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# MATCHING THEOREMS FOR COMBINATORIAL GEOMETRIES

by

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## 1. INTRODUCTION

Let $G(S)$ and $G(T)$ be combinatorial geometries [2] on sets $S$ and $T$, respectively, and let $R \subseteq S \times T$ be a binary relation between the points of $G(S)$ and $G(T)$. A matching from $G(S)$ into $G(T)$ is a triple $(A, B, f)$, where $A$ and $B$ are independent sets in $G(S)$ and $G(T)$, respectively, and $f$ is a one-one function from $A$ onto $B$ such that $(a, f(a)) \in R$ for all $a \in A$.

The present paper presents a characterization of matchings of maximum cardinality, a max-min theorem, and a number of related results. In the case where both $G(S)$ and $G(T)$ are free geometries, Theorem 4 reduces to a characterization of maximum matchings associated with the "Hungarian method" as introduced by Egerváry and Kuhn (see [1]), Theorem 5 to the König-Egerváry Theorem, Theorem 6 to a theorem of Ore [5], and Corollary 6.1 to the classical Marriage Theorem. The latter corollary for the case when $G(S)$ is free and $G(T)$ arbitrary was first obtained by Rado [6] (see also Crapo-Rota [2]). Theorem 7 is known (see [2], [3], [4]), but the proof, based on earlier results in the paper, is new.

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## 2. DEFINITIONS, NOTATION, AND TERMINOLOGY

For completeness, several definitions and results on combinatorial geometries from Crapo-Rota [2] are included in this section.

A closure relation on a set $S$ is a function $A \to \overline{A}$ defined for all subsets $A \subseteq S$, satisfying

(2.1) $\quad A \subseteq \overline{A}$,

(2.2) $\quad A \subseteq \overline{B}$ implies $\overline{A} \subseteq \overline{B}$,

for all subsets $A, B$ of $S$. A set endowed with a closure relation is a closure space. A subset $A \subseteq S$ is closed if and only if $A = \overline{A}$.

A closure relation on a set $S$ has finite basis if and only if

(2.3) Any subset $A \subseteq S$ has a finite subset $A_0 \subseteq A$ such that $\overline{A_0} = \overline{A}$.

A closure relation satisfies the exchange property if and only if

(2.4) For any elements $a, b \in S$, and for any subset $A \subseteq S$,

$$a \in \overline{A \cup b},\quad a \notin \overline{A} \text{ implies } b \in \overline{A \cup a}.$$

A pregeometry (matroid) $G(S)$ is any closure space consisting of a set $S$ and a closure relation with finite basis and the exchange property. A pregeometry $G(S)$ is a combinatorial geometry if and only if

(2.5) $\quad \overline{\phi} = \phi,\ \overline{a} = a$ for all $a \in S$.

Associated with every pregeometry $G(S)$ on a set $S$ is a unique geometry $G(S_0)$ on the set $S_0$ of equivalence classes of $S$ under the equivalence relation

$$a \sim b \quad \text{if and only if} \quad \overline{a} = \overline{b} .$$

We shall confine our attention to geometries with no loss of generality; the results may easily be extended to pregeometries.

The cardinality of a set $S$ will be denoted $\nu(S)$. In a geometry $G(S)$, all minimal subsets $A_0$ of any subset $A \subseteq S$ satisfying $\overline{A_0} = \overline{A}$ have the same cardinality, which is defined as the <u>rank</u> $r(A)$ of the set $A$. A set $A$ is independent if and only if $r(A) = \nu(A)$, i.e., if and only if no proper subset of $A$ has closure $\overline{A}$. The <u>rank function</u> $r$ of $G(S)$ satisfies the <u>semi-modular inequality</u>

$$(2.6) \qquad r(A \cup B) + r(A \cap B) \leq r(A) + r(B).$$

A combinatorial geometry $G(S)$ is <u>free</u> if $\overline{A} = A$ for all subsets $A \subseteq S$. In this case, every subset of $S$ is both independent and closed. The concept of independence is thus non-trivial only with regard to "multisets" on "lists", in which an element may occur more than once. A multiset in a free geometry is independent if and only if all its elements are distinct, i.e., if and only if it is a set.

