This research was supported by the Army Research Office, Durham, Grant No. DA-ARO-D-31, 124-G910, and the United States Air Force Office of Scientific Research, Office of Aerospace Research under Grant No. AFOSR-68-1406.

PARTIAL DIFFERENCE SETS AND PARTIALLY BALANCED WEIGHING DESINGS.

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Institute of Statistics Mimeo Series No. 600.14

OCTOBER 1969

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Abstract

Balanced weighing designs investigated by R. C. Bose and J. M. Cameron [2,3] are of special importance in some practical types of studies where one wishes to compare the value of the "unknown" objects in terms of the accepted standards. With the aim of further reduction of the number of weighings required, some study has been made by introducing "association schemes" to a weighing situation. This paper deals with the use of "difference sets and partial difference sets" in constructing this new class of designs called "Partially Balanced Weighing Designs".

I. Introduction and Summary.

There has been much work [6,8] on the construction, non-existence and applications of difference sets. Two of the main applications are in the areas of Design of Experiments and Error correcting codes. A new class of difference sets arose in the construction of association schemes and partially balanced incomplete block designs [4,5]. Such type of difference sets, called, "partial difference sets" are described in section 2.

Calibration designs or balanced weighing designs have been constructed and used by Bose and Cameron [2,3] in comparing the value of the "unknown" objects in terms of the accepted standards.

The extension of the balanced weighing designs (BWD) to a situation of "partial balance" based on "association schemes" leads to a new type of designs

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called "partially balanced weighing designs (PBWD)" [7,10,11]. These are described in section 3.

The main contribution in this paper is the construction of partial difference sets and the application of difference sets and partial difference sets in constructing partially balanced weighing designs. The methods of construction are described in section 4.

2. Perfect and Partial Difference Sets

A perfect difference set [6,8] is a set of k integers $\mathcal{D}=\{d_1, d_2,\ldots d_k\}$ modulo v such that every $a \not\equiv 0 \pmod v$ can be expressed in exactly λ ways in the form

$$d_i - d_j \equiv a \pmod v,$$

where d_i and d_j are in $\mathcal{D}$. It is easily seen that

$$\lambda = \frac{k(k-1)}{v-1} .$$

In order to exclude some degenerate configurations, we may impose the restriction

$$0 < \lambda < k < v-1.$$

$\mathcal{D}$ is called a (v, k, λ) - difference set.

A (v, k, λ) - difference set is equivalent to a cyclic (v, k, λ) symmetric balanced incomplete block design. For $\lambda = 1$, a (v, k, λ) difference set yields a cyclic projective plane.

A generalization of the idea of a difference set was noticed in Bose and Nair [4] and later in Bose and Shimamoto [5]. They did not, however, use the term "partial difference set". In [3], for defining a cyclic association scheme with two associate classes, the authors introduced a set $D = \{d_1, d_2, \ldots, d_{n_1}\}$ of n_1 integers modulo v, having the properties

(i) the d_j's are all different, and $0 < d_j < v$ $(j = 1, 2,\ldots n_1)$,

(ii) among the $n_1(n_1-1)$ differences $d_j - d_{j'}$, $(j \neq j' = 1,2,\ldots, n_1)$ (mod v) each of the integers $d_1, d_2,\ldots, d_{n_1}$ occurs g times and each of the integers $\{e_1, e_2,\ldots, e_{n_2}\}$ occurs h times, where the set $\{d_1, d_2,\ldots d_{n_1}, e_1, e_2,\ldots e_{n_2}\}$ is exactly the set of integers $\{1, 2, \ldots, v-1\}$.

D will be called a partial difference set and denoted by $D(v, n_1, n_2, g, h)$.

We note that the parameters of D satisfy the following relations.

$$n_1 + n_2 = v-1$$

$$n_1 g + n_2 h = n_1(n_1-1).$$

If $g = h$, D is a (v, n_1, g) perfect difference set.

The following is the statement of a well known theorem in difference sets [8].

Theorem 2.1. If the fourth powers of a primitive root of a finite field of order $p^n = 4f + 1$ form a perfect difference set, then $p^n = 1 + 4y^2$ where y is odd. If $n = 1$, then this condition is also sufficient. The equation $p^n = 1 + 4y^2$, $n > 1$, $y \equiv 1 \pmod 2$ do not have solutions.

Theorem 2.2 . Let $v = p^n$, where p is an odd prime. Let x be a primitive root of $GF(p^n)$. Then if $(x^{d_1}, x^{d_2}, \ldots, x^{d_k})$ forms a (v, k, λ) - difference set, then $(x^{d_1+h}, x^{d_2+h}, \ldots, x^{d_k+h})$ also forms a (v, k, λ) - difference set, for h an integer.

Proof: Easy.

Corollary 2.1 . If p is a prime of the form $4h^2 + 1$, where h is an odd integer, then $(x^2, x^6, \ldots, x^{4u-2})$ forms a perfect difference

set $(p, u, (u-1/4)$ where $u = h^2$.

Proof: From Theorem 2.1 $(x^4, x^8, \ldots x^{4u})$ forms a perfect difference set $(p, u, \frac{u-1}{4})$. Multiplying every element of this difference set by x^2 and using Theorem 2.2 we get that $(x^2, x^6, \ldots, x^{4u-2})$ is a difference set $(p, u, \frac{u-1}{4})$.

The following lemma is well known, see for instance [1]. It will be used to prove Theorem 2.3.

Lemma 2.1 : If x is a primitive root of $GF(p^n)$, p an odd prime, then among the elements $\{(x^2 - 1), (x^4 - 1), \ldots, (x^{s-3} - 1), (x^{s-1} - 1)\}$ there is one zero, $(t-1)$ quadratic residues and t non quadratic residues if $s = p^n = 4t + 1$ and one zero, $(t-1)$ quadratic residues and $(t-1)$ non quadratic residues if $s = p^n = 4t-1$.

Theorem 2.3 : Let $p^n = 4u + 1$, where p is a prime, u and n are positive integers. Let x be a primitive element of $GF(p^n)$. Then the $2u$ elements $\{x^2, x^4, x^6, \ldots, x^{4u}\}$ constitute a partial difference set with the parameters $(4u + 1, 2u, u - 1, u)$.

