

0	6	5	4	9	8	7	1	2	3	8	6	1	3
7	1	0	6	5	9	8	2	3	4	7	0	2	4
8	7	2	1	0	6	9	3	4	5	6	1	3	5
9	8	7	3	2	1	0	4	5	6	0	7	8	9
1	9	8	7	4	3	2	5	6	0	1	9	7	8
3	2	9	8	7	5	4	6	0	1	2	8	9	7
5	4	3	9	8	7	6	0	1	2	3			
2	3	4	5	6	0	1	7	8	9				
4	5	6	0	1	2	3	8	9	7				
6	0	1	2	3	4	5	9	7	8				

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A GENERALIZATION OF SOME GENERALIZATIONS OF SPERNER'S THEOREM<sup>1</sup>

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# A GENERALIZATION OF SOME GENERALIZATIONS OF SPERNER'S THEOREM<sup>1</sup>

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## INTRODUCTION

Sperner proved the following theorem [1]: Let  $A = \{A_1, \dots, A_m\}$  be a family of subsets of a set  $S$  of  $n$  elements. If no two of them possess the property  $A_i \subset A_j$  ( $i \neq j$ ), then  $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ . Erdős answered the question which is the maximum of  $m$  if no  $h+1$  different elements of the family form a chain  $A_{i_1} \subset A_{i_2} \subset \dots \subset A_{i_{h+1}}$ . The answer [2] is the sum of the  $h$  largest binomial coefficients of order  $n$ . Kleitman [3] and Katona [4] independently proved a stronger form of Sperner's theorem: Let  $S_1$  and  $S_2$  be disjoint sets of  $n_1$  and  $n_2$  elements respectively. If  $A = \{A_1, \dots, A_m\}$  is a family of subsets of  $S = S_1 \cup S_2$  and no two different  $A_i, A_j$  satisfy the properties

$$A_i \cap S_1 = A_j \cap S_1 \quad \text{and} \quad A_i \cap S_2 \subset A_j \cap S_2$$

or

$$A_i \cap S_1 \subset A_j \cap S_1 \quad \text{and} \quad A_i \cap S_2 = A_j \cap S_2$$

then  $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ , where  $n = n_1 + n_2$ . De Bruijn, Tengbergen and

Kruyswijk [5] generalized the original theorem of Sperner in the

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following manner: Let  $f_1, \dots, f_m$  be integer-valued functions defined on  $S = \{x_1, \dots, x_n\}$  such that  $0 \leq f_i(x_k) \leq \alpha_k$ , where  $\alpha_k$ 's are given positive integers. If no two different of them satisfy  $f_i(x_k) \leq f_j(x_k)$  (for all  $k$ ), then  $m \leq M$ , where  $M$  is the number of functions satisfying  $\sum_{k=1}^n f(x_k) = \left\lfloor \frac{\sum_{k=1}^n \alpha_k}{2} \right\rfloor$ . Recently, Schönheim [6] gave generalizations of both Erdős's and Kleitman-Katona's results for integer-valued functions. The aim of this paper is to give a common generalization of all these papers in a little more general language.

#### DEFINITIONS AND THE THEOREM

We will say that a directed graph  $G$  is a symmetrical chain graph if

1. There is a partition of its vertices into disjoint subsets  $K_0, K_1, \dots, K_n$  (they are called levels) of  $k_0, k_1, \dots, k_n$  elements and all the directed edges connect a vertex from  $K_i$  with a vertex from  $K_{i+1}$  ( $0 \leq i \leq n$ ).
2.  $k_0 \leq \dots \leq k_{\lfloor \frac{n}{2} \rfloor}$ ;  $k_i = k_{n-i}$  ( $0 \leq i \leq n$ ).
3. There is a partition of its vertices into disjoint symmetrical chains, where symmetrical chain is the set of vertices of a directed path and if the starting point of this directed path is in  $K_i$  then the endpoint is in  $K_{n-i}$ . (The notion of symmetrical chain is introduced in [5].) It is easy to see, 2 is a consequence of 3, that is 2 is not necessary.

