

UNIQUELY PARTIALLY ORDERABLE GRAPHS

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I. INTRODUCTION

For any binary relation R on a set A one may define a graph $G(R)$ by taking A as the vertex set of $G(R)$ and joining two vertices x, y if and only if $(x R y) \vee (y R x)$. $G(R)$ is undirected, but may have loops. A well-known type of problem is the characterization of the graphs associated in this manner with a specific kind of relation. For example, the graphs of strict (= irreflexive) partial orders (PO-graphs) have been characterized by Ghouilà-Houri [1, 2] and Gilmore-Hoffman [3], while characterizations of the graphs of semi-orders (SO-graphs) and indifference systems (I-graphs) were given by Roberts [5]. All these characterizations are of the Kuratowski-type, i.e., they are given in terms of certain subgraphs the graphs under consideration must not contain.¹ Given a PO-graph G , Gilmore and Hoffman also provide an algorithm for obtaining a partial order R for which $G(R) = G$. In Section II, we somewhat simplify this algorithm.

If R is symmetric, then $G(R)$ completely determines R . But if R is not symmetric, then not only do we have $G(R) = G(\overset{U}{R})^2$, but there may be other relations R' with $G(R') = G(R)$. We shall call a graph G uniquely partially orderable (UPO) or uniquely semi-orderable (USO) if G is a PO-graph or SO-graph, and if R and R' are two relations such that $G = G(R) = G(R')$, then $R' = R$ or $R' = \overset{U}{R}$. In Section II, we will investigate in which way directions may be forced by already existing orientations. Section III shall be

¹These three types of graphs are special cases of the class of perfect graphs, which, too, have been successfully characterized in the sense of Kuratowski [6].

² $\overset{U}{R}$ stands for the inverse relation of R .

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devoted to semi-orders and related structures, and we give a characterization of USO-graphs there. In Section IV, we discuss partial orders and present a sufficient condition for PO-graphs to be uniquely partially orderable.

NOTATION:

1. Let G be a graph, then $\underline{V(G)}$ denotes its vertex set, $\underline{E(G)}$ its edge-set.
2. Let $S \subset V(G)$, then $\underline{[S]}$ shall denote the full subgraph or subgraph induced by S , i.e., the sub graph which contains all edges of G between any two points of S .
3. $\underline{C_k}$ stands for the cycle of length k (without chords), $\underline{K_\ell}$ for the complete graph on ℓ points, $\underline{K_{m,n}}$ for the complete bipartite graph on m and n vertices.
4. An isolated point will be called trivial component.

In this paper, a graph is assumed to be finite undirected without loops or multiple edges, all sets considered are finite.

II. STRONG AND TRANSITIVE FORCINGS

Let us begin by giving the Gilmore-Hoffman characterization of PO-graphs.

Definition 1: A generalized path of length $k-1$ in a graph G is a progression of vertices a_1, a_2, \dots, a_k , $a_i \in V(G)$, $(a_i, a_{i+1}) \in E(G)$, such that no two ordered pairs (a_i, a_{i+1}) , (a_j, a_{j+1}) are the same (i.e., an edge may be traversed twice, but at most once in either direction). In case $a_1 = a_k$, we speak of a generalized cycle.

Definition 2: Given the generalized path a_1, \dots, a_k , an edge (a_i, a_j)

with $|j - i| \geq 2$ is called a chord, if $|j - i| = 2$, we speak of a triangular chord. Similar definitions hold for generalized cycles, modulo $k - 1$, of course.

THEOREM [3]. G is a PO-graph iff G contains no generalized cycle of odd length without triangular chords.

In their proof, Gilmore and Hoffman actually design an algorithm and show that one arrives at a partial ordering by assigning a direction to an arbitrary edge, orienting all edges whose direction is determined by previously oriented edges, choosing again an arbitrary undirected edge, etc. Since the direction of an edge may be forced in two possible ways, we give the following definitions.

Definition 3: The generalized path a_1, \dots, a_k is called a strong path from (a_1, a_2) if it is without triangular chords. The direction of the edge (c, d) is said to be strongly forced by the direction of (a, b) if there exists a strong path $a_1, a_2, \dots, a_{k-1}, a_k$ with $(a, b) = (a_1, a_2)$, $(c, d) = (a_{k-1}, a_k)$.

Definition 4: $(a \rightarrow b)$ means the edge (a, b) has the orientation $a \rightarrow b$.

The direction of the edge (a, c) is said to be transitively forced by the directions of (a, b) and (b, c) , if either $(a \rightarrow b), (b \rightarrow c)$ or $(b \rightarrow a), (c \rightarrow b)$.