Proof: The $2u(2u - 1)$ differences which can be formed from $(x^2, x^4, x^6, \ldots, x^{4u})$ can be shown in the form of an array:

$$x^2(x^2 - 1), \quad x^2(x^4 - 1), \ldots, \quad x^2(x^{4u-2} - 1)$$

$$x^4(x^2 - 1), \quad x^4(x^4 - 1), \ldots, \quad x^4(x^{4u-2} - 1)$$

$$\ldots \qquad\qquad\qquad\qquad 1$$

$$x^{4u}(x^2 - 1), \quad x^{4u}(x^4 - 1), \ldots, \quad x^{4u}(x^{4u-2} - 1).$$

Evidently, the hth column of the array in (1) contains all even powers of the primitive root x multiplied by $(x^{2h} - 1)$. Hence if $(x^{2h} - 1)$ is an even power of x or in other words a quadratic residue, then that entire column exhausts the even powers. Similarly, if $(x^{2h} - 1)$ is a non-Q.R., then that entire h-th column exhausts the set of all odd powers of x. So the frequencies of the sets $\{x^2, x^4, x^6, \ldots, x^{4u}\}$ and $\{x, x^3, \ldots, x^{4u-1}\}$ are the same as the number of Q.R.'s and non-Q.R.'s among the set $\{(x^2 - 1), (x^4 - 1), \ldots, (x^{4u-2} - 1)\}$. But by the lemma, these are respectively (u-1) and u. So by definition of the partial difference set, $\{x^2, x^4, x^6, \ldots, x^{4u}\}$ forms a partial difference set with parameters (4u+1, 2u, 2u, u-1, u). which proves the theorem.

Example 2.1 Let u = 3. Hence (4u+1)(=13) is a prime power. We know that 2 is a primitive root of GF(13). It can be easily verified that among the differences which can be formed from $(2^2, 2^4, 2^6, 2^8, 2^{10}, 2^{12}) =$ (4, 3, 12, 9, 10, 1), the set (4, 3, 12, 9, 10, 1) occurs 2 times and each of the other remaining 6 non-zero integers (mod 13), occurs 3 times.

Example 2.2 Let u = 4. Hence (4u+1))=17) is a prime power. We know that 3 is a primitive element of GF(17) and that (1, 2, 4, 8, 9, 13, 15, 16) is the set of quadratic residues. It can be easily verified that the frequencies are 3 and 4 respectively for the sets of Q.R.'s and non-Q.R.'s among the differences formed from (1, 2, 4, 8, 9, 13, 15, 16).

Example 2.3. Let u = 9. Hence (4u+1) = 37 is a prime power. We know that 2 is a primitive element of GF(37). It can be verified that the frequencies of Q.R.'s and non-Q.R.'s among the differences formed from the Q.R.'s (1,3,4,7,9,10,11,12,16,21,25,26,27,28,30,33,34,36) are 8 and 9 respectively.

Example 2.4. Let u = 7. Hence (4u+1) (=29) is a prime power. We know that 2 is a primitive root of GF(29). It can be verified that the frequencies of Q.R.'s and non-Q.R.'s among the differences formed from the quadratic residues (4,16,6,24,9,7,28,25,13,23,5,20,22,1), are 6 and 7 respectively.

Association shcemes have been developed [4,5], in connection with the study of partially balanced incomplete block designs. As has been mentioned already, the extensions of the balanced weighing designs going to be described in section 3, arise by the introduction of association schemes.

All the constructions which are made in section 4 are with reference to a particular case of a general type of association scheme called cyclic association scheme [5] about which a result is established in corollary 2.1.

Although a simple proof of corollary 2.2 can be given by the direct use of Theorem 2.3, it is deduced as a particular case of a Theorem due to Mesner [9], which is stated below.

Theorem 2.4 (due to Mesner): In a finite field of order V with additive group G and multiplicative group G', let m be a divisor of the order (V-1) of G' such that N=(V-1)/m is even if V is odd, and let ξ be a generator of G'. Let $\alpha_o = (0)$, let α_1 be the multiplicative sub-group of order N generated by ξ^m, and let α_i, i = 1,2,... m, be the coset of α_1 which contains ξ^{i-1}. Define an association relation F(v, m) in which two elements x, y of G are i-th associates if and only if (y - x) belongs to α_i, i = 0, 1, 2,... m. Then for i, j, k, in

the range 1, 2,... m and interpreted as taken modulo m, where necessary, F(V,m) is an m-class partially balanced association scheme with the parameters V, $n_i = N, p^i_{jk}$, $p^i_{jk} = p^i_{j+1,k+1} = p^1_{j-i+1,k-i+1}$, and p^i_{jk} is equal to the number of elements of α_{j-i+1} which occur in the set obtained by adding the unit element 1 to each element of α_{k-i+1}.

Corollary 2.1. Let x be the primitive root of the Galois field GF(4u+1), where (4u+1) is the power of an odd prime. Let $A_0 = 0$. Let $A_1 = (x^2, x^4, \ldots x^{4u})$ and $A_2 = (x, x^3, x^5, \ldots, x^{4u-1})$. Define an association relation F(4u+1,2) in which two non-zero elements x and y are i th associates if and only if (y-x) belongs to A_i, i = 0,1, 2. Then F(4u+1,2) is a two-class partially balanced association scheme with the parameters V = 4u+1, $n_1 = n_2 = 2u$ and

$$P_1 = \begin{pmatrix} u-1 & u \\ u & u \end{pmatrix} \qquad P_2 = \begin{pmatrix} u & u \\ u & u-1 \end{pmatrix}$$

Proof: The proof follows from the above Theorem, by taking V = 4u+1 and m=2.

3. Partially Balanced Weighing Designs.

Definition of Partially Balanced Weighing Designs with two association classes:

Association schemes have been defined and have been widely used. Given V treatments 1,2,... V,a relation satisfying the following conditions is said to be an association scheme with 2 classes:

(a) Any two treatments are either 1st, or 2nd associates, the relation of association being symmetrical, i.e., if the treatment α is the i-th associate of the treatment β, then β is the i-th associate of the treatment α(i=1,2).

(b) Each treatment has n_i, i-th associates, the number n_i being independent of α.

(c) If any two treatments are i-th associates then the number of treatments which are j-th associates of α and k-th associates of β is p^i_{jk} and is independent of the pair of i-th associates α and β. The parameters of the association scheme are V, n_i, p^i_{jk} (i,j,k=1,2).

A design is said to be a Partially Balanced Weighing Design (PBWD) with two association classes and with the parameters $(v,b,r,p,\lambda_{11},\lambda_{21},\lambda_{12},\lambda_{22})$ if there are v treatments arranged in b blocks, each treatment occuring in r blocks, such that the blocks are of size 2p and if each block can be subdivided into two halves with the following conditions:

(1) Any two first associates occur together in the same half block λ_{11} times and in the opposite half blocks λ_{21} times.

(2) Any two second associates occur together in the same half block λ_{12} times and in the opposite half blocks λ_{22} times.

