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ON THE PROBLEM OF CONSTRUCTION AND UNIQUENESS OF SATURATED 2_R^{K-p} DESIGNS[1]

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1. INTRODUCTION AND SUMMARY. The terminology "saturated designs in 2_R^{K-p} runs" was introduced by Box, Hunter, Draper and Mitchell [5, 6, 7]. However, the concept is not new. In other terminology, these designs are known as orthogonal arrays of strength $t = R-1$, 2^{K-p} columns, and K rows. The value p corresponds to the number of generators used for the construction. In the usual terminology of orthogonal arrays, p denotes the number of points beyond the points belonging to the identity matrix which are no-t-dependent in the projective space under consideration. The uniqueness and optimal size properties of orthogonal arrays discussed by Box, Hunter, Draper and Mitchell refer only to arrays constructable by geometrical methods. There is no assurance that they hold in general.

It is the purpose of this note to show that the construction of the arrays carried out by the above mentioned authors and the proof of uniqueness can be established using known geometrical methods without the help of the computer. The method used here is based on the evaluation of the function $m_4(r,2)$, the maximum number of no-4-coplanar points in $PG(r-1,2)$.

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The exact value of $m_4(r,2)$ is obtained for $r \le 8$ and an upper bound is computed for $r \ge 8$. The results have applications in error-correcting code problems and factorial designs.

2. THE VALUES OF $m_4(r,2)$ FOR $r \le 7$ AND A BOUND FOR $r \ge 8$.

We may assume without loss of generality that the set of maximum number of no-4-coplanar points in $PG(r-1,2)$ includes the rows of the identity matrix I_r. Furthermore, this implies that any additional point belonging to the set must consist of at least four ones. Thus we may conclude immediately that $m_4(4,2) = 5$, since one can add to the points of I_4 only the point $(1, 1, 1, 1)$. This could also be determined geometrically. Take any two points belonging to the set. Through the line determined by these points pass exactly three planes. One can choose one point on each of these planes since a line joining two points of any two planes will eliminate only one point of the third plane and leaves three more points on this plane untouched. Each of these three points could be added. Now multiplying the matrix

$$\begin{pmatrix} I_4 \\ 1\ 1\ 1\ 1 \end{pmatrix}$$

by the 4×2^4 matrix of all the distinct vectors of size four, we obtain the unique design 2^4_5. However, the complement of the vector space of all possible five-tuples also satisfies the condition of the design. Uniqueness in this case should be interpreted up to interchanging the names of the elements.

Next, we notice that $m_4(5,2) = 6$ since again we may add to I_5 only one vector, either (1,1,1,1,1) or (1,1,1,1,0). However, the uniqueness of the design does not hold in this case. Multiplying the matrix I_5 augmented by each of the vectors mentioned with the matrix of all possible five-tuples, we obtain indeed two distinct designs 2_5^5 which cannot be obtained from each other by either interchanging the rows, columns, or renaming the elements.

Proceeding in the above manner, we see that $m_4(6,2) = 8$. In this case, one can add to I_6 either the pair of vectors (1,1,1,1,1,0) and (1,1,1,0,0,0) or the pair (0,0,1,1,1,1,) and (1,1,0,0,1,1). In this case, the design 2_5^{K-p} with K=8, p=2 is unique. This follows from the fact that the line determined by the points (1,1,1,1,1,0,1,0) and (1,1,1,0,0,1,0,1) is the same as the line passing through the points (0,0,1,1,1,1,1,0) and (1,1,0,0,1,1,0,1) up to interchanging the coordinates. The uniqueness of the one-dimensional subspace of the eight-dimensional vector space implies clearly the uniqueness of the complementary subspace forming the design 2_5^6.

Presently, it will be shown that $m_4(7,2) = 11$. To prove this, let us consider all possible sets of no-4-coplanar points which can be added to I_7. It is easy to check that up to interchanging the coordinates there are exactly five such distinct sets, each consisting of four points. These are the following sets.

(I)	(II)	(III)	(IV)	(V)
1 1 1 1 1 1 1	1 1 1 1 1 1 0	1 1 1 1 1 1 0	1 1 1 1 1 0 0	1 1 1 1 1 0 0
0 0 0 1 1 1 1	1 1 1 1 0 0 1	0 0 0 1 1 1 1	1 1 1 0 0 1 1	1 0 0 0 1 1 1
0 1 1 0 0 1 1	1 1 0 0 1 1 1	0 1 1 0 0 1 1	0 1 0 1 0 1 1	0 1 0 1 1 1 0
1 0 1 0 1 0 1	0 1 0 1 0 1 1	1 1 0 1 0 0 1	1 1 0 0 1 0 1	0 0 1 1 1 0 1

This enumeration was obtained by observing the following points:

(i) If the point consisting of all ones is added, the remaining three points must have exactly four ones.