Definition 5: $G(a, b)$ denotes the set of all edges whose direction is forced by an initial orientation of (a, b) and subsequent strong and transitive forcings.

LEMMA 1. If $(c, d) \in G(a, b)$, then $G(c, d) = G(a, b)$.

Proof: The hypothesis clearly implies $G(c, d) \subset G(a, b)$, hence we only have to verify the inverse inclusion. If $(a, b) \in G(c, d)$, then we are done, so

suppose $(a, b) \notin G(c, d)$, and assume without loss of generality that $(a \rightarrow b)$ implies $(c \rightarrow d)$. Starting the Gilmore-Hoffman algorithm now with the edge (c, d) and choosing the direction $(d \rightarrow c)$ we may assign $(a \rightarrow b)$ by the nature of the algorithm, thus contradicting the hypothesis.

Lemma 1 gives a partitioning of $E(G)$ into orientation-blocks where the number of blocks represents the degree of freedom we have in constructing our partial order. Hence, we have the following

COROLLARY 1. Let G be a PO-graph with m orientation-blocks. Then the number of partial orders R with $G(R) = G$ is 2^m . G is a UPO-graph iff $m = 1$.

It is to be noted, however, that some or all of these partial orders may be isomorphic.

THEOREM 1. In an orientation-block $G(a, b)$, the direction of any edge strongly forces the direction of every other edge, in other words, transitive forcing can be replaced by strong forcing.

Proof: Since strong forcing is easily seen to be a symmetric and transitive property, we are able to partition $G(a, b)$ into classes H_i with two edges being in the same class iff their directions force each other strongly. Let $G(a, b) = H_1 + H_2 + \dots + H_t$, we then wish to show $t = 1$. Assume the opposite and let us start orienting the edges of H_1 . Among the edges in $G(a, b) - H_1$ there must be one whose direction is transitively forced by the directions of two edges in H_1 . Let $(c, e) \in H_2$ be such an edge and assume w.l.o.g. that the two corresponding edges (c, d) and (d, e) in H_1 have been directed $(c \rightarrow d)$ and $(d \rightarrow e)$.

We first prove that for any other edge $(f, g) \in H_2$ with, say, $(f \rightarrow g)$, there exists a pair of edges $(f, d), (d, g) \in H_1$ with directions $(f \rightarrow d), (d \rightarrow g)$, i.e., the direction of every edge in H_2 is transitively forced by the directions of a pair of edges in H_1 which are joined at d . We proceed by finite induction on the length of a strong path from (c, e) in H_2 . Suppose the assertion is correct for edges in H_2 that can be reached by a strong path of length $k-1$ and let $(f, g) \in H_2$ such that $a_1, a_2, \dots, a_{k-1}, a_k, a_{k+1}$ is a strong path in H_2 with $(a_1, a_2) = (c, e), (c \rightarrow e), (a_k, a_{k+1}) = (f, g), (f \rightarrow g)$. We may assume $a_k = f, a_{k+1} = g$, then by the induction hypothesis we know there are edges $(a_{k-1}, d), (f, d)$ in H_1 with $(d \rightarrow a_{k-1})$ and $(f \rightarrow d)$, hence the edge (d, g) must exist in G , since otherwise $(f \rightarrow d)$ would force $(f \rightarrow g)$, contrary to the assumption. Finally, since $(a_{k-1}, g) \notin E(G)$, we conclude that $(d \rightarrow a_{k-1})$ forces $(d \rightarrow g)$, and the assertion follows by induction.

Now, suppose we orient $G(a, b)$ by directing H_2 first. It follows from $H_2 \subsetneq G(a, b)$ that there are edges in $G(a, b) - H_2$ whose direction is forced transitively by the directions of a pair of edges in H_2 . This, however, implies the existence of two edges in H_2 with a common endpoint, say z , and transitive orientations, which in turn would imply that (d, z) is oriented both ways, a contradiction.

Using Theorem 1, we now obtain the following simplification of the Gilmore-Hoffman algorithm:

Let G be a PO-graph. We may find a partial ordering R such that $G(R) = G$ by the following inductive procedure, based on the number of orientation blocks of G .

- (1) Choose an arbitrary edge (a_1, b_1) of G and let E_1 be the set of all edges in G which can be reached by a strong path from (a_1, b_1) . Orient (a_1, b_1) and give the other edges of E_1 the

direction forced by the initial orientation of (a_1, b_1) . Proceed to (2).