Let us consider a set  $S$  of  $n$  elements. Let its subsets be the vertices of the graph  $G$  and connect two vertices  $A$  and  $B$

(from  $A$  to  $B$ ) if  $B \supset A$  and  $|B - A| = 1$ . This is the so called subset-graph. In this case,  $K_i$  is the family of all subsets of  $i$  elements,  $k_i = \binom{n}{i}$ , thus conditions 1 and 2 easily hold. Condition 3 also holds as it is proved in [5] in a more general case.

Similarly, if we consider the set of integer-valued functions satisfying  $0 \leq f(x_k) \leq \alpha_k$  as a vertex-set of a graph  $G$ , then we have to connect two vertices  $f$  and  $g$  (from  $f$  to  $g$ ) if  $f = g$  except one place  $x_k$ , where  $f(x_k) = g(x_k) - 1$ .  $K_i$  is in this case the set of functions for which  $\sum_{k=1}^n f(x_k) = i$ . It is easy to see that condition 1 is satisfied. [5] proves that condition 3 also holds. This is the function graph.

Now we define the direct sum  $G + H$  of two symmetrical chain graphs. Its vertices will be the ordered pairs  $(g, h)$  ( $g \in G, h \in H$ ) and  $(g_1, h_1)$  is connected with  $(g_2, h_2)$  (from  $(g_1, h_1)$  to  $(g_2, h_2)$ ) only if  $g_1 = g_2$  and  $h_1, h_2$  are connected in  $H$  (from  $h_1$  to  $h_2$ ) or  $h_1 = h_2$  and  $g_1, g_2$  are connected in  $G$  (from  $g_1$  to  $g_2$ ).

If  $G$  is the subset-graph of a set  $S_1$  and  $H$  is a subset-graph of a set  $S_2$  ( $S_1$  and  $S_2$  are disjoint), then  $G + H$  is the subset-graph of  $S_1 \cup S_2$ . The situation is the same in the case of function graphs; the direct sum of two function graphs is again a function graph.

The generalization of Sperner's theorem (and also of De Bruijn - Tengbergen-Kruyswijk's theorem) in this language is the following: If we have a set  $\{a_1, \dots, a_m\}$  of vertices of a symmetrical chain-graph and no two of them are connected with a directed path, then

$$m \leq k \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The generalization of Erdős's theorem would sound that if

no  $h+1$  different vertices from  $\{a_1, \dots, a_m\}$  lie in a directed path, (1)

then  $m \leq$  the sum of the  $h$  largest  $k_i$ 's. In a direct sum graph, we will use a weaker condition rather than (1):

THEOREM. Let  $G$  and  $H$  be symmetrical chain-graphs with levels  $K_0, \dots, K_n$  (or  $k_0, \dots, k_n$  elements) and  $L_0, \dots, L_p$  (of  $l_0, \dots, l_p$  elements) respectively. If we have a set  $(g_1, h_1), \dots, (g_m, h_m)$  of vertices of  $G+H$  such that

<u>no <math>h+1</math> different ones of them satisfy the conditions</u>			
$g_{i_1} = \dots = g_{i_w}$ ;		$h_{i_1}, \dots, h_{i_w}$ <u>lie in a directed</u>	} (2)
		<u>path in <math>H</math> (in this order)</u>	
$g_{i_w}, \dots, g_{i_{h+1}}$ <u>lie in a directed</u>		$h_{i_w} = \dots = h_{i_{h+1}}$	
<u>path in <math>G</math> (in this order)</u>			
<u>for some <math>w</math> (<math>1 \leq w \leq h+1</math>),</u>			

then  $m \leq$  the number of vertices of the  $h$  largest levels of  $G+H$ , that is the sum of the  $h$  largest numbers of type  $\sum_{i=0}^{\alpha} k_i l_{\alpha-i}$ .

