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TOURNAMENTS THAT ADMIT EXACTLY ONE HAMILTONIAN CIRCUIT

by

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ABSTRACT

In this paper is given a characterization of those tournaments that admit exactly one Hamiltonian circuit. In addition, we determine the number of non-isomorphic tournaments, with n vertices, that admit a unique Hamiltonian circuit.

I. INTRODUCTION

A <u>tournament</u> is a directed graph where, between each two vertices v and w, either the ordered pair (v, w) or (w, v) is an edge, but not both. Tournaments have been studied by a number of authors, and John Moon's book [3] contains a good review and an extensive bibliography on the subject. The present paper gives a solution to a problem raised by Branko Grünbaum (oral communication), namely finding a characterization of those tournaments that contain a unique Hamiltonian circuit. (All paths and circuits are simple and directed, and a path or circuit is <u>Hamiltonian</u> if it passes through all the vertices of the graph.) Frequently in combinatorial mathematics, once a characterization of a class of objects is found, an enumeration theorem follows. Such is the case here where we also give the exact number of non-isomorphic tournaments, with n vertices, which admit a unique Hamiltonian circuit.

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The analogous problem for Hamiltonian paths is easy, and is given by the following theorem due to Rédei [4].

THEOREM. The tournament T_n with n vertices has a unique Hamiltonian path $v_1, v_2, \ldots, v_n$ if and only if T_n is oriented in the following way: the directed edge (v_i, v_j) is in T_n if and only if $i < j$.

II. SOME NOTATION

If S is any set, $|S|$ will be the cardinality of S. $G: (V, E)$ will denote a graph G with vertex set V and edge set E. $T_n: (V, E)$ will denote a tournament with $|V| = n$. For $v \in V$, $\rho(v)[\sigma(v)]$ is the number of edges with v as initial [terminal] vertex. If $P: v_1, \ldots, v_k$ and $Q: w_1, \ldots, w_m$ are paths in a graph, then $P(v_i, v_j)$, for $i \leq j$ denotes the path $v_i, v_{i+1}, \ldots, v_j$, and $P + Q$ denotes the path $v_1, \ldots, v_k, w_2, \ldots, w_m$ provided $v_k = w_1$. Finally, if $G: (V, E)$ is a graph, and $V' \subseteq V$, then $S(V')$ is the section graph determined by V', i.e., the graph with V' as vertex set and whose edges are all those edges in E which connect two vertices in V'.

III. STATEMENT OF THE RESULTS

THEOREM 1. If $C: v_1, \ldots, v_k, v_{k+1}$ ($= v_1$) is a (simple) circuit in a strongly connected tournament $T_n: (V, E)$, then for any vertex $x \in V \sim C$, there exists a (simple) path $P: w_1, \ldots, w_m$ in $V \sim C$ which has x as starting vertex or terminal vertex, and such that P can be inserted between some two adjacent vertices of C to obtain a longer circuit, i.e., such that there exists an i where $(v_i, w_1) \in E$ and $(w_m, v_{i+1}) \in E$ and hence

$$v_i, w_1 + P + w_m, v_{i+1} + C(v_{i+1}, v_i)$$

is a circuit. (P may consist of only the vertex x.)

COROLLARY 1. In any strongly connected tournament T_n, every (simple) circuit C can be extended to a Hamiltonian circuit H, i.e., there exists a Hamiltonian circuit H such that the order of the vertices of C in H is the same as their order in C.

We thus obtain the following result which abounds in the literature (cf. P. Camion [1], J.D. Foulkes [2], J.W. Moon [3]):

COROLLARY 2. A tournament admits a Hamiltonian circuit if and only if it is strongly connected.

PROPOSITION. For $n \geq 5$, a tournament $T_n: (V, E)$ admits a unique Hamiltonian circuit H if and only if the following three conditions hold:

There exist vertices $\mathcal{C} = \{c_1, \ldots, c_k\} \neq \emptyset$ and $\mathcal{D} = \{d_1, \ldots, d_m\}$ (possibly empty) and a circuit

$$H: x, c_1, \ldots, c_k, y, d_1, \ldots, d_m, x$$

where

$$n = m+k+2 \text{ and } \rho(x) = 1 = \sigma(y)$$

and, for $|i - j| \geq 2$,

$$(c_i, c_j) \in E \text{ if } i > j$$

$$(d_i, d_j) \in E \text{ if } i < j \qquad (1)$$

and

$$(d_m, c_1) \in E, \quad (c_k, d_1) \in E \tag{2}$$

and

$$\left.\begin{array}{l} \underline{\text{If}} \ \ i < j, \ \ r \le s, \ \ \underline{\text{and}} \ \ (c_i, d_r) \in E, \\ \underline{\text{then}} \ \ (c_j, d_s) \in E. \end{array}\right\} \tag{3}$$

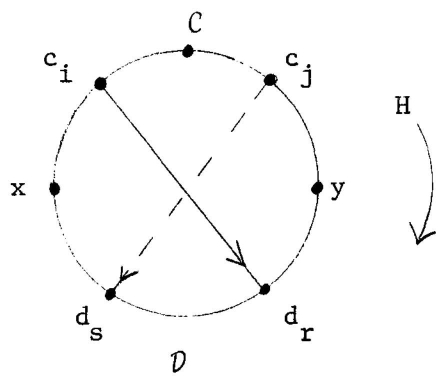

Figure 1. Condition 3

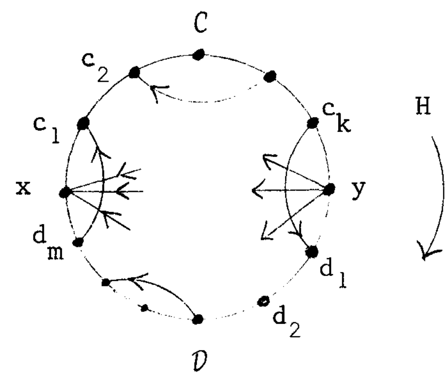

Figure 2.