Such a design is denoted by $PBWD(v,b,r,p,\lambda_{11},\lambda_{21},\lambda_{12},\lambda_{22})$.

The combinatorial properties of these designs together with the methods of construction and the analysis are described elsewhere [10]. A good account of the methods of construction of the PBWD's with the triangular, Latin square and group divisible association shcemes is given in a separate paper [11].

The next section is concerned with a method called "the cyclic generation of the PBWD's", which consists in generating all the blocks of the design with the help of one or more initial blocks. The investigation is restricted to a particular class of cyclic association schemes.

4. The construction of PBWD's from difference sets.

Let $\{d_1, d_2, \ldots, d_{n_1}\}$ be a partial difference set with parameters $(v, n_1, n_2, \alpha, \beta)$ and let $D = \{d_1, d_2, \ldots, d_{n_1}\}$ and $E = (e_1, e_2, \ldots, e_{n_2})$ define a cyclic association scheme.

Suppose $\{a_{i1}, a_{i2}, \ldots, a_{ip}\}$ and $\{b_{i1}, b_{i2}, \ldots, b_{ip}\}$ $(i=1,2,\ldots,t)$ are $2t$-sets of distinct integers, each of size p, with the following properties:

I. The sets $\{a_{i1}, a_{i2}, \ldots, a_{ip}\}$ and $\{b_{i1}, b_{i2}, \ldots, b_{ip}\}$ are disjoint.

II. Among the differences $(a_{ij} - a_{i\ell})$, $(b_{ij} - b_{i\ell})$ $(i=1,2,\ldots,t,\ j \neq \ell = 1,2,\ldots,p)$, each d_j occurs λ_{11} times.

III. Among the differences mentioned in II, each e_q $(q=1,2,\ldots,n_2)$ occurs λ_{12} times.

IV. Among the differences of the form $\pm (a_{ij} - b_{i\ell})$, each d_j occurs λ_{21} times.

V. Among the differences mentioned in IV, each e_ℓ $(\ell=1,2,\ldots,n_2)$ occurs λ_{22} times.

We treat the initial block $\{a_{i1}, a_{i2}, \ldots, a_{ip}; b_{i1}, b_{i2}, \ldots, b_{ip}\}$ of the i-th set as '0'-th block of that set and the block j $\{a_{i1} + j, a_{i2} + j, \ldots, a_{ip} + j; b_{i1} + j, \ldots, b_{ip} + j\}$, where each element must be interpreted as belonging to the residue system mod v, is called j-th block generated from the initial block. When we consider the residue '0', it will be taken as v. With this notation, we can state and prove the following theorem:

THEOREM 4.I. If the $2t$ sets $\{a_{i1}, a_{i2}, \ldots, a_{ip}\}$, $\{b_{i1}, b_{i2}, \ldots, b_{ip}\}$, $i=1,2,\ldots,t$, satisfying the properties I - V exist, then the vt blocks generated by developing the sets $\{a_{i1}, a_{i2}, \ldots, a_{ip}; b_{i1}, b_{i2}, \ldots, b_{ip}\}$ treated as the initial blocks, constitutes a partially balanced weighing design with the parameters:

$$(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}) = (v, vt, 2pt, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}) \quad (2)$$

and with the association scheme defined by the partial difference set $\{d_1, d_2, \ldots, d_{n_1}\}$ with parameters:

$$n_1,\ n_2 \text{ and } P_1 = \begin{pmatrix} \alpha & n_1-\alpha-1 \\ n_1-\alpha-1 & n_2-n_1+\alpha+1 \end{pmatrix} \quad (3)$$

$$\text{and } P_2 = \begin{pmatrix} \beta & n_1-\beta \\ n_1-\beta & n_2-n_1+\beta-1 \end{pmatrix}$$

Thus the corresponding association scheme called cyclic association scheme is specified by the parameters α, β, n_1 and n_2.

Proof: For some verifications arising in the proof, we can have an idea of the situation by restricting to a single initial block from out of the set of t -blocks.

We know that the number of blocks is vt when we start with t initial blocks. We also know that p is the half-block size.

Next, consider a treatment numbered u. In the development of $\{a_{i1}, a_{i2}, \ldots, a_{ip}; b_{i1}, b_{i2}, \ldots, b_{ip}\}$, u occurs in the block number q, if and only if we can find an a_{ij} or b_{ij} such that $a_{ij} + q = u$ or $b_{ij} + q = u$. So q can be uniquely fixed by

$q = u - a_{ij}$ or $u - b_{ij}$ as the case may be. So restricting to the blocks generated from $\{a_{i1}, a_{i2}, \ldots, a_{ip}; b_{i1}, b_{i2}, \ldots, b_{ip}\}$, the blocks in which u occurs are the ones numbered $u - a_{i1}, u - a_{i2}, \ldots u - a_{ip}, u - b_{i1}, u - b_{i2}, \ldots u - b_{ip}$. As there are t initial blocks, $r = 2pt$.

Let u and s be two treatments, which are first associates. Then the q-th block of the i-th set will contain both u and s in the first half-block if and only if we can find two integers a_{ij} and $a_{i\ell}$ such that

$$a_{ij} + q = u \qquad \text{——(4)}$$

$$a_{i\ell} + q = s \qquad \text{——(5)}$$

From (4) and (5), $a_{ij} - a_{i\ell} = u - s$.

Let the partial difference set $\{d_1, d_2, \ldots, d_{n_1}\}$ mentioned in the hypothesis be denoted by D.

If u and s are fixed, then $(u - s)$ is also fixed and it is an element of D, since u and s are first associates of each other. Thus $(u - s)$ can be identified to be d_h (say). Then it is evident that u and s occur together in the first half of the i-th set, as many times as d_h can be represented as a difference between two a_i's.

Similar argument applies to the other half blocks of the i-th set $(b_{i1}, b_{i2}, \ldots, b_{ip})$. So the number of times d_h occurs either as difference between two a_i's or as difference between two b_i's, is the same as the number of times u and s occur together. But

this number is λ_{11} by the condition II. Also this entire argument is valid as long as the pair constitutes a pair of first associates.

In the same way, it is easily seen that two treatments which are first associates will occur in opposite half blocks of the same block λ_{21} times and two treatments which are second associates will occur in the same half block λ_{12} times and in opposite half blocks of the same block λ_{22} times.

THEOREM 4.2: A ncessary set of conditions for the existence of a cyclic PBWD described in Theorem 1 is that:

(i) $n_1\lambda_{11} + n_2\lambda_{12} = 2tp(p - 1)$

(ii) $n_1\lambda_{21} + n_2\lambda_{22} = 2tp^2$.

