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CONSTRUCTION OF RULED SURFACES IN 5-DIMENSIONAL FINITE PROJECTIVE GEOMETRIES*

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1. Introduction. I want to describe, as briefly as possible, the methods for constructing one of my ruled surfaces in PG(5,q). Here q is any prime-power. However, the theory becomes trivial (and exceptional) for q=2. So assume

(1.1) $q \geqq 3$ (q a prime-power).

2. Preliminary. Nothing is said in my paper on ruled surfaces (the Luxembourg paper) about reguli. I need the following as background:

Lemma 2.1. Let S be a surface in $\Sigma = PG(5,q)$ ruled by 3 systems of planes, belonging, say, to classes I, II and III. Let L be a line lying in a ruling plane of (say) class I. Let $\mathcal{R}$ be the set of q+1 ruling planes of (say) class II each of which contains exactly one point of L. Then: (i) $\mathcal{R}$ is a regulus of planes of Σ.

(ii) Each of the q^2 and q+1 ruling planes of class I meets the q+1 planes of $\mathcal{R}$ in the q+1 points of a line -- a transversal line to $\mathcal{R}$.

(iii) Each of the q^2 and q+1 ruling planes of class III meets the q+1 planes of $\mathcal{R}$ in the q+1 points of a conic.

Proof. It is to be understood that S is constructed as in my Luxembourg paper. Set

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$$F = GF(q), \qquad K = GF(q^3),$$

and let $V = K \times K$ be the 6-dimensional vector space over F consisting of ordered pairs (x,y), $x,y \in K$, with

$$\left.\begin{aligned} (x,y) + (x',y') &= (x+x', y+y') \\ f(x,y) &= (fx, fy) \end{aligned}\right\} \begin{aligned} &\forall\ x,y,x',y' \in K \\ &\forall\ f \in F. \end{aligned}$$

Then S consists of all points $\langle(x,y)\rangle$ such that

$$N(x) = N(y);$$

equivalently

$$x^{q^2+q+1} = y^{q^2+q+1} .$$

The classes (I), (II), (III) are defined as the set of planes of the following sorts:

$$\left.\begin{aligned} &\text{(I)} & y &= kx \\ &\text{(II)} & y &= kx^q \\ &\text{(III)} & y &= kx^{q^2} \end{aligned}\right\} \quad \begin{aligned} &\text{For any fixed } k \text{ such that} \\ &N(k) = k^{q^2+q+1} = 1. \end{aligned}$$

For the proof, we may assume that L is the 2-dimensional vector space

$$L = \langle(1,1), \quad (a,a)\rangle$$

for some fixed a,

$$a \in K-F.$$

Then the $q+1$ points of L are the following: the point

$$\langle(1, 1)\rangle$$

and q points of form

$$\langle f(1,1) + (a,a)\rangle = \langle (f+a, f+a)\rangle, \quad f \in F.$$

The plane of class (II) through $\langle (1,1)\rangle$ is

$$\Pi_\infty: \; y = x^q.$$

The plane of class (II) through $\langle (f+a, f+a)\rangle$ is

$$\Pi_f: \; y = \frac{f+a}{f+a^q} x^q .$$

We may check that, for each $k \in K$, $k \neq 0$, the line

$$L_k = \langle (k,k^q), \; (ka,k^q a)\rangle$$

meets Π_∞ in $\langle (k,k^q)\rangle$ and Π_f in $\langle (k(a+f),k^q(a^q+f)\rangle$, and hence is a transversal to

$$R = \{\Pi_\infty\} \cup \{\Pi_f | f \in F\}.$$

Furthermore, L_k lies in the following plane of class I,

$$y = k^{q-1}x.$$

We also note that

$$L_k = L_{k'} \iff \frac{k'}{k} \in F,$$

so that the total number of distinct lines L_k is

$$\frac{q^3-1}{q-1} = q^2 + q + 1.$$

Thus: R is a regulus of (skew) planes of Σ:PG(5,q), with the q^2+q+1 lines L_k as its transversal lines.