REMARK 1. If  $G$  and  $H$  are the subset-graphs of the sets  $S_1$  and  $S_2$  of  $n$  and  $p$  elements, respectively, then condition (2)

becomes

there are no  $h+1$  different subsets  $A_1, \dots, A_{h+1}$  in  $S_1 \cup S_2$  such that

$$\left. \begin{aligned} A_1 \cap S_1 = \dots = A_w \cap S_1; & \quad A_1 \cap S_2 \subset A_2 \cap S_2 \subset \dots \subset A_w \cap S_2 \\ A_w \cap S_1 \subset A_{w+1} \cap S_1 \subset \dots \subset A_{h+1} \cap S_1; & \quad A_w \cap S_2 = A_{w+1} \cap S_2 = \dots = A_{h+1} \cap S_2 \end{aligned} \right\}$$

holds for some  $w$  ( $1 \leq w \leq h+1$ ).

(3)

It is clear if (3) would hold, then  $A_1 \subset A_2 \subset \dots \subset A_w \subset \dots \subset A_{h+1}$  also should hold, that is, in this case we have a weaker condition than Erdős's theorem has, but we have the same result. The relation of this special case of our theorem to Erdős's theorem is the same as the relation of Kleitman-Katona's result to Sperner's theorem.

REMARK 2. If we put  $h=1$  in the preceding example, we obtain Kleitman-Katona's result.

REMARK 3. Theorems of Schönheim can be obtained if we use our theorem for function graphs and we put  $h=1$  or we change condition (1) with the stronger condition: no  $h+1$  functions satisfy

$$f_1 \leq \dots \leq f_{h+1} \quad \text{for every } x_k.$$

REMARK 4. Let us consider now an other important special case. Let  $S_1$  be a one-element set, and let the vertices of  $G$  be the "functions"  $f$  defined on  $S_1$  such that  $0 \leq f \leq n$  and  $f$  is an integer. There is a directed edge from  $f$  to  $g$  only if  $g = f+1$ . Thus,  $G$  will be a directed path of length  $n+1$ . Let  $H$  be the same graph with  $p$  instead of  $n$ .  $G+H$  is in this case a rectangular  $(n+1) \times (p+1)$  lattice (Fig. 1).

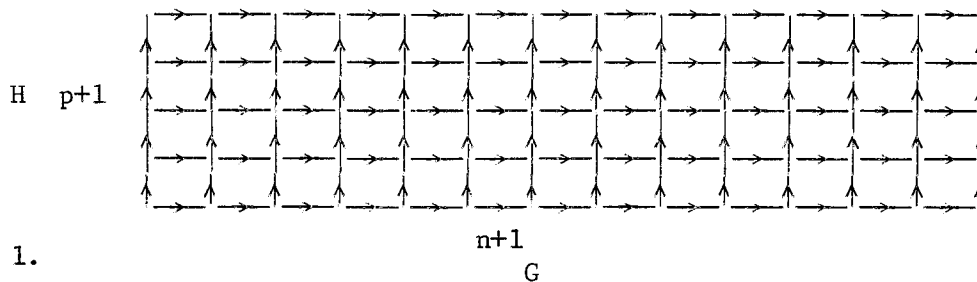


Fig. 1.

In this special case, De Bruijn-Tengbergen-Kruyswijk theorem would state that if we have a set of points of this rectangle no two of them connected with a directed path, then the maximal number of these points is the length of the maximal diagonal (a diagonal is a set of vertices with the same co-ordinate-sum) that is,  $\min(n+1, p+1)$ .

Schönheim's generalization of Erdős's theorem would state in this special case that if we have a set of vertices from this rectangle and

$$\text{no } h+1 \text{ different ones lie in one directed path,} \quad (4)$$

then the maximal number of these points is the sum of the lengths of the  $h$  largest different diagonals.