Proof:

(i) The totality of pairs which can be formed from a single initial block is $p(p-1)$ and hence the total number of all possible pairs both of whose elements belong to the same half block, is $2tp(p-1)$. Considering any pair is equivalent to considering the difference between the two integers representing the pair of treatments. But difference between any pair must belong to $D = \{d_1, d_2, \ldots, d_{n_1}\}$ or $E = \{e_1, e_2, \ldots, e_{n_2}\}$. The existence of the PBWD guarantees that each element of D occurs as a difference between the elements of such pairs λ_{11} times and that each element of E occurs λ_{12} times.

Since there are n_1 elements in D and n_2 elements in E , it follows that: $n_1\lambda_{11} + n_2\lambda_{12} = 2tp(p-1)$.
This proves (i). Similar argument establishes (ii).

Now we proceed to develop some results relevant for the construction of the partially balanced weighing designs, with the cyclic association scheme described in Corollary 2.2.

Lemma 4.1. (i) Let $v = 4u+1$ be the power of a prime (p^n), in a positive integer.

(ii) Let x be a primitive root of the Galois field $GF(p^n)$;

(iii) Let q_w be defined by $(x^{4w} - 1) = x^{q_w}$ for $w = 1,2,\ldots,(u-1)$.

Then $(x)^{q_w} = x^{4w+2u+q_a}$ where $a = (u-w)$.

Proof: Since x is a primitive element of $GF(4u + 1)$, $x^{4u} = 1$ or $x^{4u} - 1 = 0$ or $y^u - 1 = 0$, where $y = x^4$. Since $y = x^4 \neq 1$, except for the trivial case $v = 5$,

$$y^u - 1 = 0 \quad \text{implies}$$

$$y^{u-1} + y^{u-2} + \ldots + y + 1 = 0 \quad \text{or}$$

$$x^{4(u-1)} + x^{4(u-2)} + \ldots + x^4 + 1 = 0 \qquad \text{(6)}$$

$$(x^{4w} - 1) = (x^4-1)[x^{4(w-1)} + x^{4(w-2)} + \ldots + x^4 + 1]$$

$$= -(x^4-1)[x^{4(u-1)} + x^{4(u-2)} + \ldots + x^{4w}]$$

by the use of (4).

$$\therefore \quad (x^{4w} - 1) = -x^{4w}(x^4-1)[x^{4(u-1-w)} + x^{4(u-2-w)} + \ldots + 1] .$$

Noting that the product of the last two expressions is $[x^{4(u-w)}-1]$ and using the fact that $x^{2u} = -1$, we have $x^{4w}-1 = x^{4w+2u}[x^{4(u-w)}-1]$ $= x^{4w+2u+q_a}$. This proves the lemma.

LEMMA 4.2: With the same notation and conditions ((i) - (iii)) of the previous lemma, let u be of the form $(4t \pm 3)$ and let for a fixed w, the sets $\{x^{q_w+4}, x^{q_w+8}, \ldots, x^{q_w+4u}\}$ and $\{x^{q_w-(2u+4w)+4}, x^{q_w-(2u+4w)+8}, \ldots, x^{q_w-(2u+4w)+4u}\}$ be denoted by s_1 and s_2 respectively.

Then s_1 and s_2 are disjoint (i.e.) $s_1 \cap s_2 = \phi$ (empty set).

Proof: If s_1 and s_2 are not disjoint, let there exist two integers r and n $(1 \leq r, n \leq u)$ such that $x^{q_w+4r} = x^{q_w-(2u+4w)+4n}$.

This implies that $(4r + 4w - 4n) + 2u \equiv 0 \pmod{4u}$.

Since 4 must divide the left hand side of the above congruence, 4 divides $2u$. This is a contradiction, since u is of the form $(4t + 3)$. This contradiction is due to the assumption that one member of s_1 is identical with an element of s_2 . Hence it follows that s_1 and s_2 are disjoint.

LEMMA 4.3. : With the same notation as in the above lemmas, and under the same situation, where $v = p^n$ is of the form $(4u + 1)$, let us put an additional restriction that u is of the form $(4t + 3)$.

Then the number of Q.R.'s among the set of $(u-1)$ -elements $\{x^{4w}-1,\ w=1,2,\ldots,(u-1)\}$, is different from that of non-Q.R.'s among the same set.

Proof: Suppose if possible that the two numbers are the same. In that case, the number is $(\frac{u-1}{2})$. Also if $q_{u-w} = q_w$ for any $w=1,2,\ldots,(u-1)$, then $(u-w) = w$ or $u = 2w$, which is a contradiction to the assumption that $u = (4t + 3)$. Hence it follows that q_{u-w} and q_w are distinct for any fixed w , $(w = 1, 2, \ldots, (u-1))$. The conclusion $x^{q_w} = x^{4w+2u+q_a}$, of the Lemma 4.1 can be written as:

$$q_w \equiv 4w + 2u + q_a \pmod{4u}$$

$$\text{or} \quad q_w \equiv 4w + 2u + q_{u-w} \pmod{4u} \qquad \text{——— ( )}$$

$$\text{(i.e.)} \quad q_{u-w} \equiv q_w - 4w - 2u \pmod{4u}$$

$$\equiv q_w + 2u + 4(u-w) \pmod{4u}$$

$$\equiv q_w + 2u + 4a \pmod{4u} \qquad \text{——— (8)}$$

It is obvious from(7) (or (8)) that if q_{u-w} is odd, then q_w is also odd and vice versa. Similarly, if q_{u-w} is even, then q_w is also even and vice versa.

So the odd (even) powers of x , if at all there are any, among $\{x^{4w}-1,\ w=1,2,\ldots,(u-1)\}$, can be paired, and hence the number of odd (even) powers of x among this set must be of the form 2ℓ , for some suitable ℓ . But by a remark given in the beginning of the proof, the common frequency of odd or even powers of x among $\{x^{4w}-1,\ w=1,2,\ldots,(u-1)\}$ is $\frac{(u-1)}{2}$. Hence $\frac{u-1}{2} = 2\ell$ or $u = 4\ell + 1$ which is a contradiction, since $u = (4t + 3)$ by condition (1) . This contradiction is due to our assumption that the numbers of Q.R.'s and non-Q.R.'s are the same among the set under consideration. Hence the lemma follows.

Remarks: The Lemmas (4.I-- 4.3) are developed to prove the theorem, ~~viz~~ 4.3.2 which is useful for the construction of PBWD's.

THEOREM 4.3:

(i) Let p be an odd prime and let v be an integer of the form $4u + 1$, which can be expressed as $v = p^n$.