The following definitions and notation are introduced in the present paper. For a further theory of combinatorial geometries, the reader is referred to [2].

Let $G(S)$, $G(T)$ be combinatorial geometries on sets $S$, $T$, respectively, and let $R \subseteq S \times T$ be a binary relation between the points of $S$ and $T$. The system consisting of $G(S)$, $G(T)$, and $R$ will be denoted $(G(S), G(T), R)$. We shall denote the rank functions of both $G(S)$, $G(T)$

by $r$ and the closure relations by $A \to \overline{A}$, $B \to \overline{B}$, where $A \subseteq S$, $B \subseteq T$. For $A \subseteq S$, $R(A)$ denotes the set of points $b \in T$ such that $(a,b) \in R$ for some $a \in A$.

A <u>matching</u> in $(G(S), G(T), R)$ is a triple $(A, B, f)$, where $f$ is a one-one function from $A$ onto $B$ such that $(a, f(a)) \in R$ for all $a \in A$, and $A$, $B$ are independent sets in $G(S)$, $G(T)$, respectively. A matching $(A, B, f)$ is characterized by its <u>edge set</u>

$$M = \{(a, f(a)): a \in A\},$$

and we formally identify these two concepts by writing $M = (A, B, f)$. The common cardinality of $A$, $B$, $M$ is called the <u>size</u> $\nu(M)$ of the matching $M$. We shall be interested in matchings of maximum size in $(G(S), G(T), R)$.

A <u>support</u> of $(G(S), G(T), R)$ is a pair $(C, D)$ of closed sets in $G(S)$, $G(T)$, respectively, such that $(c, d) \in R$ implies at least one of $c \in C$, $d \in D$ holds. The <u>order</u> of a support $(C, D)$ is the number $\lambda(C, D) = r(C) + r(D)$.

Note that if both $G(S)$ and $G(T)$ are free geometries, then the system $(G(S), G(T), R)$ is, apart from the orientation of the edges from $S$ to $T$, a bipartite graph, and the above definitions of a matching and a support reduce to the usual ones for this case. The following definition uses the exchange property to generalize a notion associated with the "Hungarian method" for finding a maximum matching in a bipartite graph.

Let $M = (A, B, f)$ be a matching in $(G(S), G(T), R)$. A sequence

(2.7) $$(a_0', b_1'),\ (b_1, a_1),\ (a_1', b_2'),\ \ldots,\ (b_n, a_n),\ (a_n', b_{n+1}')$$

of $2n + 1$ distinct pairs $(n \geq 0)$ is an augmenting chain with respect to $M$ if and only if

(2.8) $\quad (a_i, b_i) \in M, \quad (a'_i, b'_{i+1}) \in R-M,$

(2.9) $\quad a'_0 \in S-\overline{A}, \quad b'_{n+1} \in T-\overline{B},$

(2.10) $\quad a'_i \in \overline{A}, \quad a'_i \notin \overline{(A - \bigcup_{j=1}^{i} a_j) \cup \bigcup_{j=1}^{i-1} a'_j},$

$$b'_i \in \overline{B}, \quad b'_i \notin \overline{(B - \bigcup_{j=1}^{i} b_j) \cup \bigcup_{j=1}^{i-1} b'_j},$$

for $1 \leq i \leq n$.

Note that if both $G(S)$, $G(T)$ are free geometries, (2.10) implies $a'_i = a_i$, $b'_i = b_i$ for $1 \leq i \leq n$, so that the sequence represents an ordinary augmenting chain in the bipartite graph.

## 3. MATCHING THEOREMS

THEOREM 1. If there exists an augmenting chain with respect to a matching $M = (A, B, f)$ in $(G(S), G(T), R)$, then $M$ is not of maximum size.