Our theorem says if we exclude the existency of  $h+1$  different points lying in a directed path which consists of two straight lines (Fig. 2) (instead of (4))

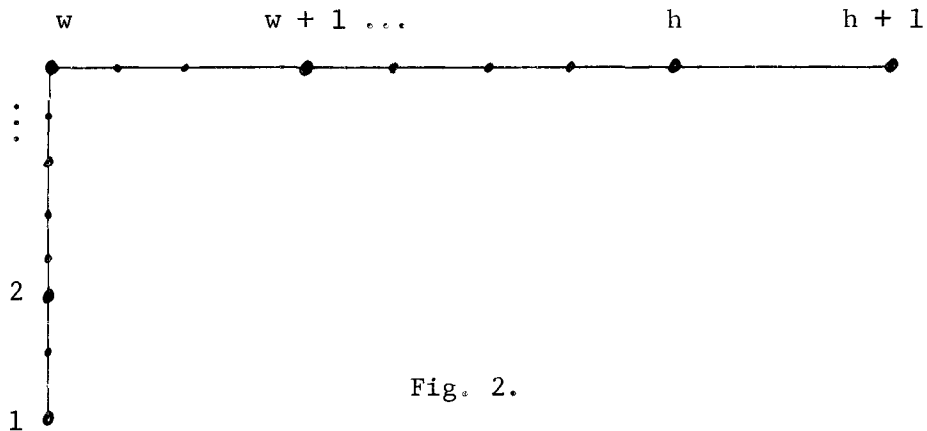


Fig. 2.

we obtain the same maximum. More exactly:

LEMMA. Let  $R$  be a graph with vertices  $(i, j)$   
 $(0 \leq i \leq a; 0 \leq j \leq b; i$  and  $j$  are integers), where there are  
directed edges from  $(i, j)$  only to  $(i, j+1)$  and  $(i+1, j)$ . If we  
have a set of vertices of  $m$  elements such that

there are no  $h+1$  different vertices  $(i_1, j_1), \dots, (i_{h+1}, j_{h+1})$   
with the property

$$\begin{aligned} i_1 = \dots = i_w & & j_1 < \dots < j_w & & (5) \\ i_w < \dots < i_{h+1} & & j_w = \dots = j_{j+1} & & \end{aligned}$$

for some  $w$   $(1 \leq w \leq h+1)$ ,

then  $m \leq$  sum of the lengths of the  $h$  largest different diagonals.

#### PROOFS

PROOF OF THE LEMMA. The set of vertices  $(i_0, j)$ , where  $i_0$  is fixed, and  $0 \leq j \leq b$  is called a column. The rows are defined similarly. Let  $V$  be the set of vertices satisfying the conditions of the lemma and denote by  $c_t$  the number of columns having exactly  $t$  vertices from  $V$ . Obviously, by (5)  $c_t = 0$  if  $t > h$ . Thus

$$\sum_{t=0}^h c_t = a + 1. \quad (6)$$

Let us count in two different ways the number of vertices which are at least  $u$ -th elements of  $V$  in any column starting from below. In a column where the number of elements of  $V$  is less than  $u$  we have to



count 0, so counting column by column, we obtain

$$c_u + 2c_{u+1} + 3c_{u+2} + \dots + (h-u+1)c_h .$$

On the other hand, counting row by row, we obtain that this number is at most  $(h-u+1)(b-u+2)$  because we have not to count the first  $u-1$  rows, and in the other rows we can have at most  $h-u+1$  such points by condition (5). Thus, we have the inequality

$$c_u + 2c_{u+1} + 3c_{u+2} + \dots + (h-u+1)c_h \leq (h-u+1)(b-u+2), \quad (1 \leq u \leq h). \quad (7)$$

We have to maximize  $\sum_{i=0}^h c_i i$  under the conditions (6) and (7). If  $a+1 \leq b-h+2$ , then obviously, the optimal solution is  $c_h = a+1$ ,  $c_{h-1} = \dots = c_1 = 0$ . Assume now that  $a+1 > b-h+2$ . First we will show that there is an optimal solution with  $c_h = b-h+2$ . If we have an optimal solution  $c'_h, c'_{h-1}, \dots, c'_1$  with  $c'_h < b-h+2$  then  $c_h = b-h+2$ ,  $c_{h-1} = c'_{h-1} - 2(b-h+2-c'_h)$ ,  $c_{h-2} = c'_{h-2} + b-h+2 - c'_h$ ,  $c_{h-3} = c'_{h-3}, \dots, c_1 = c'_1$  is also an optimal solution, because  $\sum_{i=1}^h i c_i = \sum_{i=1}^h i c'_i$ , and the validity of (7) for  $c'_i$  results that (7) is also valid for  $c_i$ :