- (2) If $E(G) - \sum_{i=1}^{n-1} E_i$ is empty, stop. If it is not empty, let (a_n, b_n) be an arbitrary edge in this set and define E_n as the set of all edges of G on a strong path from (a_n, b_n) . Give (a_n, b_n) a direction and assign to the other edges of E_n the direction forced by the orientation of (a_n, b_n) . Proceed to (2).

THEOREM 2. A graph G with at most one non-trivial component and no triangles is a UPO-graph iff it is a PO-graph.

Proof: Every two edges are on a path, and every path is a strong path.

III. USO-GRAPHS

Definition 6 [4]: The binary relation P is called a semi-order on A iff for all $x, y, z, w \in A$

- (a) $\neg x P x,$
 (b) $(x P y \wedge z P w) \Rightarrow (x P w \vee z P y),$
 (c) $(x P y \wedge y P z) \Rightarrow (x P w \vee w P z).$

A semi-order clearly is asymmetric and transitive, hence a (strict) partial order. Furthermore, it readily follows from (b) that a SO-graph has at most one non-trivial component.

As it is our object to determine the structure of USO-graphs and UPO-graphs, we note first that a graph G of either type can have at most one non-trivial component. Next, we are going to prove a theorem about the complement G^c which also applies to both classes of graphs.

THEOREM 3. Let G be a UPO- or USO-graph and assume G^c is not connected, then $G \cong K_{m,n}$ for suitable m, n .

Proof: We first show that G^c possesses at most two components. Assume the opposite and let the components of G^c have vertex sets A_1, \dots, A_n . Orient the edges of the $[A_i]$'s arbitrarily (e.g., the orders induced by, respectively, a partial ordering or semi-ordering of G). Every point of A_i is joined to every point of A_j , $i \neq j$. We orient these edges as follows: for $i < j$, $i \leq n-2$, orient the edges from $[A_i]$ to $[A_j]$. It is easy to see that up to this point we do have a partial order (semi-order). We now obtain two different orderings by directing the remaining edges (a) from $[A_{n-1}]$ to $[A_n]$, (b) from $[A_n]$ to $[A_{n-1}]$. Again it is easy to check that both orderings are well defined.

Now let G^c have two components with point-sets A_1 and A_2 . If either $[A_1]$ or $[A_2]$ contains an edge, then we obtain two different orderings by again orienting the lines of $[A_1]$ and $[A_2]$ arbitrarily, and then directing the remaining lines (a) from $[A_1]$ to $[A_2]$, and (b) from $[A_2]$ to $[A_1]$. Finally, we remark that $K_{m,n}$ clearly is both a UPO- and a USO-graph, and the proof is complete.

THEOREM 4. A SO-graph is a USO-graph if and only if it is either isomorphic to $K_{m,n}$ for some m and n or if its complement is connected.

In order to prove Theorem 4, we need the definition of an indifference system and two theorems due to Roberts, which we will state below as Lemmas 2 and 3.

Definition 7: A binary relation I on a set A is called an indifference system if there exists a real-valued function f on A and a real number $\delta > 0$ such that $x I y \iff |f(x) - f(y)| \leq \delta$. We call f a defining function for I .

We note that I is clearly symmetric and hence its graph $G(I)$ completely defines I .

LEMMA 2 [5]. G is a SO-graph iff G^c defines an indifference system I . Moreover, if P is a semi-order whose graph is G , then there exists a defining function f for I such that $a P b \implies f(a) < f(b)$.

LEMMA 3 [5]. Let I be an indifference system on a set A , and let $G(I)$ be connected. Define $a, b \in A$ to be equivalent if (1) $a I b$ and (2) $a I c \iff b I c$ for $c \neq a, b$. Then if f and g are two defining functions for I and if there exists a pair of non-equivalent elements $a, b \in A$ such that $f(a) < f(b)$ and $g(a) < g(b)$, then for every two non-equivalent elements $x, y \in A$ we have $f(x) < f(y) \iff g(x) < g(y)$.

Proof of THEOREM 4: Let G be a SO-graph. If G^c is not connected, then G is USO iff $G \cong K_{m,n}$ (by Theorem 3). If on the other hand G^c is connected, let P, P' be two semi-orders with $G = G(P) = G(P')$ and $P' \neq P, P' \neq \bigcup P$. Then there exist a, b, c, d such that $a P b, a P' b, c P d, d P' c$ and hence, by Lemma 2, two defining functions f, f' for the indifference system defined by G^c such that $f(a) < f(b), f'(a) < f'(b), f(c) < f(d), f'(d) < f'(c)$. As clearly neither a and b nor c and d are equivalent, we obtain a contradiction to Lemma 3. Hence G is a USO-graph.