PROOF. Let the augmenting chain be given by (2.7) and define

$$P = \{(a_i, b_i): \ 1 \leq i \leq n\},$$
$$P' = \{(a'_i, b'_{i+1}): \ 0 \leq i \leq n\} .$$

A straightforward inductive argument using (2.10) and the exchange property shows that

$$(A - \bigcup_{j=1}^{i} a_j) \cup \bigcup_{j=1}^{i} a'_j$$

and

$$(B - \bigcup_{j=1}^{i} b_j) \cup \bigcup_{j=1}^{i} b'_j$$

are independent sets with closures $\bar{A}$, $\bar{B}$, respectively, for all $i$, $1 \leq i \leq n$. Thus by (2.9),

$$(3.1) \qquad (A - \bigcup_{j=1}^{n} a_j) \cup \bigcup_{j=0}^{n} a'_j$$

and

$$(3.2) \qquad (B - \bigcup_{j=1}^{n} b_j) \cup \bigcup_{j=1}^{n+1} b'_j$$

are independent sets of cardinality $\nu(M) + 1$. The edges of

$$(3.3) \qquad M' = (M - P) \cup P'$$

define a one-one function $f'$ of (3.1) onto (3.2), so $M'$ is a matching in $(G(S), G(T), R)$, and $\nu(M') = \nu(M) + 1$. Thus $M$ is not of maximum size.

<u>THEOREM 2</u>. If $M = (A,B,f)$ is a matching and $(C,D)$ is a support in $(G(S), G(T), R)$, then

$$\nu(M) \leq \lambda(C,D).$$

<u>PROOF</u>. By definition, $R(S-C) \subseteq D$. Therefore

$$\begin{aligned} \nu(M) &= \nu(A) \\ &= \nu(A\cap C) + \nu(A\cap(S-C)) \\ &= \nu(A\cap C) + \nu(f(A\cap(S-C))) \end{aligned}$$

$$
\begin{aligned}
&= r(A\cap C) + r(f(A\cap(S-C))) \\
&\leq r(A\cap C) + r(R(A\cap(S-C))) \\
&\leq r(C) + r(R(S-C)) \\
&\leq r(C) + r(D) \\
&= \lambda(C, D).
\end{aligned}
$$

<u>THEOREM 3</u>. If $M = A,B,f)$ is a matching in $(G(S), G(T), R)$ and there does not exist an augmenting chain with respect to $M$, then there exists a support $(\overline{A-A_m}, \overline{f(A_m)})$, where $A_m \subseteq A$.

The proof of Theorem 3 is constructive. We require several lemmas before proceeding with the main proof.

<u>LEMMA 1</u>. If $B_1, B_2$ are independent sets in $G(T)$, and $B_1 \cup B_2$ is independent, then

$$\overline{B}_1 \cap \overline{B}_2 = \overline{B_1 \cap B_2}.$$

<u>PROOF</u>. Clearly $B_1 \cap B_2 \subseteq \overline{B}_1 \cap \overline{B}_2$, and since the latter is a closed set, $\overline{B_1 \cap B_2} \subseteq \overline{B}_1 \cap \overline{B}_2$. By the semimodular inequality (2.6) for the rank function $r$ of $G(T)$,

$$
\begin{aligned}
r(\overline{B}_1 \cap \overline{B}_2) &\leq r(\overline{B}_1) + r(\overline{B}_2) - r(\overline{B}_1 \cup \overline{B}_2) \\
&= r(\overline{B}_1) + r(\overline{B}_2) - r(\overline{B_1 \cup B_2}) \\
&= \nu(B_1) + \nu(B_2) - \nu(B_1 \cup B_2) \\
&= \nu(B_1 \cap B_2) \\
&= r(\overline{B_1 \cap B_2}),
\end{aligned}
$$

and the lemma follows.

LEMMA 2. Let $B$ be an independent set in $G(T)$ and suppose $D \subseteq \overline{B}$. Then the set

$$B_1 = \{b \in B: D \not\subseteq \overline{B-b}\}$$

is the unique minimal subset of $B$ whose closure contains $D$.

PROOF. Let $B_2$ be any subset of $B$ such that $D \subseteq \overline{B_2}$. If $B_1 \not\subseteq B_2$, then there exists $b \in B_1$ such that $B_2 \subseteq B-b$. But then $D \subseteq \overline{B-b}$, a contradiction.