(ii) Let u be of the form $(4t + 3)$.

(iii) Let x be a primitive root of GF_{p^n} and let the 4 sets $\{x^4, x^8, \ldots x^{4u}\}$, $\{x^2, x^6, x^{10}, \ldots x^{4u-2}\}$, $\{x, x^5, x^9, \ldots x^{4(u-1)+1}\}$ and $\{x^3, x^7, x^{11}, \ldots x^{4u-1}\}$ be denoted by A_1, A_2, A_3, A_4 respectively.

(iv) Among the (u-1) distinct elements $\{x^{4w} - 1,\ w = 1, 2, \ldots, (u-1)\}$, let there be g quadratic residues.

Then among the $u(u-1)$ differences formed from $\{x^4, x^8, x^{12}, \ldots, x^{4u}\}$, the frequencies of the sets A_1, A_2, A_3, and A_4 are $\frac{g}{2}$, $\frac{g}{2}$, $\frac{u-1-g}{2}$ and $\frac{u-1-g}{2}$ respectively.

Proof: All the possible differences which can be formed from $\{x^4, x^8, x^{12}, \ldots, x^{4u}\}$ can be written in the form of an array as follows:

$$\left.\begin{array}{l} x^{4+q_1}, x^{4+q_2}, \ldots\ldots x^{4+q_{u-1}} \\ x^{8+q_1}, x^{8+q_2}, \ldots\ldots x^{8+q_{u-1}} \\ \\ x^{4u+q_1}, x^{4u+q_2}, \ldots. x^{4u+q_{u-1}} \end{array}\right\} \quad \text{(9)}$$

Evidently, all the u elements in a column of this array are distinct.

Also the s-th row of this array can be written as $x^{4s+q_1}, x^{4s+q_2}, \ldots, x^{4s+q_{u-1}}$. Among these, the two elements $x^{q_{u-w}+4s}$ and x^{q_w+4s} $\left(= x^{4w+2u+q_{u-w}+4s}\right)$ by Lemma 4.1, are distinct, as it is already noted (Lemma 4.3.) for any fixed w $(1 \leq w \leq u-1)$.

If q_{u-w} is of the form $4p$, then evidently $4w + 2u + q_{u-w}$ is of the form $4h + 2$, since $u = (4t + 3)$. So we conclude that "q_{u-w} is of the form $4p$" implies "$(q_w + 4s)$ is of the form $(4c + 2)$."

If q_{u-w} is of the form $4p + 2$, then $4w + 2u + q_{u-w}$ is of the form $4h$, since $q_{u-w} \equiv 2 \pmod 4$ and $2u \equiv 2 \pmod 4$.

The same type of argument leads us to conclude that

"$q_{u-w} + 4s \equiv 1 \pmod 4$" implies

"$q_w + 4s \equiv 3 \pmod 4$" and vice versa.

So for a fixed s, the elements can be paired as $(x^{4s+q_{u-w}}, x^{4s+q_w})$ for different w's such that this unordered pair will be of the form (x^{4c+1}, x^{4b+3}) or (x^{4c+2}, x^{4b}).

This assertion is true for all s ($s = 1, 2, \ldots, u$). This together with the fact that each column exhausts all the possible 4-th powers (which are u in number), when x^{q_j} is taken out as common factor, for $j = 1, 2, \ldots, u$, lead to the conclusion that the columns of the array (9) can be so paired that $u + u = 2u$ individual elements of each pair of columns exhaust all the Q.R.'s exclusively or they exhaust all the non-Q.R.'s.

So if g and h stand for the Q.R.'s and non-Q.R.'s among $\{x^{4w}-1,\ w=1,2,\ldots,(u-1)\}$, we conclude the following:

(i) $h = u - 1 - g$.

(ii) $g \neq h$ (by Lemma 4.3.8).

(iii) The number of elements of the form x^t among those of the array (9) is $\frac{g}{2}$, or $\frac{(u-1-g)}{2}$ according as t is even or odd.

(iv) All the u elements in a column of the array (9) are distinct and each column contains elements of the form x^t, where t covers exclusively all possible elements of the form $t \equiv i \pmod 4$ for some i ($i = 0, 1, 2,$ or 3).

So with the help of array (9), it follows that among the $u(u-1)$ differences formed from $A_1 = \{x^4, x^8, \ldots, x^{4u}\}$, the frequencies of the sets A_1, A_2, A_3, A_4 are $\frac{g}{2}$, $\frac{g}{2}$, $\frac{u-1-g}{2}$, and $\frac{u-1-g}{2}$ respectively. This proves Theorem 4.3.

COROLLARY 4.1. With the same notation as in Theorem 4.3 among the differences formed from A_2 : $x^2, x^6, \ldots, x^{4u-2}$, the frequencies of A_1, A_2, A_3, A_4 are $\frac{g}{2}$, $\frac{g}{2}$, $\frac{u-1-g}{2}$ and $\frac{u-1-g}{2}$, respectively.

Proof: Since $A_2 = \{x^2, x^6, \ldots, x^{4u-2}\} = x^2\{x^{4u}, x^4, x^8, \ldots, x^{4u-4}\}$ or $x^2\{x^4, x^8, \ldots, x^{4u-4}, x^{4u}\}$, we get the same type of differences as in the theorem (for the case A_1), except for the fact that each resulting x^t will be multiplied by x^2.

So denoting the set of differences for A_1 and A_2 as D_f and D_s respectively, the assertion of the Theorem 4.3. that

$$D_f = \frac{g}{2} A_1 + \frac{g}{2} A_2 + \frac{u-1-g}{2} A_3 + \frac{u-1-g}{2} A_4$$

implies

$$D_s = \frac{g}{2} A_2 + \frac{g}{2} A_1 + \frac{u-1-g}{2} A_4 + \frac{u-1-g}{2} A_3$$

(by the remark of the above paragraph).

Hence the corollary follows.

THEOREM 4.4 Let p be an odd prime and let $v = p^n$ be of the form $(4u+1)$. Let $A_1 = \{x^4, x^8, \ldots, x^{4u}\}$ and $A_2 = \{x^2, x^6, \ldots x^{4u-2}\}$, where x is a primitive element of $GF(p^n)$.

Let $(A_1 - A_2)$ stand for the u^2 differences $(x^4-x^2, x^4-x^6, \ldots x^4-x^{4u-2}; x^8-x^2, x^8-x^6, \ldots, x^8-x^{4u-2}; \ldots, x^{4u}-x^2, x^{4u}-x^6, \ldots, x^{4u}-x^{4u-2})$. Similarly, let $(A_2 - A_1)$ be defined.