$$\begin{aligned} u = h-1: \quad c_{h-1} + 2c_h &= c'_{h-1} - 2(b-h+2-c'_h) + 2(b-h+2) = \\ &= c'_{h-1} + 2c'_h \leq 2(b-h+2), \end{aligned}$$

$$\begin{aligned} u \leq h-2: \quad c_u + \dots + (h-u-1)c_{h-2} + (h-u)c_{h-1} + (h-u+1)c_h &= \\ &= c_u + \dots + (h-u-1)(c'_{h-2} + b-h+1 - c'_h) \\ &\quad + (h-u)(c'_{h-1} - 2(b-h+2-c'_h)) + (h-u+1)(b-h+2) = \\ &= c'_u + \dots + (h-u-1)c'_{h-2} + (h-u)c'_{h-1} + (h-u+1)c'_h \\ &\leq (h-u+1)(b-u+2). \end{aligned}$$

Thus for this special optimal solution we have

$$c_u + 2c_{u+1} + \dots + (h-u)c_{h-1} \leq (h-u+1)(h-u) \quad (1 \leq u \leq h-1) \quad (8)$$

instead of (7). If  $a+1 \leq b-h+4$ , then the optimal solution is:

$c_{h-1} = 2$  or  $1$ , if  $a+1 = b-h+4$  or  $a+1 = b-h+3$ , respectively,

$c_{h-2} = \dots = c_1 = 0$ . Let us assume that  $a+1 > b-h+4$ . It is easy to

see that there exists an optimal solution for which  $c_{h-1} = 2$

(naturally  $c_h = b-h+2$  is fixed). If we have an other optimal solu-

tion  $c'_{h-1}, \dots, c'_1$  with  $c'_{h-1} < 2$ , then we can change for a more

appropriate one:  $c_{h-1} = 2$ ,  $c_{h-2} = c'_{h-2} - 2(2 - c'_{h-1})$ ,

$c_{h-3} = c'_{h-3} + (2 - c'_{h-1})$ ,  $c_{h-4} = c'_{h-4}$ ,  $\dots$ ,  $c_1 = c'_1$ , which satisfy

(8). Following this procedure, we obtain that there exists an optimal

solution of form

$$c_h^0 = b-h+2, \quad c_{h-1}^0 = 2, \dots, c_{v+1}^0 = 2, \quad c_v^0 = 1 \text{ or } 0, \quad c_{v-1}^0 = \dots = c_1^0 = 0,$$

where  $v$  is determined by (6).

The set of points  $(i, j)$  satisfying  $i+j = k$  is called the  $k$ -th diagonal of the rectangle and it is denoted by  $D_k$ . The  $h$  middle

diagonals are  $D_y, \dots, D_{y+h-1}$ , where  $y = \lfloor \frac{a+b-h+1}{2} \rfloor$ . It is easy to

see, that the number of the  $h$  middle diagonals is just  $\sum_{i=1}^h ic_i^0$ ,

that is, maximal. It means they are also the  $h$  largest diagonals,

because arbitrary  $h$  diagonals satisfy the conditions of the lemma.

The lemma is proved.

PROOF OF THE THEOREM. By point 3 of definition of the symmetrical chain-graphs, the vertices of  $G$  and  $H$  are divisible into symmetrical chains. Denote by  $G'$  and  $H'$  the graphs which have the same vertex-

set as  $G$  and  $H$ , respectively, but they have edges only along these chains. Thus,  $G'$  and  $H'$  is a subgraph of  $G$  and  $H$ , respectively. It follows that  $G' + H'$  is a subgraph of  $G+H$ . So, it is sufficient to prove the theorem for  $G' + H'$  instead of  $G+H$ . However,  $G' + H'$  consists of rectangular lattices and condition (2) means simply condition (5) for every such rectangle.