Using Lemma 1 and the definition of $B_1$, we have

$$\begin{aligned} D &\subseteq \bigcap_{b' \in B-B_1} \overline{B-b'} \\ &= \overline{\bigcap_{b' \in B-B_1} (B-b')} \\ &= \overline{B_1}. \end{aligned}$$

LEMMA 3. Suppose $B$ is an independent set in $G(T)$ and

$$B_0 \subset B_1 \subset B_2 \subset \dots \subset B_n$$

is an increasing sequence of subsets of $B$. Let $b_i$, $b_i'$, $1 \le i \le n$, be points satisfying

(i) $b_i \in B_i - B_{i-1}$,

(ii) $b_i' \in \overline{B_i} - \overline{B-b_i}$.

Then

$$b_i' \notin \overline{(B - \bigcup_{j=1}^{i} b_i) \cup \bigcup_{j=1}^{i-1} b_i'}$$

for $1 \le i \le n$.

PROOF. We first show that

$$B_i' = (B_i - \bigcup_{j=1}^{i} b_j) \cup \bigcup_{j=1}^{i} b_j'$$

is an independent set with closure $\overline{B_i}$. Now by (ii) $b_1' \not\in \overline{B_1 - b_1}$, so $b_1' \in \overline{B_1}$ implies $\overline{B_1'} = \overline{B_1}$ by the exchange property. Assuming the result true for $i-1$, let

$$C_i = (B_i - \bigcup_{j=1}^{i-1} b_j) \cup \bigcup_{j=1}^{i-1} b_j'$$

$$= (B_i - B_{i-1}) \cup B_{i-1}'.$$

Then

$$\overline{C_i} = \overline{(B_i - B_{i-1}) \cup B_{i-1}'}$$

$$= \overline{(B_i - B_{i-1}) \cup \overline{B}_{i-1}'}$$

$$= \overline{(B_i - B_{i-1}) \cup \overline{B}_{i-1}}$$

$$= \overline{(B_i - B_{i-1}) \cup B_{i-1}}$$

$$= \overline{B}_i,$$

and by a similar argument

$$\overline{C_i - b_i} = \overline{(B_i - B_{i-1} - b_i) \cup B_{i-1}'}$$

$$= \overline{B_i - b_i}.$$

It follows now from (ii) and the exchange property that

$$\overline{B_i} = \overline{C_i} = \overline{(C_i - b_i) \cup b_i'} = \overline{B_i'},$$

so by induction $B_i'$ has closure $\overline{B_i}$ for $1 \le i \le n$. Therefore

$$\overline{(B - \bigcup_{j=1}^{i} b_i) \cup \bigcup_{j=1}^{i-1} b_i'} = \overline{(B - B_{i-1} - b_i) \cup B_{i-1}'} = \overline{B - b_i},$$

and the lemma follows by (ii).

<u>LEMMA 4</u>. Suppose $A$ is an independent set in $G(S)$ and

$$A_0 \subset A_1 \subset A_2 \subset \dots \subset A_n$$

is an increasing sequence of subsets of $A$. Let $a_i, a_i'$, $1 \le i \le n$, be points satisfying

(i) $a_i \in A_i - A_{i-1}$,

(ii) $a_i' \in \overline{A-A_{i-1}} - \overline{A-a_i}$.

Then

$$a_i' \not\in \overline{(A - \bigcup_{j=1}^{i} a_j) \cup \bigcup_{j=1}^{i-1} a_j'}$$

for $1 \le i \le n$.

PROOF. By (ii), $a_1' \not\in \overline{A-a_1}$, so assume inductively that the lemma holds for i-1. Then

$$A^*_{i-1} = (A - \bigcup_{j=1}^{i-1} a_j) \cup \bigcup_{j=1}^{i-1} a_j'$$

is an independent set, and thus so also is $A^*_{i-1} - a_i$. Since

$$(A^*_{i-1} - a_i) \cup (A - A_{i-1}) = A^*_{i-1},$$

it follows from Lemma 1 that

$$\overline{A^*_{i-1}-a_i} \cap \overline{A-A_{i-1}} = \overline{A-A_{i-1}-a_i} \,.$$

Hence if $a_i' \in \overline{A^*_{i-1}-a_i}$, then by (ii),

$$a_i' \in \overline{A-A_{i-1}-a_i} \subseteq \overline{A-a_i},$$

a contradiction.