IV. UPO-GRAPHS

We begin this section by quoting a theorem due to Marley.

THEOREM [5]. H is an I-graph iff

- (a) H contains no $K_{1,3}$'s,
- (b) H contains no C_4 's,
- (c) H is the complement of a PO-graph.

In (a), (b) (and for the remainder of the paper) all subgraphs contained in a graph are full subgraphs.

This result together with Theorem 4 gives the following

COROLLARY 2. A sufficient condition for a PO-graph to be uniquely partially orderable is that its complement be connected and contain neither a C_4 nor a $K_{1,3}$.

We wish to remove the restriction that G^c does not contain a C_4 . But if we do this, we may obtain graphs G with more than one non-trivial component, and such a graph is clearly not UPO. The purpose of this section is to show that if we confine ourselves to graphs with only one non-trivial component we can indeed omit the C_4 condition. We must first prove a number of lemmas.

LEMMA 4. Let G be a PO-graph without $K_{1,3}$'s in the complement. Let $[a, b, c, d]$ be an arbitrary C_4 in G^c . If we delete the edge (a, c) (or (b, d)) from G , then the resulting graph G' is again a PO-graph and contains no $K_{1,3}$ in its complement.

Proof: Let us first verify that G' is a PO-graph. *W.l.o.g.* we delete (a, c) in G . If in the resulting graph $G' = G - (a, c)$ an odd generalized cycle C without triangular chords exists, then it must be of the form $C = \{\dots a, e, c, \dots\}$ for some $e \neq a, b, c, d$. Since G^c contains no $K_{1,3}$, the edge $(d, e) \in E(G)$. But now it is easy to see that $C' = \{\dots a, e, d, e, c, \dots\}$ constitutes an odd generalized cycle without

triangular chords in G (after deletion of possible duplicated edges), contradicting the hypothesis on G .

Next, suppose the deletion of (a, c) causes a $K_{1,3}$ in G^c with, say, $K_{1,3} = [c; a, e, f]$, i.e., c is joined to a, e, f in G^c , $e, f \neq a, b, c, d$. Looking at the subgraph induced by the set $\{b, d, c, f\}$ in G^c , we conclude that f must be joined in G^c to either b or d , suppose w.l.o.g. to d . Similarly e must be joined in G^c to b or d . If e were adjacent to d in G^c , then $[a, d, e, f]$ would be a $K_{1,3}$ in G^c . Thus e must be joined to b in G^c . By the same argument f cannot be adjacent to b in G^c , hence the set $\{a, b, c, d, e, f\}$ induces the full subgraph of fig. 1 in G .

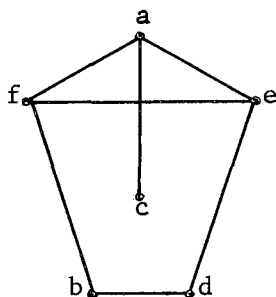


fig. 1

This graph, however, does not permit a partial ordering, in contradiction to the hypothesis on G .

LEMMA 5. Let G be a PO-graph without a $K_{1,3}$ in its complement. Let $[a, b, c, d]$ be a C_4 in G^c , and assume $G' = G - (a, c)$ is a UPO-graph. Then the reinsertion of the edge (a, c) does not destroy the UPO-property of G' .

Proof: The only way the reinsertion of (a, c) could possibly violate the unique partial ordering on G' arises when we have two edges $(a, e), (c, e)$ $e \neq a, b, c, d$, which are both oriented toward or away from e . Assume w.l.o.g. the latter. Since G^c does not contain a $K_{1,3}$, (d, e) exists in G and, since G' is a UPO-graph, (d, e) must be oriented $(e \rightarrow d)$. But now we conclude that all three edges $(a, e), (c, e), (d, e)$ are in the same orientation-block of G' , and the lemma follows.

LEMMA 6. Let G be a connected PO-graph whose complement is connected and does not contain a $K_{1,3}$. Let $[a, b, c, d]$ be a C_4 in G^c and assume $G' = G - (a, c)$ to be a UPO-graph. Then, after reinsertion, (a, c) receives a unique orientation, i.e., G is a UPO-graph.