(i) Let u be of the form $(4t+3)$ and

(ii) Let there be g -quadratic residues among the $(u-1)$ -elements $(x^{4h}-1)$, $h=1,2,\ldots,(u-1)$, then among the $2u^2$ -differences formed from $(A_1 - A_2)$ and $(A_2 - A_1)$, each quadratic residue occurs $(u-1-g)$ times and each non Q.R. occurs $(g+1)$ times.

Proof: The set $(A_1 - A_2)$ of differences can be written as an array given below:

$$\begin{array}{c} x^2(x^2-1),\ x^2(x^6-1),\ x^2(x^{10}-1),\ \ldots\ldots\ldots x^2(x^{4u-2}-1) \\ x^6(x^2-1),\ x^6(x^6-1),\ x^6(x^{10}-1),\ \ldots\ldots\ldots x^6(x^{4u-2}-1) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ x^{4u-2}(x^2-1),\ x^{4u-2}(x^6-1),\ x^{4u-2}(x^{10}-1),\ \ldots\ldots\ldots x^{4u-2}(x^{4u-2}-1). \end{array}$$

So any arbitrary element of $(A_1 - A_2)$ will be of the form x^{h_w+4s+2}, where $(x^{4w+2}-1) = x^{h_w}$. Let $h_w + 4s + 2 = R_w$ which obviously depends on the arbitrary integer s. With this notation $x^{h_w+4s+2} = x^{R_w}$.

The corresponding element (i.e. just the negative of this arbitrary element) of $(A_2 - A_1)$ can be written as $(-1)x^{R_w} = x^{R_w+2u}$, since it follows from the definition of x that $x^{2u} = (-1)$. Since

$2u = 2(4t+3) = 4(2t+1) + 2$, the comparison of R_w and R_w+2u , appearing in x^{R_w} and x^{R_w+2u} , leads us to the conclusion that $R_w+2u \equiv 0, 1, 2, 3 \pmod 4$ according as

$$R_w \equiv 2, 3, 0, 1 \pmod 4 \qquad \text{——(10)}$$

Let us write the u elements of the $(w+1)$ -th column of the above array and the u -elements corresponding to their negatives as a partitioned row vector as follows:

$$(x^{2+h_w}, x^{6+h_w}, \ldots x^{4u-2+h_w} | x^{2+h_w+2u}, x^{6+h_w+2u}, \ldots x^{4u-2+h_w+2u}) \ . \qquad \text{——(11)}$$

Now let us study the nature of these elements. Let (h_w+4s+2) and $(h_w+4c+2+2u)$ be two arbitrary elements of this partitioned row vector taken respectively from the first and second part. Comparing these two elements, it is obvious that if $h_w+4s+2 \equiv h_w+4c+2+2u \pmod{4u}$, we have to conclude that $4(s-c) \equiv 2u \pmod{4u}$ and hence that 4 divides $2u$, which is impossible.

Also, it is obvious that the first part of (11) is a set of u distinct elements and that the second part of (11) is a set of u distinct elements.

So it follows from (10) that all the $2u$ -distinct elements of (11) are quadratic residues or non-quadratic residues according as the corresponding x^{R_w} (see (10) is a quadratic residue or a non-residue, respectively. So if the array $(A_2 - A_1)$ is formed directly from $(A_1 - A_2)$ by change of sign of each element, we have the conclusion that:

(i) the $(w+1)$ -th column of the combined array $\begin{pmatrix} A_1-A_2 \\ A_2-A_1 \end{pmatrix}$ either exhausts all the $2u$ Q.R.'s or all the non-Q.R.'s.

(ii) If c is the number of quadratic residues among $\{(x^2-1), (x^6-1), \ldots, (x^{4u-2}-1)\}$, (i) implies that the set of quadratic residues repeats itself c times in the combined array.

Determination of c : By the Corollary (1.1.1) of [ ] there are (u-1) Q.R.'s among $\{x^2-1, x^4-1, x^6-1, \ldots, x^{4u-2}-1\}$ and hence by the notation of g in the hypothesis, $c = u-1-g$.

So by (i), each quadratic residue occurs (u-1-g) times in $\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$. Since the total number of rows is u in this array, each non Q.R. occurs among $\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$, $g + 1$ ($= u-(u-1-g)$) times. This proves Theorem 4.4.3.

THEOREM 4.5 Let p be an odd prime and let $v = p^n$ of the form $(4u + 1)$. Let x be a primitive root of $GF(p^n)$.

(i) Let u be of the form $(4t + 3)$.

(ii) Let there be g quadratic residues among the (u-1) -elements $\{(x^{4h}-1), \ h=1,2,\ldots,(u-1)\}$.

Then the initial block $\{x^4, x^8, \ldots, x^{4u}; x^2, x^6, \ldots, x^{4u-2}\}$, when developed, gives a partially balanced weighing design with the association scheme defined in the Corollary 4.3.1 and the parameters of the design being $(v, b, r, p) = (4u+1, 4u+1, 2u, u)$ and

$$(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (g, u-1-g; u-1-g, g+1) .$$

Proof: We follow the notation of A_1 and A_2 of the Theorem 4.4.3 and note that the initial block can be written as: $\{A_1; A_2\}$.

Step 1: By the Theorem 4.3. among the differences formed from A_1, each quadratic residue occurs $\frac{g}{2}$ times and each non Q.R. occurs $\frac{u-1-g}{2}$ times.

Step 2: By the Corollary 4.1 among the differences formed from A_2, each Q.R. occurs $\frac{g}{2}$ times and each non Q.R. occurs $\frac{u-1-g}{2}$.

Step 3: By the Theorem 4.4 among the differences $\{A_1 - A_2, A_2 - A_1\}$ which are the opposite differences arising from $(A_1; A_2)$, each Q.R. occurs $(u-1-g)$ times and each non Q.R. occurs $(g+1)$ times.

Steps 1 and 2 imply that among the differences formed from the same half block, each Q.R. occurs g times and each non Q.R. occurs $(u-1-g)$ times. Hence $(\lambda_{11}, \lambda_{12}) = (g, u-1-g)$.

Step 3 implies that $(\lambda_{21}, \lambda_{22}) = (u-1-g, g+1)$. Hence the initial block under consideration satisfies all the 5 conditions of the Theorem 4.1 with $\{d_1, d_2, \ldots, d_{2u}\} = \{x^2, x^4, x^6, \ldots, x^{4u}\}$ and $\{e_1, e_2, \ldots, e_{2u}\} = \{x, x^3, x^5, \ldots, x^{4u-1}\}$ defining the association scheme.