Next consider a typical ruling plane of class III, say

$$\alpha: \quad y = kx^{q^2} ,$$

where k is some fixed element of K with

$$k^{q^2+q+1} = 1.$$

We can choose b (in $q-1$ ways) so that

$$k = b^{q-1}.$$

Thus the equation of α becomes

$$\alpha: \quad y = b^{q-1}x^{q^2}.$$

Then we may check that

$$\Pi_\infty \cap \alpha = \langle(b^{-q^2}, b^{-1})\rangle$$

$$\Pi_f \cap \alpha = \langle(b^{-q^2}(f+a)^{-q^2}, b^{-1}(f+a)^{-1}\rangle, \qquad \forall f \in F.$$

To complete the proof, we must show that the $q+1$ points so obtained lie on a conic (in α). If q is odd, it is enough to show that no three of the points lie on a line. Consider the special case of these points

$$\Pi_{f_i} \cap \alpha, \quad i = 1, 2, 3,$$

where f_1, f_2, f_3 are distinct elements of F. If these points lie on a line, we must have

$$u(b^{-q^2}(f_1+a)^{-q^2},\ b^{-1}(f_1+a)^{-1}) + v(b^{-q^2}(f_2+a)^{-q^2},\ b^{-1}(f_2+a)^{-1})$$
$$+ w(b^{-q^2}(f_3+a)^{-q^2},\ b^{-1}(f_3+a)^{-1})$$
$$= 0$$

for elements u, v, w of F, not all zero. Considering components, we get two equations for u, v, w which (after cancelling some non-zero factors) become

$$(1) \qquad (f_1+a)^{-q^2}u + (f_2+a)^{-q^2}v + (f_3+a)^{-q^2}w = 0$$

$$(2) \qquad (f_1+a)^{-1}u + (f_2+a)^{-1}v + (f_3+a)^{-1}w = 0.$$

We note that (2) implies (1). However, (2) (with u, v, w in F, not all zero) means that a satisfies a quadratic equation with coefficients in F. Since $a \in K-F$, and K is three dimensional over F, this is false.

Similarly, three points $\Pi_\infty \cap \alpha$, $\Pi_{f_1} \cap \alpha$, $\Pi_{f_2} \cap \alpha$, cannot be collinear.

We leave the proof of Lemma 2.1 at this point.

3. Construction of surfaces S. (Some algebraic details.)

We suppose given a regulus $\mathcal{R}$ of planes of $\Sigma = PG(5,q)$, consisting of $q+1$ skew planes Π;

$$\mathcal{R} = \{\Pi_i \mid i = 1,2,\ldots,q+1\}$$

such that each of the q^2+q+1 transversal lines to Π_1, Π_2, Π_3 meets every Π_i in a point. We also suppose given one more plane, Π, disjoint from the $q+1$ planes Π_i.

We consider the problem of constructing a triply-ruled surface S of Σ such that the set

$$\mathcal{R} \cup \{\Pi\}$$

of q+2 skew planes forms part of one of the three systems of q^2+q+1 ruling planes of S.

In the light of Lemma 2.1, we may assume that the planes of $\mathcal{R}\cup\{\Pi\}$ belong to Class II, and that every plane of Class I contains a (unique) transversal line to $\mathcal{R}$.

We may suppose that $\Sigma = PG(5,q)$ is given by a six-dimensional vector space V over $F = GF(q)$ with a basis $\ell_1, \ell_2, \ell_3, \ell_1', \ell_2', \ell_3'$ chosen so that

$$\Pi_1 = J(\infty) \text{ has basis } \ell_1, \ell_2, \ell_3$$

$$\Pi_2 = J(0) \text{ has basis } \ell_1', \ell_2', \ell_3',$$

$$\Pi_3 = J(I) \text{ has basis } \ell_1+\ell_1', \ell_2+\ell_2', \ell_3+\ell_3'.$$

Then every plane skew to $\Pi_1=J(\infty)$ has form J(X) for a unique 3×3 matrix $X=(x_{i})$ over F=GF(q), where

$$J(X) \text{ has basis } \begin{aligned} &x_{11}\ell_1 + x_{12}\ell_2 + x_{13}\ell_3 + \ell_1', \\ &x_{21}\ell_1 + x_{22}\ell_2 + x_{23}\ell_3 + \ell_2', \\ &x_{31}\ell_1 + x_{32}\ell_2 + x_{33}\ell_3 + \ell_3'. \end{aligned}$$

In particular, $\mathcal{R}$ consists of $J(\infty)$ and the J(fI), $f\in F$, where

$$fI = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix} .$$

Also

$$\Pi = J(U)$$

for some irreducible 3×3 matrix U.