Notice that (1) completely determines the section graphs $S(\{x, y, c_1, \ldots, c_k\})$ and $S(\{x, y, d_1, \ldots, d_m\})$, while (2) and (3) are conditions on the edges joining the sets $\mathcal{C}$ and $\mathcal{D}$. Also notice that (3) holds if and only if (3) holds for $j = i + 1$.

COROLLARY 3. *If* T_n *is strongly connected, and* $\rho(v) \ge 2$ *for all* v, *or* $\sigma(v) \ge 2$ *for all* v, *then* T_n *admits at least two Hamiltonian circuits*.

We would like to replace the "forbidden subgraph" condition(3) in the Proposition by an explicit description of the orientation on the edges with one end point in $\mathcal{C}$ and the other end point in $\mathcal{D}$. To accomplish this, we now define (see figure 3) a class $\mathcal{T}_n$ of tournaments T_n: (V, E) on $n \geq 5$ vertices:

DEFINITION OF $\mathcal{T}_n$. Let $\mathcal{C} = \{c_1, \ldots, c_k\} \neq \phi$, $\mathcal{D} = \{d_1, \ldots, d_m\}$ (possibly empty), $V = \{x, y\} \cup \mathcal{C} \cup \mathcal{D}$, and let (1) and (2) hold. If $\mathcal{D} = \phi$, then T_n: (V, E) is completely described by (1). If $\mathcal{D} \neq \phi$, fix

$$\left.\begin{array}{l} 0 \leq p \leq \min(m-1, k-1) \\ 1 \leq i_0 < i_1 < \ldots < i_p \leq k \\ 1 = j_0 < j_1 < \ldots < j_p \leq m. \end{array}\right\} \qquad (4)$$

For $j \geq i$ define $[c_i, c_j] = \{c_i, \ldots, c_j\}$, $]c_i, c_j[= \{c_{i+1}, \ldots, c_{j-1}\}$; similar definitions are used for $]c_i, c_j]$, $[d_i, d_j]$, etc. If $S_1 \subseteq \mathcal{C}$ and $S_2 \subseteq \mathcal{D}$, define

$$(S_1, S_2) \in E$$

to mean $(s_1, s_2) \in E$ for all $s_1 \in S_1$ and $s_2 \in S_2$. (A similar definition holds for "$(S_2, S_1) \in E$".) We will inductively describe all the edges between $\mathcal{C}$ and $\mathcal{D}$. We first give the edges with one end point in $[c_1, c_{i_0}]$ and the other in $\mathcal{D}$.

<u>Let</u>

$(c_{i_0}, d_{j_p}) \in E$

$([d_1, d_{j_p}[, c_{i_0}) \in E$

$(\mathcal{D}, [c_1, c_{i_0}[) \in E$

$i_0 = 1 \implies m \neq j_p.$

<u>Also let</u> S_0 <u>be any subset of</u> $]d_{j_p}, d_m] = D_0$ (5)

<u>subject only to the condition that</u>

$i_0 = 1 \implies d_m \notin S_0,$

<u>and let</u>

$(c_{i_0}, S_0) \in E$

$(D_0 \sim S_0, c_{i_0}) \in E.$

This starts the inductive definition. Now, for $1 \le q \le p$, assume the edges with one end point in $[c_1, c_{i_{q-1}}]$ and the other end point in $\mathcal{D}$ are given. We will extend the definition to give the edges joining the sets $[c_1, c_{i_q}]$ and $\mathcal{D}$. To do this, we need to give the edges between $]c_{i_{q-1}}, c_{i_q}]$ and $\mathcal{D}$.

For $1 \le q \le p$, define

$$(]c_{i_{q-1}}, c_{i_q}], [d_{j_{p-q+1}}, d_m]) \in E$$

$$([d_1, d_{j_{p-q+1}}[,]c_{i_{q-1}}, c_{i_q}[) \in E$$

$$(c_{i_q}, d_{j_{p-q}}) \in E$$

$$([d_1, d_{j_{p-q}}[, c_{i_q}) \in E.$$

Also let S_q be any subset of $]d_{j_{p-q}}, d_{j_{p-q+1}}[= D_q$,

and write

$$(c_{i_q}, S_q) \in E$$

$$(D_q \sim S_q, c_{i_q}) \in E. \qquad (6)$$

Finally we let

$$(]c_{i_p}, c_k], \mathcal{D}) \in E. \qquad (7)$$

REMARK. For $q = 0, 1, \ldots, p$,

$$(]c_{i_q}, c_k], [d_{j_{p-q}}, d_m]) \in E.$$

With $n \ge 5$, each choice of the parameters

$$k, p, i_0, \ldots, i_p, 1 = j_0, j_1, \ldots, j_p, S_0, \ldots, S_p \qquad (8)$$

defines, by means of (1), (2), (4), (5), (6), and (7), a tournament T_n: (V, E). Also it will be shown (see the proof of Theorem 2 in Section IV) that any two different choices of parameters (8) result in isomorphically different tournaments. The collection of these tournaments is denoted by $\mathcal{T}_n$.

