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SOME FURTHER CONSTRUCTIONS FOR $G_2(d)$ GRAPHS

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ABSTRACT

A $G_2(d)$ graph is a finite, undirected graph without loops or multiple edges in which each pair of vertices is adjacent to exactly d other vertices, $d \geq 2$. An infinite family of such graphs was given in [2]. The present paper gives some further constructions for these graphs.

1. KNOWN RESULTS

We use the notation and terminology of [2] and quote some results contained in this paper.

THEOREM A. A $G_2(d)$ graph, $d \geq 2$, is regular of valence n_1 such that $v - 1 = n_1(n_1 - 1)/d$ where v is the number of vertices and there exists a positive integer m such that

(i) $n_1 = d + m^2$, and

(ii) d/m is an integer with the same parity as $v - 1 - m$.

We note that a $G_2(d)$ graph with parameters (v, n, d), $d \geq 2$, is essentially a strongly regular graph with parameters $(v, n_1, p^1_{11}, p^2_{11})$ where $p^1_{11} = p^2_{11} = d$. By a pseudo $L_r(k)$ graph we will mean a pseudo net graph $L_r(k)$ and by an $NL_r(k)$ graph, a negative Latin Square $NL_r(k)$ graph.

THEOREM B. The existence of a pseudo $L_{r_1}(2r_1)$ and a pseudo $L_{r_2}(2r_2)$ graph implies the existence of a pseudo $L_r(2r)$ graph with $r = 2r_1r_2$.

THEOREM C. The existence of a pseudo $L_{r_1}(2r_1)$ and a $NL_{r_2}(2r_2)$ graph implies the existence of a $NL_r(2r)$ graph with $r = 2r_1r_2$.

THEOREM D. Pseudo $L_r(2r)$ and $NL_r(2r)$ graphs exist for $r = 3^m.2^{m+n-1}$, where m, n are nonnegative integers $(m, n) \neq (0, 0)$.

Noting that a pseudo $L_r(k)$ graph is a strongly regular graph with parameters $(k^2, r(k-1), k - 2 + (r - 1)(r - 2), r(r - 1))$ and a $NL_r(k)$ graph is strongly regular with parameters $(k^2, r(k + 1), -k - 2 + (r + 1)(r + 2), r(r + 1))$ we have

THEOREM E. A $G_2(d)$ graph with parameters

$$\text{(i) } v = 4r^2,\ n_1 = r(2r - 1),\ d = r(r - 1),$$

$$\text{(ii) } v = 4r^2,\ n_1 = r(2r + 1),\ d = r(r + 1),$$

$$\text{(iii) } v = 4r^2 - 1,\ n_1 = 2r^2,\ d = r^2,$$

exists for all $r = 3^m.2^{m+n-1}$, where m, n are non negative integers $(m, n) \neq (0, 0)$.

2. NEW CONSTRUCTIONS

We first prove some preliminary results.

Lemma 2.1. Let v, b, k, r be non negative integers, $k < v$ and $vr = bk$. Let there exist an incomplete block design D with v symbols (treatments) in b subsets (blocks) of size k such that any pair of treatments occurs together in at most one block. Then a necessary and sufficient condition that each block in D intersects precisely $k(r - 1)$ other blocks in D is that each treatment occurs exactly r times in D.

Proof. It is obvious that any two blocks in D intersect in at most one treatment. If each treatment occurs r times in D, then any block containing, say, treatments $t_1, t_2, \ldots, t_k$ intersects $(r - 1)$ other blocks containing t_i; $i = 1, 2, \ldots, k$. The sets of $(r - 1)$ blocks containing t_i and t_j are obviously disjoint, $i \neq j$. Hence this block intersects precisely $k(r - 1)$ other blocks necessarily in one treatment. This proves the sufficiency part of the theorem.