Every vector a in $J(\infty)$ has form

$$a = a_1\ell_1 + a_2\ell_2 + a_3\ell_3$$

for unique elements a_1, a_2, a_3 in F. We define

$$a^X = (a_1x_{11} + a_2x_{21} + a_3x_{31})\ell_1 + (a_1x_{12} + a_2x_{22} + a_3x_{32})\ell_2 + (a_1x_{13} + a_2x_{32} + a_3x_{33})\ell_3$$

for every 3×3 matrix X. In this notation

$$J(X) \text{ has basis } \ell_1^X + \ell_1', \ \ell_2^X + \ell_2', \ \ell_3^X + \ell_3'.$$

Let a be a non-zero vector in $J(\infty)$, so that $\langle a\rangle$ is a point of the plane $J(\infty)$. The transversal line, L_a, to $\mathcal{R}$ through $\langle a\rangle$ is the two-dimensional vector space

$$L_a = \langle a, a'\rangle$$

where we define

$$a' = a_1\ell_1' + a_2\ell_2' + a_3\ell_3'.$$

A ruling plane, of our proposed surface S, which has Class (I) and contains L_a must (in particular) meet $\Pi = J(U)$ in a point, say in

$$\langle b^U + b'\rangle,$$

where b is some non-zero vector in $J(\infty)$. Then this ruling plane is (say)

$$\alpha = \langle a, a', b^U+b'\rangle .$$

The ruling plane α should meet each plane of R in a point, namely in a point of L_a. This puts some restrictions on the point $\langle b^U+b'\rangle$ of Π or, equivalently, the point $\langle b\rangle$ of $J(\infty)$. To see this, we note that each point of α has form $\langle v\rangle$ where v is a non-zero vector of form

$$v = fa + ga' + h(b^U+b'),$$

where f, g, h are elements of F, not all zero. Equivalently,

$$v = (fa+hb^U) + (ga+hb)'.$$

The point $\langle v\rangle$ will be in $J(\infty)$ if and only if

$$ga + hb = 0.$$

We wish this condition to imply $h = 0$. Thus we want

(1) $\qquad \langle b\rangle \neq \langle 0\rangle.$

If $t\in F$, so that $J(tI)$ is in R, the point $\langle v\rangle$ will be in $J(tI)$ if and only if

$$fa + hb^U = t(ga+hb)$$

or

$$h(b^U-tb) = (-f+tg)a$$

or

$$h\cdot b^{U-tI} = (-f+tg)a.$$

We want this equation to imply $h=0$. Thus we want

$$\langle b^{U-tI}\rangle \neq \langle a\rangle$$

or, equivalently,

(2) $\qquad \langle b\rangle \neq \langle a^{(U-tI)^{-1}}\rangle, \quad \forall\ t \in F.$

It may be shown that the point $\langle a \rangle$ of $J(\infty)$ together with the q points

$$\langle a^{(U-tI)^{-1}} \rangle, \quad t \in F$$

of $J(\infty)$ form a conic of $J(\infty)$. Hence, also, the corresponding points of Π, namely

$$\langle a^U + a' \rangle \quad \text{and} \quad \langle a^{(U-tI)^{-1}U} + (a^{(U-tI)^{-1}})' \rangle, \qquad t \in F,$$

form a conic of Π. Also the point $\langle b^U + b' \rangle$ must avoid the $q+1$ points of the latter conic. This gives

$$(q^2+q+1) - (q+1) = q^2$$

choices of the point $\langle b^U + b' \rangle$ so that

$$\alpha = \langle a, a', b^U + b' \rangle$$

meets each of the $q+2$ planes in $\mathcal{R} \cup \{\Pi\}$ in (exactly) a point.

It is easy to check that the conditions on b are equivalent to the following:

(3) $\qquad a, b, b^U$ form a basis of $J(\infty)$ over $F = GF(q)$.