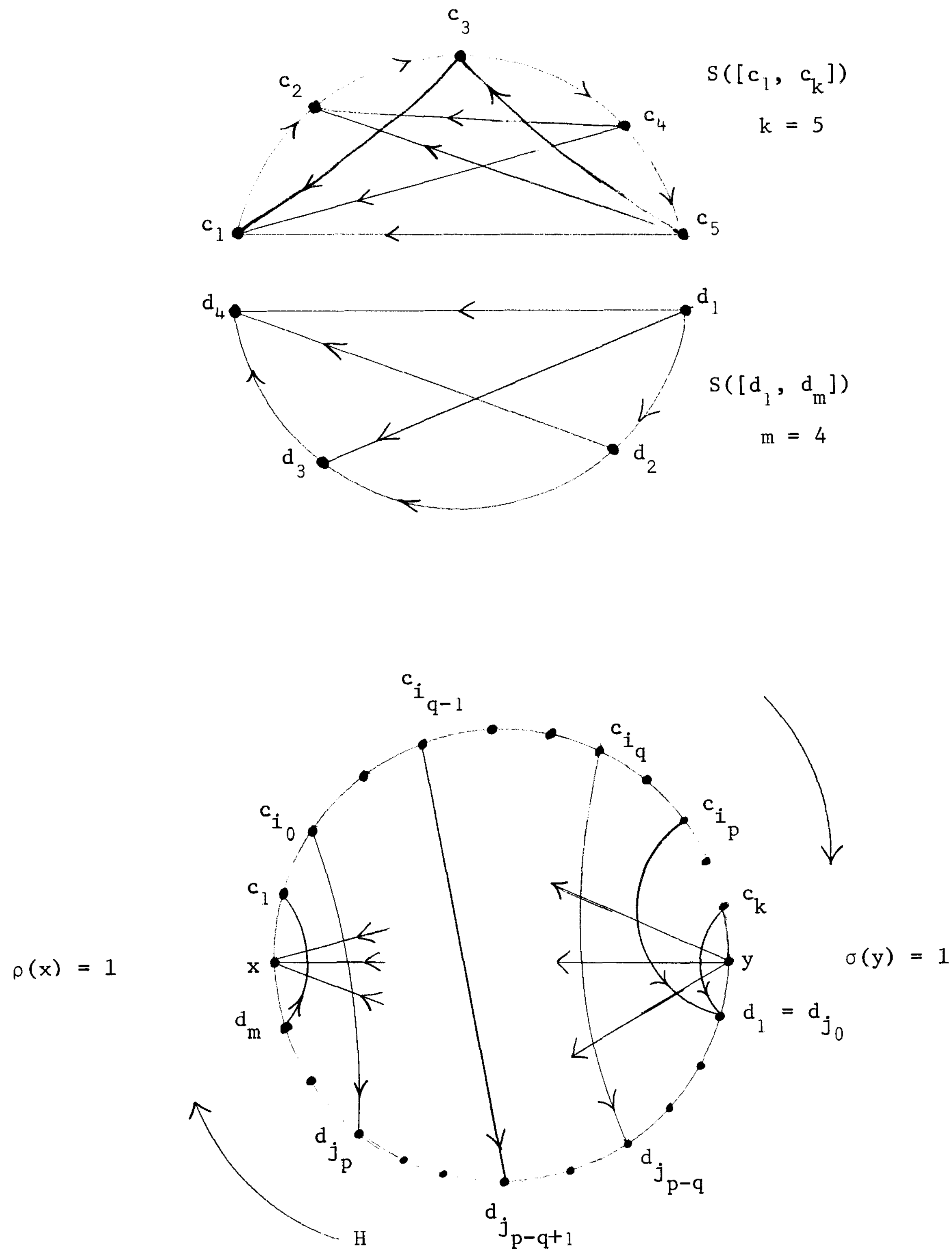
c3
c2
c4
c1
c5
S([c1, ck])
k = 5
d4
d1
S([d1, dm])
m = 4
d3
d2
c_{i_{q-1}}
c_{i_q}
c_{i_0}
c_{i_p}
c1
ck
ρ(x) = 1
x
y
σ(y) = 1
dm
d1 = d_{j_0}
d_{j_p}
d_{j_{p-q}}
H
d_{j_{p-q+1}}

Figure 3.

THEOREM 2. (MAIN THEOREM): An arbitrary tournament T_n, on $n \geq 5$ vertices, admits a unique Hamiltonian circuit H if and only if T_n is isomorphic to an element in $\mathcal{T}_n$. Furthermore, every two different choices for the parameters (8) define isomorphically different tournaments in $\mathcal{T}_n$.

THEOREM 3. For $n \geq 5$, there exist exactly

$$1 + \sum_{k=1}^{n-3} \sum_{p=0}^{\min(k-1,\, n-k-3)} 2^{n-k-p-4}\left[2\binom{n-k-3}{p}\binom{k-1}{p+1} + \binom{n-k-4}{p}\binom{k-1}{p}\right]$$

many non-isomorphic tournaments T_n, on n vertices, that admit a unique Hamiltonian circuit H. (Here $\binom{a}{b} = 0$ if $b > a$.)

IV. THE PROOFS OF THEOREM 1, THE PROPOSITION, THE REMARK, AND THEOREMS 2 AND 3.

PROOF OF THEOREM 1: Let

$$C:\ v_1, \ldots, v_k, v_1$$

be any circuit T_n: (V, E), and suppose x is a vertex of T_n not in C. For each i, $(v_i, x) \in E$ or $(x, v_i) \in E$.

Case 1: For each $i = 1, \ldots, k$, $(x, v_i) \in E$.
There exists a simple path Q in T_n from v_1 to x. Let v_i be the last vertex in Q which is also in C. Then

$$x, v_{i+1} + C(v_{i+1}, v_i) + Q(v_i, x)$$

is a circuit in T_n which is longer than C. (Here $P = Q(w, x)$ where w is the vertex in Q that follows v_i.)

Case 2: Say $(v_i, x) \in E$, $i = 1, \ldots, k$.
Reverse the orientation on all the edges of T_n, apply Case 1, and then again reverse the orientation on all the edges of T_n to obtain the required circuit.