Now suppose that each block intersects precisely $k(r - 1)$ other blocks. Let r_i be the number of times the treatment i occurs in D. Then obviously

$$\bar{r} = \sum_{1}^{v} r_{i/v} = r.$$

Let $N = (n_{ij})$ be the usual (0, 1) incidence matrix of D with v rows and b columns where $n_{ij} = 1$ or 0 according as treatment i occurs in blocks j or not. Then from our hypothesis

$$N'N = kI_b + A$$

where A is an adjacency matrix of order b which is regular and of valence $k(r - 1)$. Also

$$NN' = \mathrm{diag}(r_1, \ldots, r_v) + B$$

where B is also an adjacency matrix of order v with i-th row sum $r_i(k - 1)$. Hence

$$\text{tr}\,((NN')(NN')) = \sum_1^v r_i^2 + (k-1)\sum_1^v r_i$$
$$= \sum_1^v r_i^2 + (k-1)vr.$$

But

$$\text{tr}\,((NN')(NN')) = \text{tr}\,(N(N'N)N')$$
$$= \text{tr}\,N(kI_b + A)N'$$
$$= k\ \text{tr}\,NN' + \text{tr}\,N\,A\,N'$$
$$= k\ \text{tr}\,NN' + \text{tr}\,A\,N'N$$
$$= k\ \text{tr}\,NN' + \text{tr}\,(A(kI_b + A))$$
$$= k\sum_1^v r_i + \text{tr}\,A^2$$
$$= k\ v\,r + bk(r-1)$$
$$= v\,r(k + r - 1).$$

Hence equating the two values of the trace

$$\sum_1^v (r_i - \bar{r})^2 = \sum_1^v r_i^2 - v\,\bar{r}^2$$
$$= \sum_1^v r_i^2 - vr^2$$
$$= 0$$

which implies that each treatment occurs r times in D. This completes the proof of the lemma.

We now define the concept of an ascendent graph G* of a strongly regular graph G with parameters $(v, n_1, p_{11}^1, p_{11}^2)$. Let (V_1, V_2) be a partition of the vertex set V of G where V_1 and V_2 respectively contain n_1^* and $v - n_1^*$ vertices. Let ∞ be a vertex not in V and let G* be a graph with vertex set $(\infty \cup V)$. We define adjacency in G* as follows: The vertex ∞ is adjacent only to vertices of V_1. If x, y are in V, then they are adjacent in G* if and only if they are adjacent in G and belong both to V_1 or both to V_2, or if they are nonadjacent in G and belong one to V_1 and the other to V_2. If the graph G^* is strongly regular with parameters $(v^*, n_1^*, p_{11}^{1*}, p_{11}^{2*})$ where v^* is necessarily $v + 1$, then G^* is said to be an ascendent of G.

We derive the conditions under which a graph G with parameters $(v, n_1, p_{11}^1, p_{11}^2)$ has an ascendent G^* with parameters $(v^*, n_1^*, p_{11}^{1*}, p_{11}^{2*})$. We will assume that G is neither a void graph nor a complete graph i.e. $n_1 \neq 0$ and $n_2 = v - 1 - n_1 \neq 0$.

If G^* is an ascendent of G, then G is a descendent of G^* with respect to the vertex ∞ and hence from [2]

$$p_{11}^{1*} + p_{11}^{2*} = 2n_1^* - \frac{v^*}{2}$$

$$= 2n_1^* - \frac{v+1}{2} \quad . \tag{2.1}$$

$$n_1 = 2n_1^* - 2p_{11}^{2*} \quad . \tag{2.2}$$

$$p_{11}^1 = n_1 - n_1^* + p_{11}^{1*} \quad . \tag{2.3}$$

$$p_{11}^2 = n_1 - n_1^* + p_{11}^{2*} \quad . \tag{2.4}$$

From (2.2), (2.4)

$$2p_{12}^2 = 2(n_1 - p_{11}^2)$$

$$= 2(n_1^* - p_{11}^{2*})$$

$$= n_1 .$$

Hence from the usual parametric relation

$$n_1/2 = p_{11}^2 = p_{12}^2 \tag{2.5}$$

which implies that

$$n_2/2 = p_{12}^1 = p_{22}^1 \tag{2.6}$$

Also from (2.1), (2.3) and (2.4)