PROOF OF THEOREM 3. Let $C_0 = S - \overline{A}$. Then $R(C_0) \subseteq \overline{B}$ since there is no augmenting chain with respect to $M = (A,B,f)$. Let $B_1$ be the minimal subset of B, defined according to Lemma 2, such that $R(C_0) \subseteq \overline{B_1}$. Then let $A_1 = f^{-1}(B_1)$, $C_1 = S - \overline{A-A_1}$. In general, having constructing $C_{i-1}$, we define $B_i$ as the minimal subset of B such that $R(C_{i-1}) \cap \overline{B} \subseteq \overline{B_i}$, and set $A_i = f^{-1}(A_i)$, $C_i = S - \overline{A-A_i}$. Since $A_{i-1} \subseteq C_{i-1}$, $f(A_{i-1}) \subseteq R(C_{i-1}) \cap \overline{B}$, but $f(A_{i-1}) \not\subseteq \overline{B-b}$ for any $b \in B_{i-1}$. Thus by Lemma 2, $B_{i-1} \subseteq B_i$ and so $A_{i-1} \subseteq A_i$, $C_{i-1} \subseteq C_i$. It is, moreover, clear that each of the sequences $A_i$, $B_i$, $C_i$ is strictly increasing up to and including some index m after which the process terminates. Thus $R(C_m) \cap \overline{B} \subseteq \overline{B_m}$, but $R(C_i) \cap \overline{B} \not\subseteq \overline{B_i}$ for $0 \le i \le m$, where $B_0 = \phi$.

We shall show that $R(C_i) \subseteq \overline{B}$ for all $i$, $0 \leq i \leq m$. Assuming otherwise, let $n$ be the smallest index, $0 \leq n \leq m$, such that $R(C_n) \nsubseteq \overline{B}$. We obtain a contradiction by showing that this assumption implies the existence of an augmenting chain with respect to $M$.

Since $R(C_n) \nsubseteq \overline{B}$, but $R(C_i) \subseteq \overline{B}$ for $0 \leq i < n$, there exists an edge $(a'_n, b'_{n+1})$ such that

$$b'_{n+1} \in T-\overline{B} \ ,$$

$$a'_n \in C_n - C_{n-1} \ = \ \overline{A-A_{n-1}} - \overline{A-A_n} \ .$$

If $a'_n \in \overline{A-a_n}$ for all $a_n \in A_n$, then $a'_n \in \overline{A-A_n}$ by Lemma 1, so there exists $a_n \in A_n$ such that $a'_n \notin \overline{A-a_n}$. Since $a'_n \in \overline{A-A_{n-1}}$, $a'_n \in \overline{A-a_{n-1}}$ for all $a_{n-1} \in A_{n-1}$, and hence $a_n \in A_n - A_{n-1}$. Let $b_n = f(a_n)$, then $b_n \in B_n - B_{n-1}$. By definition of $B_n$ and Lemma 2, there exists an edge $(a'_{n-1}, b'_n)$ such that $a'_{n-1} \in C_{n-1}$, $b'_n \in \overline{B_n} - \overline{B-b_n}$. Thus $b'_n \notin \overline{B_{n-1}}$, and so $b'_n \in \overline{B_n} - \overline{B_{n-1}}$. Since $R(C_{n-2}) \subseteq \overline{B_{n-1}}$ by hypothesis, it follows that $a'_{n-1} \in C_{n-1} - C_{n-2}$ and $(a'_{n-1}, b'_n) \notin M$.

We may then continue this argument starting with $a'_{n-1} \in C_{n-1} - C_{n-2}$, and arrive finally at a sequence

(3.4) $$(b'_{n+1}, a'_n),\ (a_n, b_n),\ (b'_n, a'_{n-1}),\ \ldots,\ (a_1, b_1),\ (b'_1, a'_0),$$

where

(3.5) $$(a_i, b_i) \in M, \quad (a'_i, b'_{i+1}) \in R-M,$$

(3.6) $$a'_0 \in S-\overline{A}, \quad b'_{n+1} \in T-\overline{B},$$

(3.7) $$a_i \in A_i - A_{i-1}, \quad a'_i \in \overline{A-A_{i-1}} - \overline{A-a_i},$$

$$b_i \in B_i - B_{i-1}, \quad b'_i \in \overline{B_i} - \overline{B-b_i},$$

for $1 \le i \le n$. Now (3.5) and (3.6) are simply (2.8) and (2.9), and by Lemmas 3 and 4, (3.7) implies (2.10). It follows that the sequence (3.4) in reverse order represents an augmenting chain, which contradicts the hypothesis of Theorem 3.