Case 3: Assume there exist p, q such that $(v_p, x) \in E$ and $(x, v_q) \in E$.
Without loss of generality $p = 1$. Let i be the largest integer such that $(v_1, x), (v_2, x), \ldots, (v_i, x)$ are each in E. Hence $1 \leq i < q$ and $(x, v_{i+1}) \in E$. Consequently

$$v_i, x, v_{i+1} + C(v_{i+1}, v_i)$$

is a circuit longer than C. (Here P consists of just the vertex x.) This completes the proof of Theorem 1.

DEFINITION. If H: $v_1, \ldots, v_n, v_1$ is a Hamiltonian circuit in a tournament T_n: (V, E) we define v_i to be a type c vertex (or a c-vertex) if $(v_{i+1}, v_{i-1}) \in E$, and a type a vertex (or an a-vertex) if $(v_{i-1}, v_{i+1}) \in E$. These definitions depend on the given circuit H. ["c" is for the circuit $v_{i-1}, v_i, v_{i+1}, v_{i-1}$; "a" is for "anticircuit".] See Figure 4.

PROOF OF THE PROPOSITION:

Assume T_n: (V, E) has exactly one Hamiltonian circuit

$$H: v_1, \ldots, v_n, v_1.$$

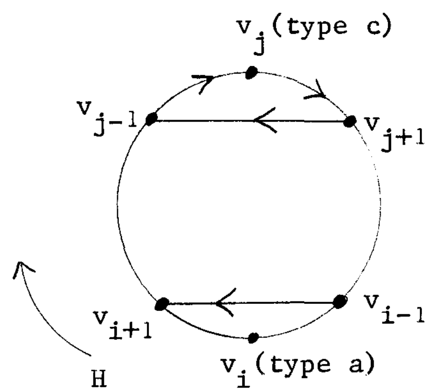

Figure 4.

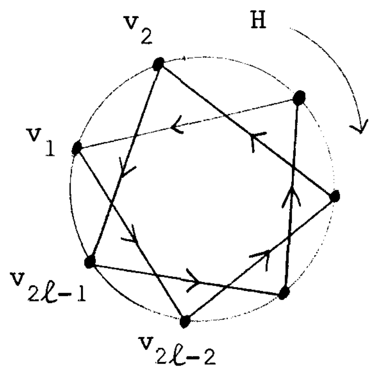

Figure 5.

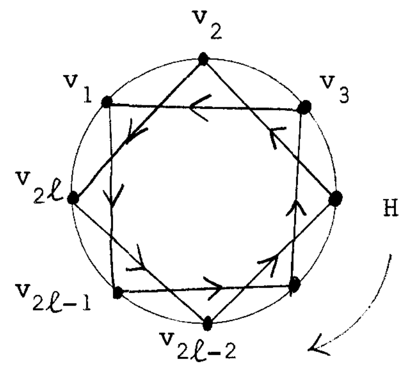

Figure 6.

LEMMA 1. H has at least one c-vertex and one a-vertex.

PROOF: If each v_i is of type c, and $n = 2\ell-1$ is odd, then (see Figure 5)

$$v_1, v_{2\ell-2}, v_{2\ell-4}, \ldots, v_2, v_{2\ell-1}, v_{2\ell-3}, \ldots, v_3, v_1$$

is a Hamiltonian circuit different from H, a contradiction. If each v_i is of type c and $n = 2\ell$ is even, then (see Figure 6) the circuit

$$v_1, v_{2\ell-1}, v_{2\ell}, v_{2\ell-2}, v_{2\ell-4}, \ldots, v_2, v_3, v_1$$

can be extended, by Corollaries 1 and 2, to a Hamiltonian circuit different from H. Now assume each v_i is of type a, $n = 2\ell$, and see Figure 7. If

$$(v_1, v_{2\ell-2}) \in E,$$

then

$$v_{2\ell-3},\ v_{2\ell-1},\ v_1,\ v_{2\ell-2},\ v_{2\ell},\ v_2 + C(v_2, v_{2\ell-3})$$

is another Hamiltonian circuit; while if

$$(v_{2\ell-2},\ v_1) \in E,$$

then

$$v_1,\ v_3,\ v_5,\ \ldots,\ v_{2\ell-1},\ v_{2\ell},\ v_2,\ v_4,\ v_6,\ \ldots,\ v_{2\ell-2},\ v_1$$

is a Hamiltonian circuit different from H. Finally, if each v_i is of type a, and $n = 2\ell-1$, then

$$v_1,\ v_3,\ \ldots,\ v_{2\ell-1},\ v_2,\ v_4,\ \ldots,\ v_{2\ell-2},\ v_1$$

is another Hamiltonian circuit. (See Figure 8.) <u>This final contradiction proves Lemma 1</u>.

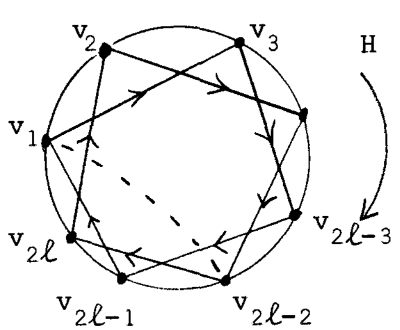

Figure 7.

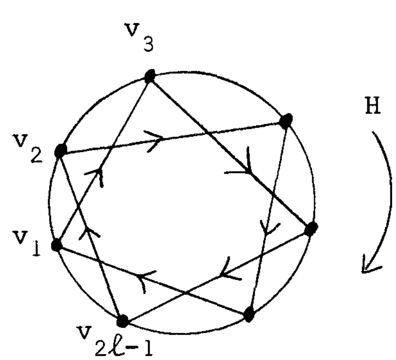

Figure 8.