However, since U is irreducible and $b \neq 0$, the vectors b, b^U, b^{U^2} form a basis of $J(\infty)$ over $F = GF(q)$. Hence (3) means that

$$\alpha = fb + gb^U + hb^{U^2}$$

for some f, g, h in F with $h \neq 0$. Equivalently (the case $h=1$)

(3') $\qquad \langle a \rangle = \langle f_0 b + g_0 b^U + b^{U^2} \rangle$

for some f_0, g_0 in F. (Note that the ordered pair f_0, g_0 can be chosen in q^2 ways.)

Assuming (3'), we can write the plane α as the plane $\alpha(b)$ where

$$(4) \qquad \alpha(b) = \langle f_0 b + g_0 b^U + b^{U^2},\ f_0 b' + g_0 (b^U)' + (b^{U^2}),\ b^U + b' \rangle .$$

Here b can be any non-zero vector in $J(\infty)$. If, now, for fixed f_0, g_0 in F, we consider all the planes $\alpha(b)$, we check easily that they form a set of

$$q^2 + q + 1$$

distinct, mutually skew, planes, each of which meets every plane of $R \cup \{\Pi\}$ in exactly a point, and each of which contains a unique transversal line to R, namely the line

$$\langle f_0 b + g_0 b^U + b^{U^2},\ f_0 b' + g_0 (b^U)' + (b^{U^2})' \rangle .$$

Although we are not supplying a proof here, these q^2+q+1 planes $\alpha(b)$ constitute the desired (complete) collection of planes of Class I.

We still have to supply the

$$q^2 + q + 1 - (q+2) = q^2 - 1$$

missing planes of Class II, and the q^2+q+1 planes of Class III.

4. Geometric approach. The material of Section 3 may be summarized as follows:

Suppose given a regulus, R, of $q+1$ skew planes of $\Sigma = PG(5,q)$ and a plane Π disjoint from each of the planes in R. Then, in precisely q^2 distinct ways, we can set up a one-to-one correspondence

$$(4.1) \qquad L \leftrightarrow P$$

between the q^2+q+1 transversal lines L to R are the q^2+q+1 points P of Π such that:

(i) If L, P are a corresponding pair, the plane $L+P$ intersects Π in P and intersects each plane Π_i of R in the point $L\cap\Pi_i$.

(ii) The q^2+q+1 planes $L+P$ obtained by letting L, P range over all corresponding pairs are structually skew.

This is essentially all we need to complete the construction of the surface S.

We assume that some fixed correspondence (4.1) has been chosen. Consider any corresponding pair L, P. We note first that the projective space

(4.2) $$H_p = L + \Pi$$

is four-dimensional and hence is a hyperplane of $\Sigma = PG(5,q)$. Next we construct a projective 3-space,

$$T_p,$$

in the following manner: Let Π_1, Π_2 be two distinct planes of R and let M be the unique transversal line through P to Π_1, Π_2, meeting Π_1, Π_2 in points P_1, P_2 respectively. Let L_1, L_2 be the transversal lines to R through P_1, P_2 respectively. Note that $L_1 \neq L_2$, else the point P of Π would lie on a transversal to R and hence on one of the planes of R, a contradiction. Since $L_1 \neq L_2$, then the space

(4.3) $$T_p = L_1 + L_2$$

is a projective 3-space. Since M contains P, P_1, P_2 and since P_1 is on L_1, P, is on L_1, then T_p contains P and M is the transversal through P to L_1, L_2 in T_p.

We need to note the following:

(a) The 3-space T_p depends only on $\mathcal{R}$ and P, not on Π or on the choice of the planes Π_1, Π_2 of $\mathcal{R}$.

(b) T_p meets each plane of $\mathcal{R}$ in a line, giving q+1 such lines, and contains precisely q+1 transversals to $\mathcal{R}$, namely the transversals to the first set of lines. These two sets of q+1 lines form the two sets of rulings of a doubly-ruled quadric

$$\mathcal{Q}_p$$

of T_p; and $\mathcal{Q}_p$ depends only on P and $\mathcal{R}$.

(c) P lies on exactly q+1 planes to $\mathcal{Q}_p$ in T_p. Each of these contains exactly one transversal line to $\mathcal{R}$ and meets exactly one plane of $\mathcal{R}$ in a line. Each of the remaining q^2 planes of T_p through P meets $\mathcal{Q}_p$ in a conic, contains no transversal line to $\mathcal{R}$, and meets no plane of $\mathcal{R}$ in a line.

