Moore geometry $MG(r, k, 2)$ by identifying points with vertices, and edges with pairs of points incident with the same line. The set of points incident with a fixed line clearly constitute a 'maximal clique'. The axioms (1),(2),(3),(4) correspond to the conditions (Q1),(Q2),(Q3),(Q4) of Theorem 2.

Properties (P3) and (P4) of Section 3, then show that there exist no triangles and quadrilaterals in $MG(r, k, 2)$.

The geometry dual to $MG(r,k,2)$ defines a graph $G^*(k,r)$ whose vertices are the points of the dual geometry with collinearity defining the adjacency relation. The 'dual graph' $G^*(k,r)$ is not strongly regular except when $r=k=2$. It is, however, metrically regular of diameter three. The parameters of the corresponding three-class association scheme can be determined directly in terms of r and k . While this fact might conceivably yield new necessary conditions for the existence of $G(r,k)$, in terms of parametric identities and eigenvalue multiplicities valid for three-class schemes, our calculations (not explicitly reproduced here) show that no new necessary conditions are obtained beyond those derived in Section 5 from analogous arguments for the original two-class scheme.

In general, one may define a 'weak geometry' of type r, k by axioms (1), (2), (3). A weak geometry may be said to be of diameter d if a sequence of at most d lines is required to join any two points subject to consecutive lines of the sequence intersecting. The number

of points in a weak geometry of diameter d is then bounded above by

$$(6.1) \quad 1 + r(k-1) + r(r-1)(k-1)^2 + \dots + r(r-1)^{d-1}(k-1)^d .$$

We can define a Moore geometry $MG(r, k, d)$ of diameter d as a weak geometry of diameter d for which the number of points attains the upper bound (6.1). The graph of $MG(r, k, d)$ is a graph whose vertices are points of $MG(r, k, d)$ and whose edges are pairs of collinear points of $MG(r, k, d)$. Then the Moore graphs of diameter d are the graphs of Moore geometries $M(r, 2, d)$, and the strongly regular graphs $G(r, k)$ are the graphs of $MG(r, k, 2)$. Moore geometries of diameter d and the associated graphs will be studied in a subsequent communication.

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