Let $t(v_i)$ be the type (with respect to H) of the vertex v_i.

LEMMA 2. If $t(v_i) = c$ and $t(v_{i+1}) = a$, then $\sigma(v_{i+1}) = 1$. (See Figure 9.)

Proof: Without loss of generality $i = 2$. Say $\sigma(v_3) > 1$; so there is a j such that $n \geq j > 4$ and $(v_j, v_3) \in E$. But then (see Figure 10) the circuit

$$v_j, v_3, v_1, v_2, v_4 + H(v_4, v_j)$$

can be extended, by Corollaries 1 and 2, to a Hamiltonian circuit different from H, and the contradiction proves Lemma 2.

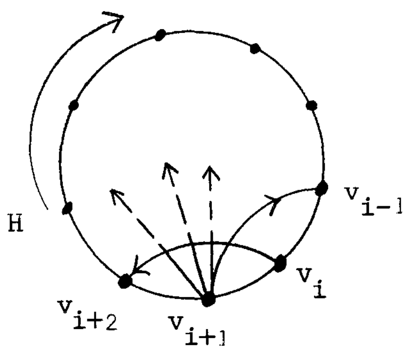

Figure 9.

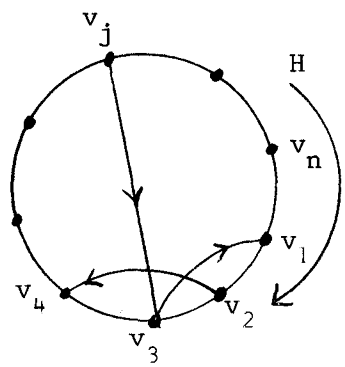

Figure 10.

As a consequence of Lemmas 1 and 2, we have that there exists exactly one i such that $t(v_i) = c$ and $t(v_{i+1}) = a$; furthermore, $t(v_{i+2}) = a$. Without loss of generality, we can, therefore, say that the following situation holds:

$$\left.\begin{array}{l} \text{For some } k, \ 1 \le k \le n-2, \\ t(v_j) = c \quad \text{for} \quad j = 1, \ldots, k \\ t(v_j) = a \quad \text{for} \quad j = k+1, \ldots, n. \end{array}\right\} \tag{9}$$

Since $t(v_n) = a$ and $t(v_1) = c$, it follows from Lemma 2 that

$$\rho(v_n) = 1. \tag{10}$$

To see this, reverse the orientation on all the edges of T_n, apply Lemma 2, and then again reverse all orientations.

Therefore, using (9), (10) and Lemma 2, we can write H in the following way:

$$H: \ x, c_1, \ldots, c_k, y, d_1, \ldots, d_m, x$$

where

$$\rho(x) = 1 = \sigma(y)$$

and $C = \{c_1, \ldots, c_k\} \neq \emptyset$ is the set of all the c-vertices, and $\mathcal{D} \cup \{x, y\}$ is the set of all the a-vertices, where $\mathcal{D} = \{d_1, \ldots, d_m\}$ (which may be empty). So $k \ge 1$ and $m + k + 2 = n$. (See Figure 2.)

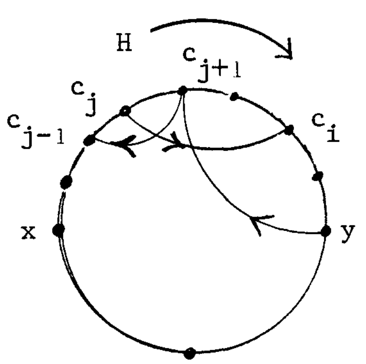

Figure 11.

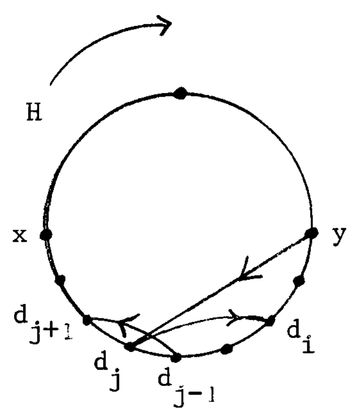

Figure 12.

Hence (2) holds, and (1) will hold if for $|i-j| \geq 3$ we can prove that

$$(c_i, c_j) \in E \quad \text{for} \quad i > j$$

$$(d_i, d_j) \in E \quad \text{for} \quad i < j .$$

Assume there exist i and j such that $i - j \geq 3$ and $(c_j, c_i) \in E$; see Figure 11. Then

$$y,\ c_{j+1},\ c_{j-1},\ c_j,\ c_i + H(c_i, y)$$

is a circuit which can be extended to a Hamiltonian circuit different from H, a contradiction. (Here we assumed $j > 1$; if $j = 1$, replace c_{j-1} by x in the above.) Now say there exist i and j such that $j - i \geq 3$ and $(d_j, d_i) \in E$; see Figure 12. Then

$$y,\ d_j,\ d_i + H(d_i, d_{j-1}) + d_{j-1},\ d_{j+1} + H(d_{j+1}, y)$$

is a circuit which can be extended to a Hamiltonian circuit different from H. (If $j = m$, replace d_{j+1} by x in the above.) *This contradiction proves* (1).

We now will show that (3) holds. Say this is not the case, i.e., there exist $i < j$ and $r \leq s$ where $(c_i, d_r) \in E$, yet $(d_s, c_j) \in E$.

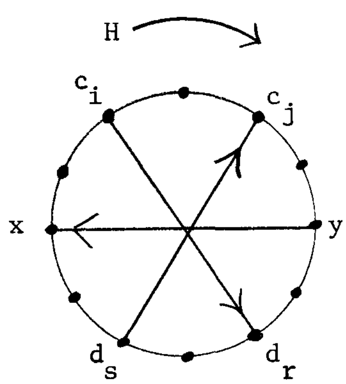

Figure 13.