Proof: According to Theorem 1, the only way the lemma could be false, arises when the edge (a, c) is contained in triangles only, i.e., whenever a vertex x is joined to either a or c , it is joined to both. We show that this assumption contradicts the connectivity of G^c . Since G is connected, there exists at least one point x which is adjacent to a and c . Considering the subgraphs $[a, b, c, x]$ and $[a, c, d, x]$, respectively, we infer the existence of the edges (b, x) and (d, x) in G . Denote by $D(x)$ the component of G^c containing x . We wish to show that $D(x)$ contains none of the points a, b, c, d . Suppose we have already verified that all vertices of distance $\leq k$ in G^c from x are adjacent to all four points a, b, c, d in G . Let y be a point of distance $k + 1$ in G^c from x , then y is adjacent in G^c to some point z at distance k from x in G^c . The subgraph $[a, c, z, y]$ tells us that y must be adjacent to a and c in G . Finally, the subgraphs $[a, b, c, y]$ and $[a, c, d, y]$ establish the existence of (b, y) and (d, y) in G , and our lemma follows by induction.

It is worth noting that the following theorem can also be derived by a direct approach involving a case by case investigation of certain subgraphs. Since that method is rather lengthy and does not throw light on related concepts, we have adopted the present development.

THEOREM 5. Let G be a connected PO-graph whose complement is connected and does not contain a $K_{1,3}$, then G is uniquely partially orderable.

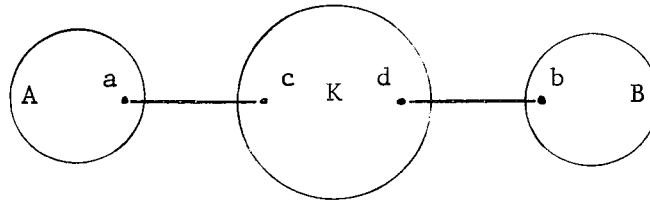
Proof: We proceed by induction on the number of edges in G , keeping the number of points p fixed throughout. The smallest possible number of edges is $p-1$, in which case G is a tree and hence a UPO-graph. Assume then the theorem holds for all graphs G' with $|V(G')| = p$, $|E(G')| \leq e$. Let G satisfy the hypotheses of the theorem with $|E(G)| = e+1$. If there are no C_4 's in G^c , we are finished by Corollary 2. So let us assume the opposite, then two possibilities arise:

- (a) There exists at least one $C_4 = [a, b, c, d]$ in G^c such that at least one of the edges (a, c) , (b, d) is not a cut-edge in G .
- (b) There is no such C_4 in G^c .

Case (a). Suppose w.l.o.g. (a, c) is not a cut-edge in G . We delete (a, c) and invoking Lemma 4, we note that $G' = G - (a, c)$ satisfies the induction hypotheses. Lemmas 5 and 6 now settle this case.

Case b. Let $[a, b, c, d]$ be a C_4 in G^c . Assume w.l.o.g. that we are given the following structure of G as displayed in fig. 2, where the only edges between A, K, B are (a, c) and (b, d) .

fig. 2



If there are no triangles in G , we are finished (Theorem 2). If there were a triangle in A (or B), then the three points making up the triangle plus b (or a) would induce a $K_{1,3}$ in G^c . Finally, a triangle in K contains at most one of the two vertices c, d , hence the triangle plus b or a again induces a $K_{1,3}$ in G^c , contradicting the hypothesis on G .

COROLLARY 3. Let G be a PO-graph with at most one non-trivial component. Let every triangle of G be contained in a connected full subgraph H of G , such that H^c (i.e., the complement with respect to itself) is connected and contains no $K_{1,3}$. Then G is a UPO-graph.

Proof: Take an arbitrary line (x, y) of G . We want to show that $G(x, y) = E(G)$. Assume the opposite, then because G contains at most one non-trivial component there exists a line (a, b) , $(a, b) \notin G(x, y)$, which is incident with some point of $G(x, y)$. If there exists an edge in $G(x, y)$ which is incident with only one of a, b , then clearly the direction of this edge forces the direction of (a, b) , and $(a, b) \in G(x, y)$. If, on the other hand, every point of $G(x, y)$, adjacent to a or b , is adjacent to both, then there exists a triangle $[a, b, c]$, (a, c) or $(b, c) \in G(x, y)$. But by the hypothesis this triangle is contained in a full subgraph H as specified in the theorem. By Theorem 5, H is a UPO-graph, hence both (a, c) and (b, c) force (a, b) , and we have arrived at a contradiction.

We conclude with an example of a UPO-graph which is covered by the corollary, but not by the main theorem.

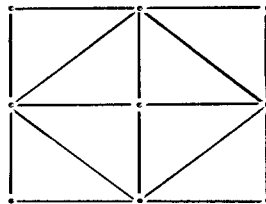


fig. 3

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