The parameter sets (1) and (2) of the Theorem 4.1 are given by:

$$(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22})$$

$$= (4u+1, 4u+1, 2u, u; g, u-1-g, u-1-g, g+1) \quad \text{----(12)}$$

and $n_1 = n_2 = 2u$, $P_1 = \begin{pmatrix} u-1 & u \\ u & u \end{pmatrix}$ and $P_2 = \begin{pmatrix} u & u \\ u & u-1 \end{pmatrix}$,

as noted in the Corollary 2.1. ----- -(13)

EXAMPLE 1: $u = 3$; $v = 4 \times 3 + 1 = 13$. 2 is a primitive root of GF(13) . $(2^4, 2^8, 2^{12}) = (3, 9, 1)$, $(2^2, 2^6, 2^{10}) = (4, 12, 10)$, $(2^4-1, 2^8-1) = (2, 2^3)$. $\therefore$ $g = 0$.

By Corollary 2.1 , (4, 3, 12, 9, 10, 1) defines a cyclic association scheme and by Theorem 4.5 (3, 9, 1; 4, 12, 10) gives a PBWD with the parameters of the association scheme and the design being given by (12) and (13).

$$(n_1, n_2) = (6, 6) ; \qquad P_1 = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} , \qquad P_2 = \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix} \qquad (12)$$

$$(v, b, r, p; \lambda_{11}, \lambda_{21}; \lambda_{12}, \lambda_{22}) = (13,13,6,3;0,2,2,1) \qquad (13)$$

This can be seen by the direct verification.

EXAMPLE 2: $u = 7$; $v = 4 \times 7 + 1 = 29$. 2 is a primitive root of GF(29) and

$$(2^4, 2^8, 2^{12}, 2^{16}, 2^{20}, 2^{24}, 2^{28}) = (16, 24, 7, 25, 23, 20, 1)$$

and $(x^{4w}-1, \ w=1,2,\ldots,6) = (15,23,6,24,22,19) = (2^{27},2^{20},2^6,2^8,2^{26},2^9)$

$$\therefore \quad g = 4 .$$

$\therefore$ By the Corollary 2.1, (4, 16, 6, 24, 9, 7, 28, 25, 13, 23, 5, 20, 22, 1) defines a cyclic association scheme and by Theorem 4.5 (16, 24, 7, 25, 23, 20, 1; 4, 6, 9, 28, 13, 5, 22) gives a PBWD with the

parameters of the association scheme and the design being given by (using (12) and (13)

$$(n_1, n_2) = (14, 14) \; ; \quad P_1 = \begin{pmatrix} 6 & 7 \\ 7 & 7 \end{pmatrix} \; ; \quad P_2 = \begin{pmatrix} 7 & 7 \\ 7 & 6 \end{pmatrix}$$

$$(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}) = (29,29,14,7; 4,2,2,5) .$$

This is verified separately by the direct method.

EXAMPLE 3: $u = 15$, $v = 4 \times 15 + 1 = 61$. 2 is a primitive root of GF(61) . The powers $\{x^{4w}\}$, $\{x^{4w+2}\}$ and $\{(x^{4w}-1),\ w=1,2,\ldots,14\}$ can be seen to be $\{16,12,9,22,47,20,15,57,58,13,25,34,56,42,1\}$, $\{4,3,48,36,27,5,19,60,45,49,52,39,14,41,46\}$ and $(2^{28},2^{15},2^{3},2^{55},2^{58},2^{26},2^{50},2^{52},2^{32},2^{8},2^{9},2^{21},2^{37},2^{54})$ respectively. So evidently $g = 8$. So the parameter sets of the corresponding association scheme and the PBWD are given by:

$$(n_1, n_2) = (30, 30) \; ; \quad p_1 = \begin{pmatrix} 14 & 15 \\ 15 & 15 \end{pmatrix} \; , \quad p_2 = \begin{pmatrix} 15 & 15 \\ 15 & 14 \end{pmatrix}$$

and $(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}) = (61,61,30,15;8,6,6,9)$.

THEOREM 4.6: Let $v = 4u+1$ be the power p^n of an odd prime, p . If $\{x^4, x^8, \ldots, x^{4u}\}$ forms a perfect difference set, then $\{x^4, x^8, \ldots, x^{4u}; x^2, x^6, x^{10}, \ldots, x^{4u-2}\}$ generates a PBWD, with $\{d_1, d_2, \ldots, d_{2u}\} = \{x^2, x^4, x^6, \ldots, x^{4u}\}$ defining the cyclic association scheme and with the following paramter sets of the association scheme and the design:

$$n_1 = n_2 = 2u\ , \quad P_1 = \begin{pmatrix} u-1 & u \\ u & u \end{pmatrix}\ , \quad P_2 = \begin{pmatrix} u & u \\ u & u-1 \end{pmatrix}$$

$$(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}) = (4u+1, 4u+1, 2u, u; \tfrac{u-1}{2}, \tfrac{u-1}{2}, \tfrac{u-1}{2}, \tfrac{u+1}{2})\ .$$

Proof: By Theorem 2.2 both half blocks of the initial block are difference sets, with $\lambda = \frac{u-1}{4}$ and hence $(\lambda_{11}, \lambda_{21}) = (\frac{u-1}{2}, \frac{u-1}{2})$. By Theorem 2.3, the elements $\{x^2, x^4, x^6, \ldots, x^{4u}\}$ form a partial difference set with $\{4u+1, 2u, 2u, u-1, u\}$. Hence among the opposite differences arising from the initial block, there are $u-1-(\frac{u-1}{2}) = \frac{u-1}{2}$ Q.R.'s and there are $u-(\frac{u-1}{2}) = \frac{u+1}{2}$ non Q.R.'s. Hence by the theorem (4.1)and the corollary (2.1), it follows that the initial block generates a PBWD and that $\{x^2, x^4, \ldots, x^{4u}\}$ defines the cyclic association scheme. Hence the theorem follows.

COROLLARY 4.2 If $v = 4u+1$ is a prime, where u is of the form $(2\ell+1)^2$, then $\{x^4, z^8, \ldots, x^{4u}; x^2, x^6, x^{10}, \ldots, x^{4u-2}\}$ generates a PBWD with the same association scheme as in the Theorem (4.6).

Proof: This follows from the Theorem 4.6 , using the result that under the above circumstances, $\{x^4, x^8, \ldots, x^{4u}\}$ forms a perfect difference set.

EXAMPLE 1: $u = 9$, $v = 4 \times 9 + 1 = 37$. 2 is a primitive root of 37.