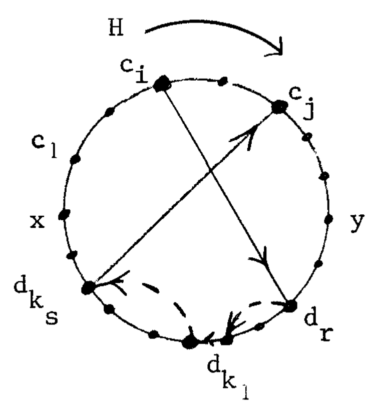

Figure 14.

Then (see Figure 13) the circuit

$$c_i, d_r + H(d_r,d_s) + d_s, c_j + H(c_j,y) + y, x + H(x, c_i)$$

can be extended to another Hamiltonian circuit; <u>hence</u> (3) <u>is</u> <u>proved</u>.

To complete the proof of the Proposition, we must show that if a tournament T_n, on n vertices, satisfies (1), (2), and (3), then H is the only Hamiltonian circuit in T_n. Assume H' is a Hamiltonian circuit different from H. Edge $(x, c_1) \in H'$ as $\rho(x) = 1$. Let $i \geq 1$ be the largest integer such that the path

$$P: \; x, c_1, \ldots, c_i$$

is a subpath of H', yet $(c_i, c_{i+1}) \notin H'$. (Such an i exists as if $x, c_1, \ldots, c_k$ is a path in H', then $H = H'$ as $\sigma(y) = 1$ and because there do not exist r an s such that $s > r$ and $(d_s, d_r) \in E$.) Let $(c_i, d_r) \in H'$ and see Figure 14. By (3), there does not exist a $j > i$ with $(d_r, c_j) \in H'$. As $\sigma(y) = 1$ and because, by (1), there do not exist u and v such that $v > u$ and $(d_v, d_u) \in E$,

$$P + c_i, d_r, d_{k_1}, d_{k_2}, \ldots, d_{k_s}, c_j$$

must be a path in H' for some s where $r < k_1 < k_2 < \ldots < k_s$ and $j > i$. But then we have $(c_i, d_r) \in E$ and $(d_{k_s}, c_j) \in E$ which contradicts (3). <u>This proves the Proposition</u>.

<u>PROOF OF THE REMARK</u>: Let

$$A_q =]c_{i_q}, c_k] \quad \text{and} \quad B_q = [d_{j_{p-q}}, d_m].$$

Now $(A_p, B_p) \in E$ by (7), so it suffices to show that $(A_q, B_q) \in E$ implies $(A_{q-1}, B_{q-1}) \in E$. Note that $A_q \subseteq A_{q-1}$ and $B_{q-1} \subseteq B_q$, whence $(A_q, B_{q-1}) \in E$, and we are done as $(A_{q-1} \sim A_q, B_{q-1}) \in E$ by (6).

PROOF OF THEOREM 2:

We notice that by the Proposition we are to show that a tournament belongs to $\mathcal{T}_n$ if and only if (1), (2), and (3) are true. First we assume (1), (2), and (3) hold for a tournament T_n, and we will show that $T_n \in \mathcal{T}_n$. We are done if $\mathcal{D} = \phi$, so say $\mathcal{D} \neq \phi$. Let i_0 be the smallest index such that $(c_{i_0}, d_j) \in E$ for some j, and define t_0 as the smallest index such that $(c_{i_0}, d_{t_0}) \in E$. Certainly i_0 exists as $(c_k, d_1) \in E$. Given i_{q-1} and t_{q-1}, let i_q be the smallest index $> i_{q-1}$ such that $(c_{i_q}, d_j) \in E$ for some $j < t_{q-1}$, and define t_q as the smallest index $< t_{q-1}$ such that $(c_{i_q}, d_{t_q}) \in E$. Continue this procedure until we are forced to stop, say with i_p and t_p. For $q = 0, 1, \ldots, p$, define

$$j_{p-q} = t_q,$$

and notice that $j_0 = 1$ as $(c_k, d_1) \in E$. Clearly (4) holds. (See Figure 3.) Now

$$i_0 = 1 \implies j_p \neq m$$

as $(c_{i_0}, d_{j_p}) \in E$ and $(d_m, c_1) \in E$. Define

$$S_0 = \{d_j\colon\ j_p < j \le m \quad \text{and} \quad (c_{i_0}, d_j) \in E\},$$

and for $q = 1, \ldots, p$, we define

$$S_q = \{d_j\colon\ j_{p-q} < j < j_{p-q+1} \quad \text{and} \quad (c_{i_q}, d_j) \in E\},$$

and note that

$$i_0 = 1 \implies d_m \notin S_0$$

as $(d_m, c_1) \in E$ by (2). From the above definitions, we immediately obtain (5), and, except for

$$(]c_{i_{q-1}}, c_{i_q}], [d_{j_{p-q+1}}, d_m]) \in E \tag{11}$$

we also obtain (6). But so far we have not used the assumption that (3) holds for T_n. Using (3), we conclude that (7) and (11) hold, and it therefore follows, using the above parameters, that $T_n \in \mathcal{T}_n$.

We next show that if $T_n \in \mathcal{T}_n$, then T_n satisfies (1), (2), and (3). The only thing to prove is that if $\mathcal{D} \neq \emptyset$, then (3) holds. Let

$$i < j, \quad r \le s, \quad \text{and} \quad (c_i, d_r) \in E.$$

Case 1: For some q, $1 \le q \le p$, $c_i \in [c_{i_{q-1}}, c_{i_q}[$.