$$v + 1 = 4n_1 - 2p_{11}^1 - 2p_{11}^2 .$$

or

$$v = 6\,p_{11}^2 - 2p_{11}^1 - 1. \tag{2.7}$$

Thus (2.7) is a necessary condition for G to have an ascendent. It is easy to see that for a graph G with parameters $(v, n_1, p_{11}^1, p_{11}^2)$ each of (2.5), (2.6), (2.7) implies the other two. It also follows from [2] that the vertex set V of G can be partitioned into (V_1, V_2) with n_1^* vertices in V_1 and $v - n_1^*$ vertices in V_2, where the set V_1 is the set of vertices in G* which are adjacent to ∞ . Further, each vertex in V_1 is adjacent to $p_{11}^{1*} = n_1^* - n_1 + p_{11}^1$ vertices of V_1 in G and each vertex in V_2 is adjacent to

$$\begin{aligned} p_{12}^{2*} &= n_1^* - p_{11}^{2*} \\ &= n_1 - p_{11}^2 \\ &= p_{11}^2 \end{aligned}$$

vertices of V_2 in G.

Conversely, suppose (2.7) is satisfied for G and further there exists a partition (V_1, V_2) of the vertex set V of G with n_1^* vertices in V_1 and $v - n_1^*$ vertices in V_2, such that each vertex in V_1 is adjacent to

$n_1^* - n_1 + p_{11}^1$ vertices in V_1 and each vertex in V_2 is adjacent to p_{11}^2 vertices in V_2. Then by using arguments similar to those in [2], it can be shown that G^* is a strongly regular graph with parameters (2.2), (2.3), (2.4) provided p_{11}^{1*}, p_{11}^{2*} are non negative integers. From the relation

$$n_1^* \; p_{12}^{1*} = n_2^* \; p_{11}^{1*}$$

it follows that n_1^* satisfies the equation

$$f(x) \equiv x^2 + x(p_{11}^1 - 5p_{11}^2) + v\, p_{11}^2 = 0.$$

The above equation is easily seen to have real positive roots. Since n_1^* is necessarily an integer a further necessary condition for G to have an ascendent G^* is that the above equation has an integral solution.

We easily verify that

$$f'(n_1 - p_{11}^1) = -(p_{11}^1 + p_{11}^2) < 0 .$$

$$f'(n_1 - p_{11}^2) = -(1 + p_{12}^1 + p_{11}^2) < 0 .$$

$$f(n_1 - p_{11}^2) = p_{11}^2 \, p_{12}^1 > 0 .$$

$$f(n_1 - p_{11}^1) = p_{11}^2 \, (p_{11}^1 - 1) .$$

Further, it is easily seen that if $p_{11}^1 = 0$, then $f(x) = 0$ has no integral solution. Hence, if the equation has an integral solution then we can assume that $p_{11}^1 \geq 1$ and then the above relations imply that

$$p_{11}^{1*} = n_1^* - n_1 + p_{11}^1 \geq 0.$$

$$p_{11}^{2*} = n_1^* - n_1 + p_{11}^2 > 0 .$$

Thus the condition that $f(x) = 0$ has an integral solution n_1^* is necessary and sufficient for nonnegativeness of p_{11}^{1*}, p_{11}^{2*}. We can, therefore, state the following theorem.

THEOREM 2.1. Let G be a strongly regular graph with parameters $(v, n_1, p_{11}^1, p_{11}^2)$. Then G has an ascendent G* with parameters $(v^*, n_1^*, p_{11}^{1*}, p_{11}^{2*})$ if and only if the following parametric and structural conditions (P) and (S) are satisfied in G.

(P) $v = 6\, p_{11}^2 - 2\, p_{11}^1 - 1$.

(S) The equation

$$x^2 + x\,(p_{11}^1 - 5\, p_{11}^2) + v\, p_{11}^2 = 0.$$

has an integral solution n_1^* and there exists a partition (V_1, V_2) of the vertex set V of G with n_1^* vertices in V_1 and $v - n_1^*$ vertices in V_2 such that every vertex in V_1 has $n_1^* - n_1 + p_{11}^1$ adjacent vertices in V_1 and every vertex in V_2 has p_{11}^2 adjacent vertices in V_2.