$\{(x^4, x^8, \ldots, x^{36}), (x^2, x^6, \ldots, x^{34})\}$ can be written as

{16,34,26,33,10,12,7,1; 4,27,25,30,36,21,3,11,28}

$$n_1 = n_2 = 18\ ; \quad P_1 = \begin{pmatrix} 8 & 9 \\ 9 & 9 \end{pmatrix}\ ; \quad P_2 = \begin{pmatrix} 9 & 9 \\ 9 & 8 \end{pmatrix}$$

and $(v, b, r, p; \lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22}) = (37,37,18,9; 4,4; 4,5)$

which is true by direct verification also.

Remarks: The Section 4 can be concluded as an attempt to construct partially balanced weighing designs, with cyclic association schemes, under two different methods which form particular cases of two rules, mentioned below as Rule I and Rule II.

Rule I:

1. Association scheme:

   (a) G is a set of v elements $\alpha_0 \{= e\}$,

   $\alpha_1, \alpha_2, \cdots, \alpha_{v-1}$.

   (b) $G - \{e\} = E_1 \cup E_2$.

   (c) E_1 forms a partial difference set.

2. Design:

   (a) D_1 and D_2 are two disjoint sets, each containing p distinct elements from G .

   (b) If $D_i D_j^{-1}$ stands for the set of all differences formed by $a_{i\ell} - b_{jm}$, $a_{i\ell} \in D_i$ and $b_{jm} \in D_j$, then

   $$D_1 D_1^{-1} + D_2 D_2^{-1} = \{\lambda_{11} E_1 \cup \lambda_{12} E_2 + 2p\{e\}\} .$$

   (c) $D_1 D_2^{-1} + D_2 D_1^{-1} = \{\lambda_{21}(E_1)\} \cup \{(\lambda_{22})E_2\}$.

Rule I says that if such a pair D_1, D_2 can be found, then $(D_1; D_2)$ leads to a PBWD, provided (E_1, E_2) can be found with conditions described in 1.

Rule II:

1. Association scheme:

(a), (b), (c) are the same as in Rule I.

2. Design

(a) D_1 and D_2 are disjoint.

(b) D_1 and D_2 are perfect difference sets separately.

(c) $|D_1| = |D_2|$.

(d) $\{(\ell_1)E_1\} \cup \{(\ell_2)E_2\} = D_1 \cup D_2$, with $\ell_1 \neq \ell_2$.

Rule II says that if such a pair D_1, D_2 can be found, $(D_1; D_2)$ leads to a PBWD, with the association scheme given by (E_1, E_2) .

The parameter sets of the designs constructed under Rules I and II can be described in Tables 4.1 and 4.2.

Both of the Tables 4.1 and 4.2 describe some examples of partially balanced weighing designs with cyclic association scheme. But they describe constructions based on Rules I and II respectively. Tables 4.1(A) and 4.2(A) describe the parameter-sets for a few of the PBWD's of this type, whereas 4.1(B) and 4.2(B) describe the actual plans of such designs.

TABLE 4.1.(A)

S.No.	v	r	p	b	n_1	n_2	λ_{11}	λ_{21}	λ_{12}	λ_{22}
	4u+1	2u	u	4u+1	2u	2u	g	u-1-g	u-1-g	g+1
C 1	13	6	3	13	6	6	0	2	2	1
C 2	29	14	7	29	14	14	4	2	2	5
C 3	61	30	15	61	30	30	8	6	6	9

TABLE 4.2.(A)

S.No.	v	r	p	b	n_1	n_2	λ_{11}	λ_{21}	λ_{12}	λ_{22}
	$4x^2+1$	$2x^2$	x^2	$4x^2+1$	$2x^2$	$2x^2$	$\frac{x^2-1}{2}$	$\frac{x^2-1}{2}$	$\frac{x^2-1}{2}$	$\frac{x^2+1}{2}$
	37	18	9	37	18	18	4	4	4	5

TABLE 4.1.(B)

Design C1 : $v = 13,\ r = 6,\ p = 3,\ b = 13,\ n_1 = 6,\ n_2 = 6$

$\lambda_{11} = 0,\ \lambda_{21} = 2,\ \lambda_{12} = 2,\ \lambda_{22} = 1$

$P_1 = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$; $P_2 = \begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$

Association Scheme		Initial Block
Variety	1st Associate	(3,9,1; 4,12,10)
1	5,4,13,10,11,2	

Design C2 : $v = 29,\ r = 14,\ p = 7,\ b = 29,\ n_1 = 14,\ n_2 = 14$

$\lambda_{11} = 4,\ \lambda_{21} = 2,\ \lambda_{12} = 2,\ \lambda_{22} = 5$

$P_1 = \begin{matrix} 6 & 7 \\ & 7 \end{matrix}$; $P_2 = \begin{matrix} 7 & 7 \\ & 6 \end{matrix}$

Association Scheme		Initial Block
Variety	1st Associate	16,24,7,25,23,20,1; 4,6,9,28,13,5,22
1	5,17,7,25,10,8,29,26,14,24,6 21,23,2	

Design C3 : $v = 61,\ r = 30,\ p = 15,\ b = 61,\ n_1 = 30,\ n_2 = 30$

$\lambda_{11} = 8,\ \lambda_{21} = 6,\ \lambda_{12} = 6,\ \lambda_{22} = 9$

$P_1 = \begin{pmatrix} 14 & 15 \\ & 15 \end{pmatrix}$; $P_2 = \begin{pmatrix} 15 & 15 \\ & 14 \end{pmatrix}$

Association Scheme		
Variety	1st Associates	Initial Block
1	17,13,10,23,48,21,16,58,59	16,12,9,22,47,20,15
	14,26,35,57,43,2,5,4,49,37	57,58,13,25,34,56,42,1; 4,3,48,36,27,5,19,60,
	28,6,20,61,46,50,53,40,15	45,49,52,39,14,41,46
	42,47	

TABLE 4.2.(B)

Design C4 : $v = 37$, $r = 18$, $p = 9$, $b = 37$, $n_1 = 18$, $n_2 = 18$

$\lambda_{11} = 4$, $\lambda_{21} = 4$, $\lambda_{12} = 4$, $\lambda_{22} = 5$

$$P_1 = \begin{pmatrix} 8 & 9 \\ & 9 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 9 & 9 \\ & 8 \end{pmatrix}$$

Association Scheme

Variety	1st Associate
1	17,35,27,10,34,11,13, 8,2,5,28,26,31,37,22, 4,12,29

Initial Block

16,34,26,9,33,10,12, 7,1; 4,27,25,30,36,21 3,11,28

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