By our definition of $\mathcal{T}_n$, $d_r \in [d_{j_{p-q+1}}, d_m]$. Now $r \le s$ implies $d_s \in [d_{j_{p-q+1}}, d_m]$, and $i < j$ implies $c_j \in]c_{i_{q-1}}, c_k]$. Hence, by the Remark, $(c_j, d_s) \in E$.

Case 2: $c_i \in [c_{i_p}, c_k]$.

Thus $c_j \in]c_{i_p}, c_k]$, whence, by (7), $(c_j, d_s) \in E$.

Case 3: $c_i \in [c_1, c_{i_0}[$.

By definition of $\mathcal{T}_n$, $(\mathcal{D}, [c_1, c_{i_0}[) \in E$, whence $(d_r, c_i) \in E$, which shows that the hypothesis of (3) cannot hold in Case 3.

Since these three cases exhaust all the possibilities, we have proved the first part of Theorem 2.

Turning to the second part of Theorem 2, we first note that, by the definition of $\mathcal{T}_n$, (1) and (2) hold for any $T_n \in \mathcal{T}_n$, and from (1) and (2) we see that the collection $\mathcal{C}$ in (1) is the set of all c-vertices in T_n (with respect to H) and $\{x, y\} \cup \mathcal{D}$ is the set of a-vertices (with respect to H). Now let T_n^1 and T_n^2 be isomorphic tournaments in $\mathcal{T}_n$. For $i = 1, 2$, let

$$H^i: v_1^i, \dots, v_n^i, v_1^i$$

be the Hamiltonian circuit in T_n^i given by (1). The image of H^1 under the isomorphism must be H^2 because T_n^2 has only one Hamiltonian circuit. Without loss of generality, we assume v_j^1 and v_j^2 correspond, for all j. Then, for all j, v_j^1 is a c-vertex (with respect to H^1) if and only if v_j^2 is a c-vertex (with respect to H^2). Using this, it easily follows that the corresponding parameters for T_n^1 and T_n^2 are equal, which proves Theorem 2.

PROOF OF THEOREM 3:

Fix $n \geq 5$. By Theorem 2, we are to determine $|\mathcal{T}_n|$. Define $\mathcal{T}'_n[\mathcal{T}''_n]$ as the set of all those tournaments $T_n \in \mathcal{T}_n$ such that $\mathcal{D} \neq \emptyset$ and $i_0 > 1[i_0 = 1]$.

We first determine $|\mathcal{T}'_n|$:

We have $k \geq 1$, $\mathcal{D} \neq \emptyset$, and $m = n - k - 2 \geq 1$. As noted in Section III, the section graphs $S(\{x, y, c_1, \ldots, c_k\})$ and $S(\{x, y, d_1, \ldots, d_m\})$ are uniquely determined by n and k. We have

$$1 < i_0 < i_1 < \ldots < i_p \leq k$$

$$1 = j_0 < j_1 < \ldots < j_p \leq n\text{-}k\text{-}2$$

where

$$0 \leq p \leq \min(k-1, n-k-3).$$

Also, for $q = 1, \ldots, p$,

$$S_0 \subseteq]d_{j_p}, d_m] \quad \text{and} \quad S_q \subseteq]d_{j_{p-q}}, d_{j_{p-q+1}}[.$$

Hence, for fixed k and p, there exist $\binom{k-1}{p+1}$ choices for the sequence $i_0, i_1, \ldots, i_p$, $\binom{n-k-3}{p}$ choices for the sequence $j_0, j_1, \ldots, j_p$, 2^{m-j_p} choices for S_0, and $2^{j_{p-q+1}-j_{p-q}-1}$ choices for each S_q $(q = 1, \ldots, p)$. Therefore we have the following:

$$|\mathcal{T}'_n| = \sum_{k=1}^{n-3} \sum_{p=0}^{\min(k-1,n-k-3)} \sum_{\substack{1=j_0<j_1<\ldots<j_p\leq n-k-2 \\ 1<i_0<\ldots<i_p\leq k}} \phi(n,k,p) \tag{12}$$

where the inner sum, for fixed k and p, is taken over all sequences $i_0, \ldots, i_p, j_0, \ldots, j_p$ such that the indicated equality and inequalities hold, and where

$$\phi \equiv \phi(n, k, p) = 2^{m-j_p} \cdot \prod_{q=1}^{p} 2^{j_{p-q+1}-j_{p-q}-1}$$

is the number of choices for the sequence $S_0, S_1, \ldots, S_p$ for a given $j_0, j_1, \ldots, j_p$. But ϕ reduces to

$$2^{m-j_p} \cdot 2^{\sum_{q=1}^{p}(j_{p-q+1}-j_{p-q}-1)} = 2^{m-j_p} \cdot 2^{j_p-j_0-p} = 2^{n-k-p-3},$$

and hence is independent of the choice of $j_0, \ldots, j_p$ (and of course $i_0, \ldots, i_p$). Thus (12) reduces to

$$|T'_n| = \sum_{k=1}^{n-3} \sum_{p=0}^{\min(k-1,n-k-3)} 2^{n-k-p-3} \cdot \binom{n-k-3}{p}\binom{k-1}{p+1} .$$

<u>We now determine</u> $|T''_n|$:

If $i_0 = 1$, then $j_p \neq m$ and $d_m \notin S_0$. Hence $S_0 \subseteq]d_{j_p}, d_m[$, and there exist 2^{m-j_p-1} choices for S_0, $\binom{k-1}{p}$ choices for $1 = i_0 < \ldots < i_p \leq k$, and $\binom{n-k-4}{p}$ choices for $1 = j_0 < \ldots < j_p < n-k-2$. With these changes, we proceed as in the first case to obtain:

$$|T''_n| = \sum_{k=1}^{n-3} \sum_{p=0}^{\min(k-1,n-k-3)} \sum_{\substack{1=j_0<\ldots<j_p<n-k-2 \\ 1=i_0<\ldots<i_p\leq k}} 2^{m-j_p-1} \cdot 2^{j_p-j_0-p} =$$

$$= \sum_{k=1}^{n-3} \sum_{p=0}^{\min(k-1,n-k-3)} 2^{n-k-p-4} \binom{n-k-4}{p} \binom{k-1}{p} .$$

Finally, as there exists exactly one $T_n \in \mathcal{T}_n$ with $\mathcal{D} = \emptyset$, we have

$$|\mathcal{T}_n| = 1 + |\mathcal{T}'_n| + |\mathcal{T}''_n| ,$$

<u>which proves Theorem 3</u>.