We know that an optimal set of points in every rectangle is the union of the  $h$  middle diagonals. Define the levels of  $G' + H'$ , or  $G+H$ , in the following manner.  $(g,h) \in M_j$  iff  $g \in K_i, h \in L_{j-i}$  for some  $i$ . By the definition of the direct sum, it is easy to see that  $M_j$ 's satisfy the 1 point of the definition of a symmetrical chain graph. The  $h$  middle levels of  $G' + H'$  are  $M_z, \dots, M_{z+h-1}$ , where  $z = \lfloor \frac{n+p-h+1}{2} \rfloor$ . We will show that the union of the  $h$  middle diagonals for all the rectangles is just the  $h$  middle levels in  $G' + H'$ .

First we verify that an element of the  $h$  middle diagonals in a rectangle is an element of the  $h$  middle levels in  $G' + H'$ . Let us consider a fixed rectangle which is a direct sum of two symmetrical chains from  $G'$  and  $H'$  with vertices  $g_0, \dots, g_a$  and  $h_0, \dots, h_b$ , respectively. If  $g_0 \in K_i$ , then by the symmetricity  $g_a \in K_{n-i}$  and thus  $i+a = n-i$  or

$$i = \frac{n-a}{2}. \quad (9)$$

(Obviously,  $n$  and  $a$  have the same parity.) Similarly, if  $h_0 \in L_j$  then

$$j = \frac{p-b}{2}. \quad (10)$$

If a point  $(g_k, h_\ell)$  is in  $D_r$ , in one of the  $h$  largest diagonal of the rectangle, then

$$k + \ell = r, \quad (11)$$

further,  $g_k \in K_{i+k}$ ,  $h_\ell \in L_{j+\ell}$ , thus  $(g_k, h_\ell) \in M_{i+j+k+\ell}$ , or using (9), (10) and (11)

$$(g_k, h_\ell) \in M_{\frac{n+p-a-b}{2} + r}. \quad (12)$$

Since  $y = \lceil \frac{a+b-h+1}{2} \rceil \leq r \leq y+h-1$ , thus  $z = \lceil \frac{n+p-h+1}{2} \rceil \leq \frac{n+p-a-b}{2} + r \leq z+h-1$ , and (12) means that  $(g_k, h_\ell)$  is in one of the  $h$  middle levels of  $G' + H'$ .

Conversely, let  $(g, h)$  be an element of  $M_s$ , where  $z \leq s \leq z+h-1$ .  $(g, h)$  is contained by one rectangular which is a direct sum of two symmetrical chains, say,  $g_0, \dots, g_a$  and  $h_0, \dots, h_b$ . Then, by (9) and (10)

$$g_0 \in K_{\frac{n-a}{2}} \quad h_0 \in L_{\frac{p-b}{2}}.$$

If  $(g, h) = (g_k, h_\ell)$  then  $\frac{n-a}{2} + \frac{p-b}{2} + k + \ell = s$ , that is, for  $r = k + \ell = s - \frac{n-a}{2} - \frac{p-b}{2}$  the following inequality holds:

$$\lceil \frac{a+b-h+1}{2} \rceil = z - \frac{n+p-a-b}{2} \leq r \leq \lceil \frac{a+b-h+1}{2} \rceil + n-1.$$

$(g, h)$  is really an element of a diagonal from the  $h$  middle ones.

Thus we proved that the points of the  $h$  middle levels form an optimal set. For the union of  $h$  arbitrarily chosen levels of  $G'+H'$ , the conditions of the theorem are satisfied, so the  $h$  middle levels

must be the  $h$  largest ones (but there may be different  $h$  levels with the same size-sum). The number of elements in  $M$  is obviously

$\sum_{i=0}^{\alpha} k_i \ell_{\alpha-i}$ , thus the optimal number is the sum of the  $h$  largest

ones of these numbers. The proof is completed.

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