Thus $R(C_{i-1}) \subseteq \overline{B_i}$ for $1 \le i \le m$, and $R(C_m) \subseteq \overline{B_m}$. Since $C_m = S - \overline{A-A_m}$, the pair $(\overline{A-A_m}, \overline{B_m})$ is a support, and the proof of Theorem 3 is complete.

<u>THEOREM 4</u>. A matching $M = (A,B,f)$ in $(G(S), G(T), R)$ is of maximum size if and only if there does not exist an augmenting chain with respect to $M$.

<u>PROOF</u>. The necessity of the condition follows by Theorem 1. If there does not exist an augmenting chain, then the support given by Theorem 3 has order equal to the size of $M$, which together with Theorem 2 implies that $M$ is of maximum size.

<u>COROLLARY 4.1</u>. If $M = (A,B,f)$ is a matching not of maximum size, there exists a matching $M' = (A' \cup a,\ B' \cup b,\ f')$ such that $\overline{A'} = \overline{A}$, $\overline{B'} = \overline{B}$.

<u>PROOF</u>. By Theorem 4, there exists an augmenting chain with respect to $M$, and the required matching $M'$ is constructed as in the proof of Theorem 1.

<u>THEOREM 5</u>. The maximum size of a matching in $(G(S), G(T), R)$ is equal to the minimum order of a support.

PROOF. If $M = (A,B,f)$ is a matching of maximum size, then by Theorem 4 there does not exist an augmenting chain with respect to $M$. A support of order $\nu(M)$ therefore exists by Theorem 3, and this support is necessarily of minimum order by Theorem 2.

Following Ore [5] for the case of a bipartite graph, we define the deficiency $\delta_S(A)$ of a subset $A \subseteq S$ by

$$\delta_S(A) = r(S) - r(S-A) - r(R(A)),$$

and let

$$\delta_S = \max_{A \subseteq S} \delta_S(A).$$

Note that $\delta_S \geq 0$ since $\delta_S(\phi) = 0$.

THEOREM 6. In the system $(G(S), G(T), R)$,

$$\max_{M \text{ matching}} \nu(M) = \min_{(C,D) \text{ support}} \lambda(C,D) = r(S) - \delta_S .$$

PROOF. Note that every support of minimum order is necessarily of the form $(C, D)$, where $D = \overline{R(S-C)}$. Now

$$r(S) - \delta_S = r(S) - \max_{A \subseteq S} [r(S) - r(S-A) - r(R(A))]$$

$$= \min_{A \subseteq S} [r(S-A) + r(R(A))]$$

$$= \min_{A \subseteq S} [r(A) + r(R(S-A))],$$

and the latter minimum is clearly attained when $A$ is a closed set.

<u>COROLLARY 6.1</u>. There exists a matching of size $r(S)$ in $(G(S), G(T), R)$ if and only if

$$r(S) - r(S-A) \leq r(R(A))$$

for every subset $A \subseteq S$.

<u>THEOREM 7</u>. (See also [2], [3], [4].) If the geometry $G(S)$ is free in the system $(G(S), G(T), R)$, then the subsets $S'$ of $S$ for which there exists a matching $(S', T', f)$ for some $T'$ and $f$, are the independent sets of a pregeometry, the <u>transversal pregeometry</u> on $S$.

<u>PROOF</u>. Let $I$ be the family of subsets $S'$ of $S$ for which there exists a matching $(S', T', f)$ for some $T'$ and $f$. Given any subset $S' \subseteq S$, let $G(S')$ be the free subgeometry on $S'$. Applying Theorem 6 to the system $(G(S'), G(T), R \cap (S' \times T))$, we have $S' \in I$ if and only if $\delta_{S'} = 0$. Equivalently, $S' \in I$ if and only if $\nu(S') \leq \nu(S') - \delta_{S'}$. The theorem will therefore follow from Proposition 7.3 of [2] if the function

$$r^*(S') = \nu(S') - \delta_{S'}$$

is increasing and semimodular. We proceed to establish these properties for $r^*$.