(d) $T_p \cap \Pi = P$.

Next we need the following:

(4.4) $T_p \cap L$ is the empty point-set.

(4.5) $III_p = T_p \cap H_p$ is a plane (disjoint from L).

To prove (4.4), first suppose that L is contained in T_p. Then P+L is a tangent plane to $\mathcal{Q}_p$ and hence meets some plane of $\mathcal{R}$ in a line. This is a contradiction. Hence L is not in T_p. Next suppose that L meets T_p in a point P'. (Since P is not on L, necessarily $P' \neq P$.) Since P' is on L, then P' is on some plane, say Π_i, of $\mathcal{R}$. Since P' is in $\Pi_i \cap T_p$, the P' is on $\mathcal{Q}_p$. In particular, the transversal line through P' to $\mathcal{R}$ is a ruling of $\mathcal{Q}_p$, and is in T_p. But this transversal, being the unique transversal line through P' to

$\mathcal{R}$, must be L. Hence L is in T_p, a contradiction. Therefore, (4.4) must be true.

In view of (4.4), the projective space T_p+L has dimension $3+1-(-1) = 5$. Therefore,

$$(4.6) \qquad T_p + L = \Sigma.$$

Since H_p contains L, we see from (4.6) that

$$T_p + H_p = \Sigma.$$

Hence

$$\dim(III_p) = \dim T_p + \dim H_p - \dim \Sigma = 3 + 4 - 5 = 2.$$

Thus III_p is a plane. Moreover

$$III_p \cap L = T_p \cap H_p \cap L$$

is empty by (4.4). This proves (4.5). From (4.3) we get

$$(4.7) \qquad III_p + L = H_p.$$

Indeed, the left-hand side of (4.7) is contained in the right-hand side, and both sides have projective dimension 4. From (4.7) and the fact that P is in III_p we get

$$III_p + (P+L) = H_p$$

and hence

$$(4.8) \qquad III_p \cap (P+L) = P.$$

Next we need

(4.9) If Σ_4 is a projective 4-space of Σ containing Π, then Σ_4 contains one and only one transversal line to $\mathcal{R}$.

To see this, we note that there are precisely q^2+q+1 distinct transversal lines to $\mathcal{R}$ and precisely q^2+q+1 distinct projective 4-spaces of Σ containing Π. If (4.9) is false, there must be a Σ_4 which contains Π but contains no transversal line to $\mathcal{R}$. Let Σ_4 be such a 4-space, and let Π_1 be one of the planes of $\mathcal{R}$. Then Σ_4 intersects Π_1 is a line, say M. The q+1 transversal lines to $\mathcal{R}$ through the points of M, each meet Σ_4 in a single point, manely a point of M_1. The same process, carried out for the q+1 distinct planes of $\mathcal{R}$, shows that there must be at least

$$(q+1)^2 > q^2+q+1$$

distinct transversal lines to $\mathcal{R}$. This is a contradiction. Hence (4.9) is true. As a special case of (4.9),

(4.10) L is the only transversal line to $\mathcal{R}$ contained in $H_p=L+\Pi$.

By (4.10) and (4.4), (4.5),

(4.11) The plane III_p contains no transversal line to $\mathcal{R}$.

In view of (4.11), III_p meets the q+1 planes of $\mathcal{R}$ in the points of a conic. (The conic lies on $\mathcal{Q}_p$.)

The q^2+q+1 planes III_p, one for each point P of Π, are our candidates for the ruling planes of Class III. (Compare Lemma 2.1.) We omit the proof that every two of these are disjoint.

Of course, the planes of class I are the planes

$$I_p = P + L$$

where L, P is a corresponding pair in the sense of (4.1). It has to be shown that every plane of class I meets every plane of class III in a point.

In addition, we have in class II only the set

$$\mathcal{R} \cup \{\Pi\}$$

of $q+2$ planes. But it should be clear at this point that the missing q^2-1 planes are uniquely determined by the planes of Classes I and III. We have only to consider Lemma 2.1 with classes I, II, III replaced (for example) by classes III, I, II respectively.

I will stop here. Note that one must prove that the construction can actually be completed consistently. (This is, in fact, true.)