(ii) If a point having exactly one zero is added, then the remaining three points may consist of one point having five ones and two points having four ones or all three having four ones.

(iii) No more than two points can have five ones.

(iv) The maximum number of points in any subspace cannot exceed the previously established bounds.

The automorphism groups of the three-dimensional projective subspaces generated by each of the sets of the four non-coplanar points was computed. It may be interesting to notice that the automorphism group of the first set is the simple group of order 168.

It may also be worthwhile to exhibit algebraic representations of the five sets of four non-coplanar points; they are as follows:

I

$$\begin{aligned} x_4 + x_5 + x_6 + x_7 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 + x_2 + x_4 + x_7 &= 0 \end{aligned}$$

II

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_3 + x_6 + x_7 &= 0 \\ x_4 + x_5 + x_7 &= 0 \end{aligned}$$

III

$$\begin{aligned} x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 + x_6 + x_7 &= 0 \\ x_2 + x_5 + x_7 &= 0 \end{aligned}$$

IV

$$\begin{aligned} x_1 + x_4 + x_7 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_3 + x_4 + x_6 &= 0 \end{aligned}$$

V

$$\begin{aligned} x_1 + x_3 + x_7 &= 0 \\ x_1 + x_2 + x_6 &= 0 \\ x_2 + x_5 + x_7 &= 0 \end{aligned}$$

where x_i denotes the i-th coordinate of the point.

Next we shall establish the uniqueness of the design 2^7_5. This is equivalent to establishing the uniqueness of the orthogonal three-dimensional projective subspaces generated by four points obtained from each of the sets after adjoining I_4 to them and thus obtaining four eleven-dimensional points in each case. An algebraic representation of the first set can be described as follows:

$$\begin{aligned} x_4 + x_5 + x_6 + x_7 &= 0 \\ x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 + x_2 + x_4 + x_7 &= 0 \\ x_1 + x_8 + x_{11} &= 0 \\ x_2 + x_8 + x_{10} &= 0 \\ x_6 + x_7 + x_{11} &= 0 \end{aligned}$$

It is easy to check that the remaining four sets can be obtained from the first set using the following mappings:

II	(1) (4) (6,7) (3,9,2,8,10,5,11)
III	(2) (4) (6,7) (3,10) (5,9) (1,8,11)
IV	(3) (5) (1,9) (2,4) (8,10) (6,7,11)
V	(1,7,6,10,2,4,8,11,9,3,5)

Hence the uniqueness of the design is established.

We shall now make use of the enumeration of the sets belonging to PG(6,2) to show that $m_4(8,2) = 17$ and that

$$m_4(r,2) \le 3(2^{r-6}-1) + 8 \quad \text{for} \quad r \ge 8.$$

This result follows from the following proposition.

All the enumerated sets of four points added to I_7 include a five-dimensional projective space containing the maximum number of no-4-coplanar points, i.e., 8 points. Even if any three points of each of the sets of four points is added to I_7 the same situation will prevail. If any two independent points are added to I_7, the resulting set will contain a five-dimensional projective space containing 7 no-4-coplanar points.

We are now ready to establish the bound for $m_4(r,2)$, $r \geq 8$.

We may assume because of the stated proposition that PG(r-1,2) includes a five-dimensional projective subspace containing 8 points. Through this five-dimensional subspace pass $2^{r-6}-1$ six-dimensional projective subspaces of which none can include more than 11 no-4-coplanar points. Hence

$$m_4(r,2) \leq 3(2^{r-6}-1) + 8 \quad \text{for} \quad r \geq 8.$$

This gives, for $r=8$, $m_4(8,2) = 17$.

We shall exhibit next a set of 17 no-4-coplanar points in PG(7,2). They are:

$$I_8$$

```
1 1 1 1 1 1 1 0
1 1 1 1 1 0 0 1
0 0 0 1 1 1 1 1
1 1 0 0 0 1 1 1
1 0 1 0 1 0 1 1
0 1 1 1 0 1 1 1
1 0 0 1 1 0 1 0
0 0 1 0 1 1 0 1
0 1 0 0 1 0 1 1
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The properties of this set are under investigation in order to obtain, if possible, a characterization of the maximum number of no-4-coplanar points in PG(7,2) and, hopefully, in higher spaces. Such a characterization would also yield a better bound for $m_4(r,2)$, $r \geq 9$.

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