V REMARKS

From Theorem 3, we obtain the following table:

T(n), the number of non-isomorphic tournaments, with n vertices, that admit a unique Hamiltonian circuit

n	3	4	5	6	7	8	9	10
T(n)	1	1	3	8	21	55	144	377

The three non-isomorphic tournaments in $\mathcal{T}_5$:

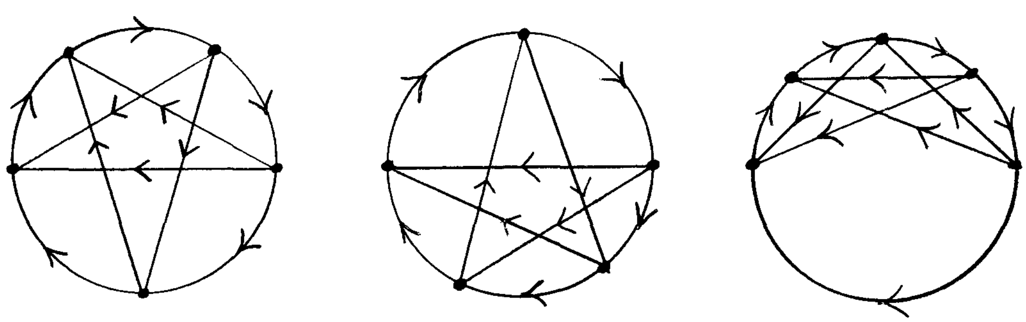

Figure 15.

The eight non-isomorphic tournaments in T_6:

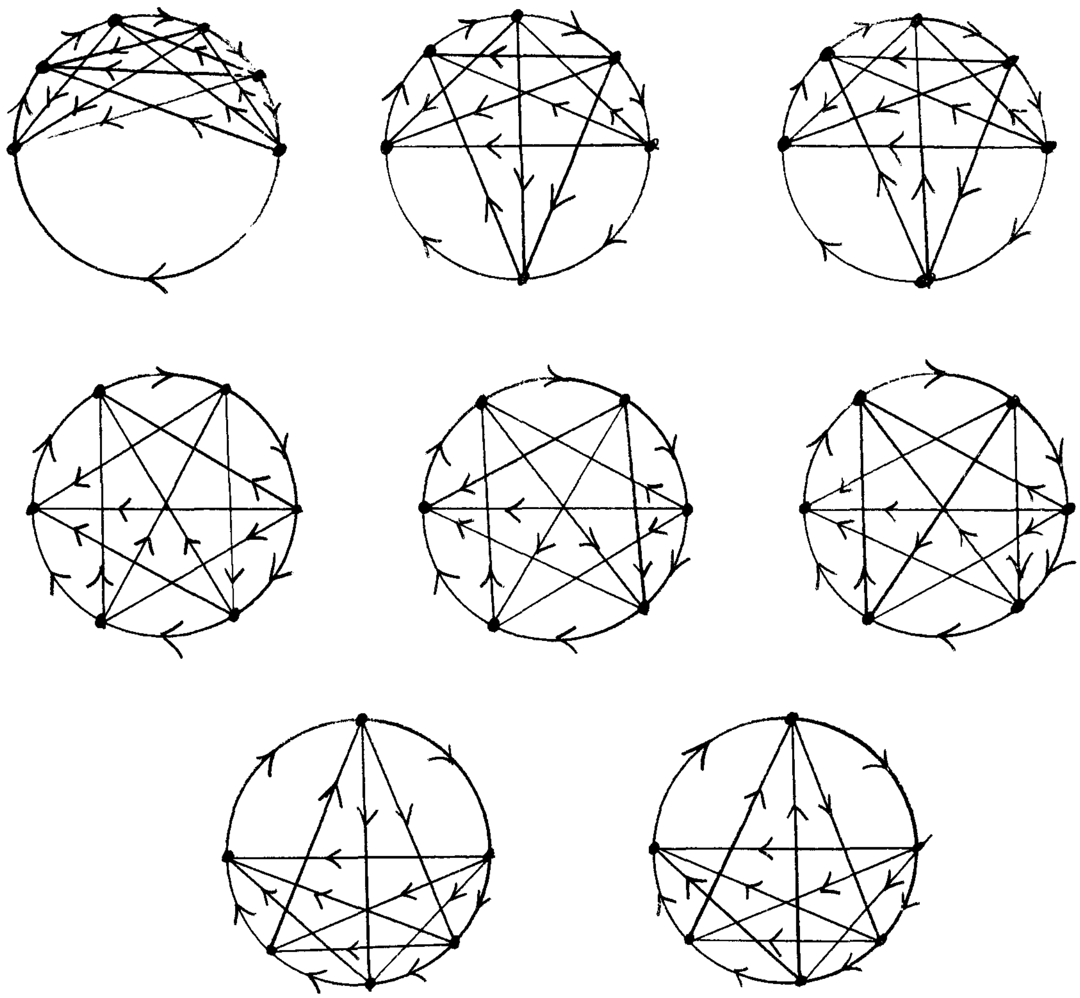

Figure 16.

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