The parameters of G* are then given by

$$v^* = v + 1, \quad n_1^*, \quad p_{11}^{1*} = n_1^* - n_1 + p_{11}^1, \quad p_{11}^{2*} = n_1^* - n_1 + p_{11}^2.$$

It is obvious that any two blocks of a BIBD with $\lambda = 1$ have at most one treatment in common. Consider a BIBD with $r = 2k + 1$, $\lambda = 1$. Then the values of v and b are given by $v = 2k^2 - k$ and $b = 4k^2 - 1$. Consider the blocks as vertices of a graph G and define two blocks as adjacent or nonadjacent according as they have a treatment in common or not. Then [2] G is strongly regular with parameters $(4k^2 - 1, 2k^2, k^2, k^2)$ and satisfies the condition (P) of the above theorem. Also the equation $f(x) = 0$ has integral solutions $k(2k - 1)$ and $k(2k + 1)$. Take $n_1^* = k(2k - 1)$. If the $4k^2 - 1$ blocks can be partitioned into sets V_1 and V_2 of $k(2k - 1)$ and $(k + 1)(2k - 1)$ blocks respectively such that each block in V_1 is adjacent to $k^2 - k$ blocks in V_1 and each block in V_2 is adjacent to k^2 blocks in V_2, then the condition (S) is also satisfied. From Lemma 2.1 this means that the set V_1 (respectively V_2) contains each of the $2k^2 - k$ treatments exactly k (respectively k + 1) times.

We note that a BIBD with $r = 2k + 1$, $\lambda = 1$ is a partial geometry $(r, k, t) = (2k + 1, k, k)$. The graph G is then [1] the graph of the dual configuration and is also a partial geometry $(k, 2k + 1, k)$. We can, therefore, state the following theorem.

THEOREM 2.2. Let G be the graph of the dual of a BIBD with $r = 2k + 1$, $\lambda = 1$. Then G has an ascendent G* which is a pseudo $L_k(2k)$ graph if and only if the $4k^2 - 1$ blocks of the BIBD can be partitioned into sets V_1 and V_2 of $k(2k - 1)$ and $(k + 1)(2k - 1)$ blocks respectively such that each of the $2k^2 - k$ treatments of the BIBD occur k times in V_1 and k + 1 times in V_2.

BIBD's having the structure of the above theorem exist for k = 5 and 7. See for example Appendix I in [4]. Hence we have the following result.

COROLLARY. Pseudo $L_5(10)$ and pseudo $L_7(14)$ graphs exist.

Goethals and Seidel [3] have constructed a pseudo $L_5(10)$ graph in precisely the same manner.

Using the Corollary and Theorems B, C and D we have the following theorem

THEOREM 2.3. (I) pseudo $L_r(2r)$ graphs exist for all $r = 3^m \; 5^a \; 7^c \; 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers $(m, n, a, c) \neq (0, 0, 0, 0)$.

(II) $NL_r(2r)$ graphs exist for $r = 5^a \; 7^c \; 2^{a+c}$ where, a, c are non negative integers and for $r = 3^m \; 5^a \; 7^c \; 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers and $(m, n) \neq (0, 0)$.

The proof is similar to that of Theorem 9.3 and 9.5 in [2] and is omitted.

Finally, noting that pseudo $L_r(2r)$ and $NL_r(2r)$ graphs are $G_2(d)$ graphs we have

THEOREM 2.4. $G_2(d)$ graphs with the following parameters exist

(i) $v = 4r^2, \quad n_1 = r(2r - 1), \quad d = r(r - 1);$

(ii) $v = 4r^2 - 1, \quad n_1 = 2r^2, \quad d = r^2;$

for all $r = 3^m \; 5^a \; 7^c \; 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers $(m, n, a, c) \neq (0, 0, 0, 0)$.

(iii) $v = 4r^2, \quad n_1 = r(2r + 1), \quad d = r(r + 1);$

with $r = 5^a \; 7^c \; 2^{a+c}$, where a, c are non negative integers and with $r = 3^m \; 5^a \; 7^c \; 2^{m+a+c+n-1}$ where m, n, a, c are non negative integers and $(m, n) \neq (0, 0)$.

We remark that since our construction is essentially by a composition method, any new $G_2(d)$ graph can be utilised in conjunction with the above theorem to enlarge such a family considerably.

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