Let $S' \subseteq S$ and $A \subseteq S'$. Since $G(S')$ is free, the deficiency

$$\delta_{S'}(A) = \nu(A) - r(R(A))$$

is independent of $S'$, so we may omit the subscript. Then

$$\delta_{S'} = \max_{A \subseteq S'} \delta(A).$$

Given $A_1, A_2 \subseteq S'$, it follows from the relations

$$R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$$

$$R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$$

and the semimodular inequality (2.6) for the rank function $r$ of $G(T)$ that

$$(3.8) \qquad \delta(A_1 \cup A_2) \cup \delta(A_1 \cap A_2) \geq \delta(A_1) + \delta(A_2).$$

Let $F(S')$ be the family of subsets $A$ of $S'$ for which $\delta(A) = \delta_{S'}$. Then by (3.8), $F(S')$ is closed under the operation of intersection, and therefore contains a minimal set, which we denote by $A_{S'}$.

Clearly if $a \in S'-A_{S'}$, then $A_{S'-a} = A_{S'}$ and $\delta_{S'-a} = \delta_{S'}$. Suppose that $a \in A_{S'}$. Since $A_{S'}$ is minimal, $\delta_{S'-a} \leq \delta_{S'}-1$. But

$$
\begin{aligned}
\delta(A_{S'} - a) &= \nu(A_{S'} - a) - r(R(A_{S'} - a)) \\
&\geq \nu(A'_S) - r(R(A_{S'})) - 1 \qquad (3.9) \\
&= \delta(A_{S'}) - 1,
\end{aligned}
$$

so $\delta_{S'-a} = \delta_{S'}-1$. We have therefore that

$$r^*(S' - a) = \begin{cases} r^*(S') - 1, & a \in S'-A_{S'}, \\ r^*(S'), & a \in A_{S'}. \end{cases}$$

Thus the function $r^*$ is not only increasing, but unit-increasing.

Note from the argument above that if $a \in S'$, then $A_{S'}-a \in F(S'-a)$ and $A_{S'-a} \subseteq A_{S'}-a$. We conclude that if $S_1 \subset S_2 \subseteq S$, then $A_{S_1} \subseteq A_{S_2} \cap S_1$ and $\delta(A_{S_1}) = \delta(A_{S_2} \cap S_1)$.

Now let $S_1$, $S_2$ be any two subsets of $S$, and let

$$A_1 = A_{S_1 \cup S_2} \cap (S_1 - S_2),$$

$$A_2 = A_{S_1 \cup S_2} \cap (S_2 - S_1),$$

$$A_3 = A_{S_1 \cup S_2} \cap (S_1 \cap S_2).$$

Then by (3.8) and the above remark we have

$$\begin{aligned} \delta_{S_1 \cup S_2} + \delta_{S_1 \cap S_2} &= \delta(A_{S_1 \cup S_2}) + \delta(A_{S_1 \cap S_2}) \\ &= \delta(A_1 \cup A_2 \cup A_3) + \delta(A_3) \\ &\geq \delta(A_1 \cup A_3) + \delta(A_2 \cup A_3) \\ &= \delta(A_{S_1}) + \delta(A_{S_2}) \\ &= \delta_{S_1} + \delta_{S_2}. \end{aligned}$$

Thus

$$r^*(S_1 \cup S_2) + r^*(S_1 \cap S_2) \leq r^*(S_1) + r^*(S_2)$$

and the proof is complete.

It should be noted that Theorem 7 is false if the geometry $G(S)$ is arbitrary. The function

$$r^*(S') = r(S') - \delta_{S'}$$

is unit-increasing, but not semimodular in general, so that Proposition 5.7 of [2] cannot be applied. For the same reason, Theorem 6 cannot be proved by extending Ore's inductive argument [5] for the case of a bipartite graph to the general case, although this approach works when G(S) is free and G(T) arbitrary.

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